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New sampling formulae for the fractional Fourier transform

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Abstract

In this note we obtain two new sampling formulae for reconstructing signals that are band limited or time limited in the fractional Fourier transform sense. In both cases, we use samples from both the signal and its Hilbert transform, but each taken at half the Nyquist rate. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

The fractional Fourier transform (FRFT) has been investigated in a number of papers [1–3,12,17] and has proved to be useful in solving some problems in quantum physics, optics, and signal processing [4,7–13]. The operational properties of the FRFT have also been the subject of some recent papers [3,17].

The Hilbert transform is also known to play an important role in signal analysis and optics. An optical implementation of the Hilbert transform was introduced in 1950 by Kastler [6], who used it for image processing, especially for edge enhancement. In 1996, Lohmann et al. [7] generalized the Hilbert transform by introducing two different definitions of what they called the *fractional Hilbert transform*. The two definitions are not equivalent. The first is a modification of the spatial filter with a fractional parameter, while the second is based on the author's work on the fractional Fourier transform. They also showed how these fractional Hilbert transforms can be easily implemented optically. In [18], Zayed introduced another generalization of the Hilbert transform in order to obtain the analytic part of a signal that is associated with the signal's FRFT, i.e., that part of the signal that is obtained by suppressing the negative frequencies of the signal's FRFT.

Sampling expansions for the fractional Fourier transform of band-limited and time-limited signals have been derived in [11,16] and they can be used to reconstruct the signal or its fractional Fourier transform from their samples at a discrete set of points satisfying the Nyquist rate.

The purpose of this letter is to derive two new sampling expansions to reconstruct the fractional Fourier transform of a time-limited or band-limited signal using samples of the signal and its Hilbert transform, each at half the Nyquist rate. This is an analogue of Goldman's classical result on reconstructing a band-pass signal using samples of the signal and its Hilbert transform, each taken at half the Nyquist rate [5]; see also [15, pp. 66–67].

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2. Preliminaries

The fractional Fourier transform with angle α of a signal $f(t)$ is defined as

$$\mathscr{F}_\alpha[f](u) = F_\alpha(u) = \int_{-\infty}^{\infty} f(t) K_\alpha(u,t)\,\mathrm{d}t, \tag{1}$$

where

$$K_\alpha(u,t) = \frac{c(\alpha)}{\sqrt{2\pi}} \exp\{\,\mathrm{j}a(\alpha)[(t^2+u^2) - 2b(\alpha)ut]\}$$

if $\alpha \neq 0, \pi/2, \pi$, with $a(\alpha) = (\cot\alpha)/2$, $b(\alpha) = \sec\alpha$, and $c(\alpha) = \sqrt{1 - \mathrm{j}\cot\alpha}$, and

$$K_0(u,t) = \delta(t-u), \qquad K_{\pi/2}(u,t) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\mathrm{j}ut},$$

$$K_\pi(u,t) = \delta(t+u).$$

Hence these special values of α yield the following FRFT of f: $\mathscr{F}_0[f](u) = f(u)$, $F_{\pi/2}[f](u) = \hat{f}(u)$, $\mathscr{F}_\pi[f]\,(u) = f(-u)$, where $\hat{f}$ denotes the ordinary Fourier transform of f. Therefore, from now on we shall confine our attention to F_α for $\alpha \neq 0, \pi/2, \pi$.

The inversion formula of the FRFT is

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\alpha(u) K_{-\alpha}(u,t)\,\mathrm{d}u. \tag{2}$$

The Hilbert transform of a signal $f(t)$ is defined as (see [14])

$$\mathscr{H}[f](t) = \tilde{f}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x)}{t-x}\,\mathrm{d}x \tag{3}$$

and the analytic part of f is defined as $F(t) = f(t) + \mathrm{j}\tilde{f}(t) = f(t) + \mathrm{j}\mathscr{H}[f](t)$. One of the most important properties of analytic signals is that they contain no negative frequency components. This fact is used to derive sampling expansions for bandpass signals using samples from both the signal and its Hilbert transform, but each taken at half the Nyquist rate (see [15, p. 67]).

