Nonuniform Sampling of Bandlimited Signals with Polynomial Growth on the Real Axis

Ahmed I. Zayed and Antonio G. García, *Member, IEEE*

***Abstract*— We derive a sampling expansion for bandlimited signals with polynomial growth on the real axis. The sampling expansion uses nonuniformly spaced sampling points. But unlike other known sampling expansions for such signals, ours converge uniformly to the signal on any compact set. An estimate of the truncation error of such a series is also obtained.**

***Index Terms*— Bandlimited signals, nonuniform sampling, sampling theorems.**

I. Introduction

The Shannon sampling theorem, also known as the Whittaker–Shannon–Kotel'nikov sampling theorem [12], asserts that if F is a signal bandlimited to $[-\pi, \pi]$, i.e., it can be written in the form

$$F(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(w)e^{-itw}\,dw$$

for some $f \in L^2(-\pi, \pi)$, then it can be reconstructed from its samples at the points $t_n = n$, $n \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, by means of the formula

$$F(t) = \sum_{n=-\infty}^{\infty} F(t_n)\frac{\sin \pi(t-t_n)}{\pi(t-t_n)}. \tag{1}$$

If the sampling points $\{t_n\}_{n\in\mathbb{Z}}$ are not equidistant, still F can be reconstructed from its values at these points by using a generalization of Shannon's theorem, known as the Paley–Wiener–Levinson sampling theorem. This theorem states that if $\{t_n\}_{n\in\mathbb{Z}}$ is a sequence of real numbers such that

$$|t_n - n| \le D < \tfrac{1}{4}, \qquad \text{for all } n \in \mathbb{Z} \tag{2}$$

then any signal bandlimited to $[-\pi, \pi]$ can be reconstructed from its values at the points $\{t_n\}_{n\in\mathbb{Z}}$ by means of the formula

$$F(t) = \sum_{n=-\infty}^{\infty} F(t_n)\frac{G(t)}{(t-t_n)G'(t_n)} \tag{3}$$

where

$$G(t) = (t-t_0)\prod_{n=1}^{\infty}\left(1-\frac{t}{t_n}\right)\left(1-\frac{t}{t_{-n}}\right). \tag{4}$$

Let us denote the class of signals bandlimited to $[-\sigma, \sigma]$ by B_σ^2. It is worth mentioning that

$$\psi_n(t) = G(t)/[(t-t_n)G'(t_n)] \in B_\sigma^2$$

for all n. Another famous theorem of Paley and Wiener [7] asserts that if $F \in B_\sigma^2$, it is an entire function of exponential type at most σ whose restriction to the real axis is square-integrable, i.e., F is an entire function that satisfies

$$|F(z)| \le Ae^{\sigma|z|}, \qquad \text{for some } A > 0$$

and

$$\int_{-\infty}^{\infty} |F(x)|^2\,dx < \infty, \qquad \text{where } x = \operatorname{Re} z.$$

For such a signal F, its energy E is represented by the L^2-norm, e.g.,

$$E^2 = \int_{-\infty}^{\infty} |F(x)|^2\,dx.$$

It is also known that bandlimited signals are bounded on the real axis. But since many signals in practical applications do not have finite energy according to the above definition, such as power signals, which are signals satisfying the condition

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|F(t)|^2\,dt < \infty$$

it becomes evident that the class B_σ^2 of bandlimited signals is not large enough to include many signals of practical applications.

A number of generalizations of the class B_σ^2 have been introduced. For example, the class $B_\sigma^p(1 \le p < \infty)$ is defined as the class of all entire functions of exponential type at most σ that belong to $L^p(\mathbb{R})$ when restricted to the real axis. Likewise, the class B_σ^∞ is defined as the class of all entire functions of exponential type at most σ that are bounded on the real axis. It is known that $B_\sigma^p \subset B_\sigma^q \subset B_\sigma^\infty$, for $1 \le p \le q$; see [12, p. 24].

More generalizations of the class B_σ^2 have also been considered. Zakai [11], for example, introduced a class of bandlimited signals F in which the signals are entire functions of exponential type that are not necessarily bounded on the real axis. In fact, they satisfy the relation

$$\int_{-\infty}^{\infty}\frac{|F(x)|^2}{1+x^2}\,dx < \infty.$$

Not only the class of bandlimited signals B_σ^2, but also its associated sampling theorems have been the focus of many generalizations; see [12, ch.2] for details.

Unlike the class B_σ^2, the class B_σ^p for $p > 2$ is not defined in terms of the Fourier transform of its members, unless the Fourier transform is taken in the sense of generalized functions, since functions in $L^p(\mathbb{R})$, for $p > 2$, do not in general have Fourier transforms in the classical sense. If we allow the Fourier transform to be taken in the sense of generalized functions or Schwartz distributions, then the class of bandlimited signals can be enlarged tremendously, allowing bandlimited signals to be not only unbounded but also of polynomial growth on the real axis. The constant function, which represents a power signal, can be regarded as a bandlimited signal since its Fourier transform is essentially the Dirac delta function, $\delta(x)$, which is a generalized function with compact support.

Sampling theorems for signals that are the Fourier transforms of generalized functions with compact support have been studied by many people. To the best of our knowledge, the first to do so was Campbell [1], who derived sampling expansions for signals that are the Fourier transform of generalized functions with compact support using uniformly distributed sampling points. His expansion converges pointwise, but lacks the familiar appearance of the Shannon sampling expansion (1). Pfaffelhuber [8] obtained a sampling expansion that

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A. I. Zayed is with the Department of Mathematics, University of Central Florida, Orlando, FL 32816 USA.

A. G. García is with the Departamento de Matemáticas, Universidad Carlos III de Madrid, 28911 Leganés, Spain.

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restored the form (1), but its convergence was weakened from pointwise to convergence in the sense of tempered distributions. His results were generalized further by Lee [5], and Hoskins and De Sousa Pinto [4]. In [9], Walter introduced a convergence factor in the sampling expansions obtained by Lee and Pfaffelhuber, and also examined the (C, α) summability of these expansions. He showed that if F is a bandlimited signal with polynomial growth on the real axis, then the series (1) is (C, α) summable to F provided that F is oversampled and α is chosen sufficiently large.

In [10], Walter extended some of his results in [9] to nonuniform sampling expansions. Among other things, he showed that for bandlimited signals with polynomial growth on the real axis, the series (3) converges to F in the sense of ultradistributions, or more precisely in the sense of $\mathcal{Z}_\pi^*$, where $\mathcal{Z}_\pi^*$ is the topological dual space of the space $\mathcal{Z}_\pi$. The space $\mathcal{Z}_\pi$ is the image under the Fourier transformation of the space $\mathcal{D}_\pi$ consisting of all infinitely differentiable functions with support in $(-\pi, \pi)$, and provided with its usual topology described in [13].

In this correspondence, we study the nonuniform sampling expansions of bandlimited signals with polynomial growth on the real axis in a way parallel to that of Walter [10]. But unlike Walter and the others mentioned above, we obtain a nonuniform sampling expansion of a signal F that converges uniformly to F on compact sets. The price we pay for obtaining uniform convergence is the lack of the familiar form (3) in our sampling series. Our sampling functions are given in a form of a convolution involving the function $G(t)$ defined in (4). This convolution structure of the sampling functions is not unusual; similar forms have already appeared in the work of Feichtinger and Gröchenig [2], [3], where the sampling expansions of a large class of functions are shown to converge uniformly on compact sets.