The Hilbert transform (3) can be considered as a convolution transform,

$$\tilde{f}(x) = \frac{1}{\pi}(f * g)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) g(x-t)\,\mathrm{d}t,$$

where $g(t) = 1/t$. Thus, the system transfer function of the Hilbert filter is readily seen to be

$$H(\omega) = \hat{g}(\omega) = \sqrt{\frac{\pi}{2}}\,\mathrm{j}\,\mathrm{sgn}(\omega)$$

$$= \sqrt{\frac{\pi}{2}}\{\mathrm{e}^{\mathrm{j}\pi/2} S(\omega) + \mathrm{e}^{-\mathrm{j}\pi/2} S(-\omega)\}, \tag{4}$$

where $S(\omega)$ denotes the Heaviside function. The first fractional Hilbert transform introduced in [7] is defined as the Hilbert transform that is implemented by the filter whose system transfer function is given by $H_\alpha(\omega) = \mathrm{e}^{\mathrm{j}\phi} S(\omega) + \mathrm{e}^{-\mathrm{j}\phi} S(-\omega)$ where $\phi = (\pi/2)\alpha$, and α is a real number. This can be also written in the form $H_\alpha(\omega) = \cos\phi H_0(\omega) + \sin\phi H_1(\omega)$. When $\alpha = 1$, this definition coincides with (4) upto a multiplicative constant.

3. The sampling formulae

Let $f(t)$ be signal band-limited to $[-\delta, \delta]$ in the FRFT sense, i.e., the support of $F_\alpha(u)$ is $[-\delta, \delta]$. It follows from the sampling expansion of band-limited FRFT signals (see [11–16]), that

$$f(t) = \mathrm{e}^{-\mathrm{j}a(\alpha)t^2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}a(\alpha)t_k^2} f(t_k) \frac{\sin[(\delta/\sin\alpha)(t-t_k)]}{(\delta/\sin\alpha)(t-t_k)}, \tag{5}$$

where the sampling points are $t_k = k\pi\sin\alpha/\delta$, $k \in Z$, where Z is the set of integers.

It now follows (see [15, p. 17]) that if $f(t)$ is band-limited to $[\omega_0 - \delta, \omega_0 + \delta]$ in the FRFT sense, i.e., the support of $F_\alpha(u)$ is $[\omega_0 - \delta, \omega_0 + \delta]$, then

$$f(t) = \mathrm{e}^{-\mathrm{j}a(\alpha)t^2} \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{j}a(\alpha)t_k^2} f(t_k) \frac{\sin(\delta/\sin\alpha)(t-t_k)}{(\delta/\sin\alpha)(t-t_k)} \times \exp\left\{\mathrm{j}\frac{\omega_0}{\sin\alpha}(t-t_k)\right\}, \tag{6}$$

where the sampling points are $t_k = k\pi\sin\alpha/\delta$, $k \in Z$, where Z is the set of integers.

We can state the sampling formulae as a theorem.

Theorem 1. *Let $f(t)$ be a real signal and suppose that $e^{-ja(\alpha)t^2}f(t)$ is band limited to $[-\sigma,\sigma]$ in the sense of the FRFT. Then the following sampling expansion for $f(t)$ holds:*

$$f(t)=\sum_{k=-\infty}^{\infty}\{f(t_k)\cos[\beta(t-t_k)]-\tilde{f}(t_k)\sin[\beta(t-t_k)]\}\frac{\sin[\beta(t-t_k)]}{\beta(t-t_k)},\qquad(7)$$

where $\beta=\sigma/(2\sin\alpha)$, $t_k=2k\pi\sin\alpha/\sigma$, $k\in Z$ and $\tilde{f}$ denotes the Hilbert transform of f.