II. Preliminaries

We define the Fourier transform of a function $f(t)$ as

$$\hat{f}(w) = \int_{-\infty}^{\infty} f(t) e^{iwt}\, dt$$

so that its inverse transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) e^{-iwt}\, dw.$$

The convolution of two functions f and g is defined as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t) g(x-t)\, dt$$

hence

$$(\widehat{f * g})(w) = \hat{f}(w)\hat{g}(w).$$

For any open interval K, we define the space $\mathcal{D}_K$ as the space of all infinitely differentiable functions $\varphi(x)$ with compact support in K. The support of φ is defined as the smallest closed set containing the set $\{x : \varphi(x) \neq 0\}$. The support of φ will be denoted by $\operatorname{supp} \varphi$. If $K = (a, b)$, then the support of any $\varphi \in \mathcal{D}_K$ is contained in $[a+\epsilon, b-\epsilon]$ for some $\epsilon > 0$. A sequence $\{\varphi_n\}_{n=0}^{\infty}$ in $\mathcal{D}_K$ is said to converge to zero in $\mathcal{D}_K$ if there is a compact set $K_1 \subset K$ such that $\operatorname{supp} \varphi_n \subset K_1$ for all n and $\varphi_n^{(p)} \to 0$ uniformly as $n \to \infty$ for all $p = 0, 1, 2, \cdots$. The topological dual space of $\mathcal{D}_K$ will be denoted by $\mathcal{D}_K^*$. A sequence $\{f_n\}_{n=0}^{\infty}$ in $\mathcal{D}_K^*$ is said to converge to 0 in $\mathcal{D}_K^*$ if for any $\varphi \in \mathcal{D}_K$, we have $\langle f_n, \varphi \rangle \to 0$ as $n \to \infty$, where $\langle f, \varphi \rangle$ is the number that the functional f assigns to φ.

In this article we shall be concerned mainly with the space $\mathcal{D}_K$ where $K = (-\pi, \pi)$ and to simplify the notation we shall denote this space by $\mathcal{D}_\pi$ and its dual by $\mathcal{D}_\pi^*$.

To prove our results, we shall need some facts from the theory of nonharmonic analysis. For the reader's convenience, we have collected in the following theorem the most important results needed for our forthcoming discussion. We call this theorem the Paley–Wiener–Levinson (PWL) theorem. The proofs of these results can be found in [6, ch.IV] and [7, ch.VII].

Theorem 2.1 (PWL Theorem): Let $\{t_n\}$ be a sequence of real numbers such that

$$|t_n - n| \leq D < \tfrac{1}{4}, \qquad \text{for all } n \in \mathbb{Z}.$$

Then the following are true:

1) The set of functions $\{e^{it_n x}\}$ is complete in $L^2[-\pi, \pi]$ and possesses a unique, complete biorthonormal set $\{h_n(x)\}$ such that

$$\int_{-\pi}^{\pi} h_m(x) e^{it_n x}\, dx = \delta_{m,n}$$

where $\delta_{m,n}$ is the Kronecker symbol.

2) If $f \in L^2[-\pi, \pi]$, then the series

$$\sum_{n=-\infty}^{\infty} (\hat{f}_n e^{inx} - f_n e^{it_n x}) \tag{5}$$

converges uniformly to zero over any interval $[-\pi + \epsilon, \pi - \epsilon]$, for $\epsilon > 0$, where

$$\hat{f}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx \tag{6}$$

and

$$f_n = \int_{-\pi}^{\pi} f(x) h_n(x)\, dx. \tag{7}$$

Moreover

$$(1 - \pi\sqrt{D})^2 \sum_{n=-\infty}^{\infty} |f_n|^2 \leq ||f||^2 = \int_{-\pi}^{\pi} |f(x)|^2\, dx \tag{8}$$

(see [7, eqs. (29.04) and (30.07)]).

3)

$$h_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{G(t)}{(t - t_n) G'(t_n)} e^{-ixt}\, dt$$

where

$$G(t) = (t - t_0) \prod_{n=1}^{\infty} \left(1 - \frac{t}{t_n}\right)\left(1 - \frac{t}{t_{-n}}\right).$$

4) If $f \in L^2[-\pi, \pi]$, then

$$f(x) = \sum_{n=-\infty}^{\infty} \tilde{f}_n h_n(x)$$

where

$$\tilde{f}_n = \int_{-\pi}^{\pi} f(x) e^{it_n x}\, dx. \tag{9}$$

III. The Sampling Theorem

To prove our sampling theorem, the following lemmas will be needed. For simplicity, we shall assume from now on that $t_0 = 0$ so that

$$\int_{-\pi}^{\pi} h_m(x)\, dx = \delta_{m,0}$$

and denote the set of all functions orthogonal to $h_0(x)$ by $H_0^{\perp}$.

Lemma 3.1: Let φ have p continuous derivatives and assume that the support of φ is contained in $(-\pi, \pi)$. If

$$\varphi^{(i)} \in H_0^{\perp}, \qquad i = 0, 1, \cdots, p$$

then the sequences $\{\varphi_n\}_{n=-\infty}^{\infty}$ and $\{\tilde{\varphi}_n\}_{n=-\infty}^{\infty}$, defined by (7) and (9), satisfy

$$\varphi_n, \tilde{\varphi}_n = O\left(\frac{1}{n^p}\right), \qquad \text{as } |n| \to \infty.$$

Proof: We begin with the sequence $\{\tilde{\varphi}_n\}$. By integrating by parts p times, we obtain

$$\tilde{\varphi}_n = \frac{(i)^p}{(t_n)^p} \int_{-\pi}^{\pi} \varphi^{(p)}(x) e^{it_n x}\, dx$$

which implies

$$|\tilde{\varphi}_n| \le \frac{C}{|t_n|^p}, \qquad \text{for all } n \text{ and } t_n \neq 0.$$

For $t_n = 0$, we have $|\tilde{\varphi}_0| \le C$. But since $t_n \sim n$ as $|n| \to \infty$, the result follows.

As for the sequence $\{\varphi_n\}$, we have in view of (5) that the series

$$\sum_{n=-\infty}^{\infty} (\widehat{\varphi_n}^{(p)} e^{inx} - \varphi_n^{(p)} e^{it_n x})$$

converges uniformly in $[-\pi + \epsilon, \pi - \epsilon]$, in particular, it converges at $x = 0$, where $\{\hat{\varphi}_n^{(p)}\}$ and $\{\varphi_n^{(p)}\}$ are the coefficients of $\varphi^{(p)}(x)$ as given by (6) and (7). Thus the series

$$\sum_{n=-\infty}^{\infty} (\hat{\varphi}_n^{(p)} - \varphi_n^{(p)})$$

converges, which implies that

$$\lim_{|n| \to \infty} (\varphi_n^{(p)} - \hat{\varphi}_n^{(p)}) = 0$$

and hence for any $\epsilon > 0$ there exists $N > 0$ such that

$$|\varphi_n^{(p)}| \le |\hat{\varphi}_n^{(p)}| + \epsilon, \qquad \text{for all } |n| \ge N.$$

But it is easy to see that $\hat{\varphi}_n^{(p)} = (in)^p \hat{\varphi}_n$ and $\varphi_n^{(p)} = (it_n)^p \varphi_n$. To show the latter, let us note that if $\varphi(x)$ and $\varphi^{(1)}(x)$ are in $H_0^{\perp}$, then

$$\varphi^{(1)}(x) = \sum_{n \neq 0} \varphi_n^{(1)} e^{it_n x}$$

and by integrating this series, we obtain

$$\varphi(x) = \sum_{n \neq 0} \varphi_n e^{it_n x} = \sum_{n \neq 0} \frac{\varphi_n^{(1)}}{(it_n)} e^{it_n x} + C$$

which implies that $C = 0$. This leads to $\varphi_n^{(1)} = (it_n)\varphi_n$ for $n \neq 0$, and the result now follows by induction. Therefore,

$$|t_n|^p |\varphi_n| \le |n|^p |\hat{\varphi}_n| + \epsilon, \qquad \text{for all } |n| \ge N.$$

Because $\{\hat{\varphi}_n\}$ are the Fourier coefficients of a p-differentiable function with compact support, we have that $|n|^p |\hat{\varphi}_n| \le A$ for all n. Thus since $t_n \sim n$ as $|n| \to \infty$, we have

$$|\varphi_n| \le \frac{B}{|t_n|^p} = \frac{B}{|n|^p} \frac{|n|^p}{|t_n|^p} \le \frac{C}{|n|^p}, \qquad \text{for all } n \neq 0$$

where $A, B,$ and C are constants. □

Corollary 3.1: Let $\varphi(x)$ be an infinitely differentiable function with support in the open interval $(-\pi, \pi)$ such that φ and all its derivatives are in $H_0^{\perp}$. Then the sequences $\{\varphi_n\}_{n=-\infty}^{\infty}$ and $\{\tilde{\varphi}_n\}_{n=-\infty}^{\infty}$ are rapidly decreasing, i.e.,

$$\varphi_n, \tilde{\varphi}_n = O\left(\frac{1}{n^p}\right), \qquad \text{as } |n| \to \infty \quad \text{for all } p \ge 0.$$

Let

$$\mathcal{A} = \{\varphi \in L^2[-\pi, \pi]: \{\varphi_n\}_{n=-\infty}^{\infty} \text{ is rapidly decreasing}\}.$$

The set $\mathcal{A}$ is nonempty as can be seen from Corollary 3.1, and it is also closed under differentiation. Let $\tilde{\mathcal{D}}_\pi$ be the set of all functions in $\mathcal{D}_\pi$ that belong to $\mathcal{A}$.

Lemma 3.2: Let $\tilde{\mathcal{D}}_\pi^*$ be the topological dual space of the space $\tilde{\mathcal{D}}_\pi$ and $\mathcal{B}_\pi^*$ be that subspace of $\tilde{\mathcal{D}}_\pi^*$ consisting of all generalized functions with support in $(-\pi, \pi)$. Then the nonharmonic Fourier series,

$$\sum_{n=-\infty}^{\infty} \varphi_n e^{it_n x}$$

of any $\varphi \in \tilde{\mathcal{D}}_\pi$ converges to φ in $\tilde{\mathcal{D}}_\pi$, and for any $f \in \mathcal{B}_\pi^*$ and $\varphi \in \tilde{\mathcal{D}}_\pi$, we have

$$\langle f, \varphi \rangle = \sum_{n=-\infty}^{\infty} \varphi_n \tilde{f}_n$$

where $\tilde{f}_n = \langle f, e^{it_n x} \rangle$ and φ_n are defined by (7).

Proof: Since f has compact support and $e^{it_n x}$ is infinitely differentiable for each n, the coefficients $\tilde{f}_n$ are well defined. The Fourier series of any $\varphi \in \tilde{\mathcal{D}}_\pi$ converges to the 2π-periodic extension, $\varphi_{\rm per}$, of φ, in the topology of $\mathcal{E}$, the space of all infinitely differentiable functions. But on $(-\pi, \pi), \varphi_{\rm per} = \varphi$; therefore, the Fourier series of φ converges to it in the topology of $\tilde{\mathcal{D}}_\pi$. Thus by Theorem 2.1, the nonharmonic Fourier series

$$\sum_{n=-\infty}^{\infty} \varphi_n e^{it_n x}$$

of φ converges to φ in $\tilde{\mathcal{D}}_\pi$ too. Now by the continuity of f, we obtain

$$\begin{aligned} \langle f, \varphi \rangle &= \left\langle f, \sum_{n=-\infty}^{\infty} \varphi_n e^{it_n x} \right\rangle = \sum_{n=-\infty}^{\infty} \varphi_n \langle f, e^{it_n x} \rangle \\ &= \sum_{n=-\infty}^{\infty} \varphi_n \tilde{f}_n \end{aligned}$$

which shows that the series converges. □

Some kind of a converse of this lemma also exists.

Lemma 3.3: Let f be a generalized function with support in $[-\pi, \pi]$ and define $\tilde{f}_n$ by $\tilde{f}_n = \langle f, e^{it_n x} \rangle$ for all n.

Then the series

$$\sum_{n=-\infty}^{\infty} \tilde{f}_n h_n(x)$$

converges in $\tilde{\mathcal{D}}_\pi^*$ to f and $\tilde{f}_n$ are of polymonial growth as $|n| \to \infty$.

Proof: As in the proof of Lemma 3.2, the coefficients $\tilde{f}_n$ are well-defined. Since f has compact support, there exist a continuous function g and a nonnegative integer q such that $f(x) = D^q g(x)$ in the sense of generalized functions, where $D = d/dx$; see [13, p. 93]. Thus

$$\tilde{f}_n = \langle f, e^{it_n x} \rangle = \langle D^q g, e^{it_n x} \rangle = (-it_n)^q \langle g, e^{it_n x} \rangle$$

and hence

$$|\tilde{f}_n| \le C_1 |t_n|^q \le C|n|^q, \qquad \text{for all } n \neq 0$$

for some constants C_1 and C.

It is evident that

$$S_N(x) = \sum_{|n| \le N} \tilde{f}_n h_n(x)$$

defines a generalized function with support in $[-\pi, \pi]$. Therefore, for any $\varphi \in \tilde{\mathcal{D}}_\pi$, we have

$$\langle S_N(x), \varphi(x) \rangle = \sum_{|n| \le N} \varphi_n \tilde{f}_n.$$

But since $\{\tilde{f}_n\}$ is of polynomial growth and $\{\varphi_n\}$ is of rapid decay, we have

$$\lim_{N\to\infty} \langle S_N(x), \varphi(x) \rangle = \lim_{N\to\infty} \sum_{|n|\le N} \varphi_n \tilde{f}_n = \sum_{n=-\infty}^{\infty} \varphi_n \tilde{f}_n$$

exists, but on the other hand, as in Lemma 3.2

$$\langle f, \varphi \rangle = \left\langle f, \sum_{n=-\infty}^{\infty} \varphi_n e^{it_n x} \right\rangle = \sum_{n=-\infty}^{\infty} \varphi_n \tilde{f}_n.$$

Therefore,

$$f(x) = \sum_{n=-\infty}^{\infty} \tilde{f}_n h_n(x)$$

in $\tilde{\mathcal{D}}^*_\pi$. □

It should be noted that Lemma 3.3 is not exactly the converse of Lemma 3.2 since the series

$$\sum_{n=-\infty}^{\infty} \tilde{f}_n h_n(x)$$

does not, in general, define a generalized function with support in the interval $(-\pi, \pi)$.

Now we are able to state and prove our sampling theorem.

Theorem 3.1: Let $f \in \mathcal{B}^*_\pi$ and choose an even function $\gamma(x)$ such that $\hat{\gamma} \in \mathcal{C}^\infty(-\pi, \pi)$ with $\operatorname{supp} \hat{\gamma} \subset (-\pi, \pi)$, $\hat{\gamma}(x) = 1$ on the support of f, and $e^{itx}\hat{\gamma}(x) \in \mathcal{A}$.

Then

$$F(t) = \langle f(x), e^{itx} \rangle$$

is an entire function of exponential type $\pi - \epsilon$ for some $\epsilon > 0$ that grows no faster than a polynomial on the real axis as $|t| \to \infty$. Moreover, F can be reconstructed from its samples at the points $\{t_n\}$ via

$$F(t) = \sum_{n=-\infty}^{\infty} F(t_n) S_n(t) \tag{10}$$

where

$$S_n(t) = (\psi_n * \gamma)(t)$$

with

$$\psi_n(t) = \frac{G(t)}{(t - t_n)G'(t_n)}$$

and the points $\{t_n\}$ are any points satisfying (2).

The series converges uniformly on any compact subset of the real axis.

Proof: That F is an entire function of exponential type $\pi - \epsilon$ with the prescribed growth rate is a known fact in the theory of generalized functions; see [13, ch.7].

Now

$$F(t) = \langle f(x), e^{itx} \rangle = \langle f(x), e^{itx}\hat{\gamma}(x) \rangle.$$

But since $\eta(x,t) = e^{itx}\hat{\gamma}(x)$ is an infinitely differentiable function in x with support in $(-\pi, \pi)$, it belongs to the space $\mathcal{D}_\pi$, and hence it has an expansion of the form

$$\eta(x,t) = \sum_{n=-\infty}^{\infty} \eta_n(t) e^{it_n x} \tag{11}$$

which converges to it in the sense of $\tilde{\mathcal{D}}_\pi$; see Lemma 3.2. Thus

$$F(t) = \sum_{n=-\infty}^{\infty} \eta_n(t) \langle f, e^{it_n x} \rangle = \sum_{n=-\infty}^{\infty} F(t_n) \eta_n(t) \tag{12}$$

with

$$\eta_n(t) = \int_{-\pi}^{\pi} \eta(t,x) h_n(x)\, dx = \int_{-\pi}^{\pi} \hat{\gamma}(x) h_n(x) e^{itx}\, dx.$$

Recognizing $\eta_n(t)$ as the Fourier transform of $\hat{\gamma}(x)h_n(x)$, we can, with the aid of the fact that γ is even, write $\eta_n(t)$ in the form

$$\eta_n(t) = \int_{-\infty}^{\infty} \psi_n(u)\gamma(t-u)\, du = (\psi_n * \gamma)(t) = S_n(t)$$

which upon its substitution in (12) yields (10).

To show that the series in (10) converges uniformly on compact sets, we recall that $F(t_n) = O(|t_n|^q)$ for some $q \ge 0$. The sampling functions $S_n(t)$ are the coefficients of an infinitely differentiable function $\eta(t,x) \in \mathcal{A}$, with support in $(-\pi, \pi)$ when expanded in the nonharmonic Fourier series (11), and therefore they are rapidly decreasing in n for each fixed t. This means that $|S_n(t)| \le C(t)/|t_n|^p$ for sufficiently large n and each fixed t, for all $p \ge 0$.

To complete the proof, we must show that for all t in some compact set $K, |C(t)| \le C_K$, independent of t. First, we differentiate (11) p times with respect to x to obtain

$$\frac{\partial^p \eta(t,x)}{\partial x^p} = \sum_{n=-\infty}^{\infty} (it_n)^p \eta_n(t) e^{it_n x}$$

then apply (8) to obtain

$$(1 - \pi\sqrt{D})^2 \sum_{n=-\infty}^{\infty} |t_n|^{2p} |\eta_n(t)|^2 \le \int_{-\pi}^{\pi} \left| \frac{\partial^p \eta(t,x)}{\partial x^p} \right|^2 dx. \tag{13}$$

By Leibniz rule, we have

$$\begin{aligned} \left| \frac{\partial^p \eta(t,x)}{\partial x^p} \right| &= \left| \sum_{k=0}^{p} \binom{p}{k} (it)^k e^{itx} \hat{\gamma}^{(p-k)}(x) \right| \\ &\le \sum_{k=0}^{p} \binom{p}{k} |t|^k |\hat{\gamma}^{(p-k)}(x)| \le 2^p \sum_{k=0}^{p} |t|^k |\hat{\gamma}^{(p-k)}(x)|. \end{aligned}$$

Let K be a compact set and choose A such that $K \subset [-A, A]$. Then

$$\left| \frac{\partial^p \eta(t,x)}{\partial x^p} \right| \le 2^p A^p \sum_{k=0}^{p} |\hat{\gamma}^{(p-k)}(x)| = 2^p A^p \rho(x)$$

which, upon its substitution in (13), yields

$$(1 - \pi\sqrt{D})^2 \sum_{n=-\infty}^{\infty} |t_n|^{2p} |\eta_n(t)|^2 \le 2^{2p} A^{2p} C^2_{\hat{\gamma}}$$

where $C_{\hat{\gamma}} = ||\rho||_2$, and

$$\rho(x) = \sum_{k=0}^{p} |\hat{\gamma}^{(p-k)}(x)|.$$

Thus

$$|S_n(t)| = |\eta_n(t)| \le \frac{2^p A^p C_{\hat{\gamma}}}{(1 - \pi\sqrt{D})|t_n|^p}, \qquad \text{for all } p \ge 0.$$

Hence, having in mind that $t_n \sim n$ as $|n| \to \infty$ and taking $p > q + 1$, the uniform convergence of the series (10) is proven. □

IV. TRUNCATION ERRORS

As a byproduct of Theorem 3.1 we can give an estimate of the truncation error which arises if one ignores all the samples outside a finite interval. More precisely, we have the following corollary.

Theorem 4.1: Let us define the truncation error $E_N(t)$ as follows:

$$E_N(t) = F(t) - \sum_{|n|\le N} F(t_n)S_n(t).$$

Then

$$|E_N(t)| \le \frac{2^{2p-q} b A^p C_{\hat{\gamma}}}{(1-\pi\sqrt{D})(p-q-1)N^{p-q-1}}, \quad t \text{ in the compact set } K \tag{14}$$

where $p > q+1$, and q the polynomial order of growth of $F(t)$.

Proof: We know from Theorem 3.1 that $|F(t_n)| \le b|t_n|^q$ for some $q \ge 0$ and a constant b, and

$$|S_n(t)| \le \frac{2^p A^p C_{\hat{\gamma}}}{(1-\pi\sqrt{D})|t_n|^p} = \frac{C_K}{|t_n|^p}, \quad \text{for all } p \ge 0 \text{ and } t \text{ in the compact set } K.$$

Hence, if we take any $p > q+1$, we will have

$$\begin{aligned}|E_N(t)| &\le \sum_{|n|>N} |F(t_n)|\,|S_n(t)| \\ &\le bC_K \sum_{|n|>N} \frac{1}{|t_n|^{p-q}} \\ &\le bC_K \left(\sum_{n=-\infty}^{-(N+1)} \left(\left| n + \frac{1}{4} \right| \right)^{q-p} + \sum_{n=N+1}^{\infty} \left(n - \frac{1}{4} \right)^{q-p} \right) \\ &\le 2bC_K \int_{N-(1/4)}^{\infty} \frac{1}{(x-\frac{1}{4})^{p-q}}\, dx.\end{aligned}$$

If we use the change of variable, $x - 1/4 = Nt$, and note that $1/2 \le [N-(1/2)]/N, \forall N \ge 1$, we obtain

$$\begin{aligned}|E_N(t)| &\le 2bC_K \int_{N-(1/4)}^{\infty} \frac{1}{(x-\frac{1}{4})^{p-q}}\, dx \le \frac{2bC_K}{N^{p-q-1}} \int_{1/2}^{\infty} \frac{dt}{t^{p-q}} \\ &= \frac{2bC_K}{2^{q-p+1}(p-q-1)N^{p-q-1}}.\end{aligned}$$

Therefore, replacing the constant C_K by its value, we obtain the desired result (14). □

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Covering Numbers for Real-Valued Function Classes

P. L. Bartlett, *Member, IEEE*,
S. R. Kulkarni, *Senior Member, IEEE*,
and S. E. Posner

***Abstract*—We find tight upper and lower bounds on the growth rate for the covering numbers of functions of bounded variation in the $\mathcal{L}_1$ metric in terms of all the relevant constants. We also find upper and lower bounds on covering numbers for general function classes over the family of $\mathcal{L}_1(dP)$ metrics in terms of a scale-sensitive combinatorial dimension of the function class.**

***Index Terms*— Bounded variation, covering numbers, fat-shattering dimension, metric entropy, scale-sensitive dimension, VC dimension.**

I. INTRODUCTION

Covering numbers have been studied extensively in a variety of literature dating back to the work of Kolmogorov [10], [12]. They play a central role in a number of areas in information theory and statistics, including density estimation, empirical processes, and machine learning (see, for example, [4], [8], and [16]). Let $\mathcal{F}$ be a subset of a metric space $(\mathcal{X}, \rho)$. For a given $\epsilon > 0$, the metric covering number $\mathcal{N}(\epsilon, \mathcal{F}, \rho)$ is defined as the smallest number of sets of radius ϵ whose union contains $\mathcal{F}$. (We omit ρ if the context is clear.)

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P. L. Bartlett is with the Department of Systems Engineering, Australian National University, Canberra 0200, Australia.

S. R. Kulkarni is with the Department of Electrical Engineering, Princeton University, Princeton, NJ 08544 USA.

S. E. Posner is with ING Baring Securities, New York, NY 10021 USA.

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