Similarly if $f(t)$ is time limited to $[-\sigma,\sigma]$, then the following sampling expansion holds for the FRFT, $F_\alpha(u)$:

$$F_\alpha(u)=\sum_{k=-\infty}^{\infty}\{F_\alpha(t_k)\cos[\beta(t-t_k)]-\tilde{F}_\alpha(t_k)\sin[\beta(t-t_k)]\}\frac{\sin[\beta(t-t_k)]}{\beta(t-t_k)}.\qquad(8)$$

Proof. Let $g(t)$ be the analytic part of $f(t)$, i.e., $g(t)=f(t)+j\tilde{f}(t)$. Multiplying by $e^{-ja(\alpha)t^2}$ and taking the FRFT of both sides, we have

$$F_\alpha[e^{-ja(\alpha)t^2}g(t)](u)=F_\alpha[e^{-ja(\alpha)t^2}f(t)](u)+jF_\alpha[e^{-ja(\alpha)^2}\tilde{f}(t)](u).\qquad(9)$$

But

$$F_\alpha[e^{-ja(\alpha)t^2}\tilde{f}(t)](u)=\int_{-\infty}^{\infty}e^{-ja(\alpha)t^2}\tilde{f}(t)K_\alpha(u,t)\,dt,$$

where $K_\alpha(u,t)$ is defined in (1). Thus,

$$\begin{aligned}F_\alpha[e^{-ja(\alpha)t^2}\tilde{f}(t)](u)&=\int_{-\infty}^{\infty}e^{-ja(\alpha)t^2}K_\alpha(u,t)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{t-x}\,dx\right)dt\\&=\frac{c(\alpha)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ja(\alpha)u^2}e^{-j(\csc\alpha)ut}}{t-x}\,dt\right)dx\\&=\frac{c(\alpha)e^{ja(\alpha)u^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\mathscr{H}[e^{-j(\csc\alpha)ut}](x)\,dx,\end{aligned}$$

where $\mathscr{H}[\cdot]$ denotes the Hilbert transform. But in view of the fact that

$$\mathscr{H}[e^{-j(\csc\alpha)ut}](x)=(-j)\mathrm{sgn}(u)e^{-j(\csc\alpha)ut}\quad\text{if } 0<\alpha<\pi,$$

we have, for $0<\alpha<\pi$,

$$\begin{aligned}F_\alpha[e^{-ja(\alpha)t^2}\tilde{f}(t)](u)&=\frac{c(\alpha)e^{ja(\alpha)u^2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)[(-j)\mathrm{sgn}(u)]e^{-j(\csc\alpha)ux}\,dx\\&=-j\,\mathrm{sgn}\,u\int_{-\infty}^{\infty}f(x)e^{-ja(\alpha)x^2}K_\alpha(u,x)\,dx\\&=-j\,\mathrm{sgn}\,u\,F_\alpha[e^{-ja(\alpha)x^2}f(x)](u),\end{aligned}\qquad(10)$$

where we have used $2a(\alpha)b(\alpha)=\csc\alpha$. Therefore, by substituting (10) into (9) we obtain

$$F_\alpha[e^{-ja(\alpha)t^2}g(t)](u)=(1+\mathrm{sgn}\,u)F_\alpha[e^{-ja(\alpha)t^2}f(t)](u),$$

which implies that $F_\alpha[e^{-ja(\alpha)t^2}g(t)](u)$ has support in $[0,\sigma]$. If we denote $e^{-ja(\alpha)t^2}g(t)$ by $h(t)$, then it follows by setting $\omega_0=\sigma/2$ and $\delta=\sigma/2$ in formula (6) that

$$h(t)=e^{-ja(\alpha)t^2}\sum_{k=-\infty}^{\infty}e^{ja(\alpha)t_k^2}h(t_k)\frac{\sin[(\sigma/(2\sin\alpha))(t-t_k)]}{(\sigma/(2\sin\alpha))(t-t_k)}\times\exp\left\{j\frac{\sigma}{2\sin\alpha}(t-t_k)\right\},$$

which reduces to

$$g(t)=\sum_{k=-\infty}^{\infty}g(t_k)\frac{\sin[\beta(t-t_k)]}{\beta(t-t_k)}\exp\{j\beta(t-t_k)\},$$

where $\beta=\sigma/(2\sin\alpha)$ and $t_k=2k\pi\sin\alpha/\sigma$, $k\in Z$.

Taking the real part of the above equation we obtain

$$f(t)=\sum_{k=-\infty}^{\infty}(f(t_k)\cos[\beta(t-t_k)]-\tilde{f}(t_k)\sin[\beta(t-t_k)])\frac{\sin[\beta(t-t_k)]}{\beta(t-t_k)},$$

which is Eq. (7).

If $f(t)$ is a time-limited signal to $[-\sigma,\sigma]$, then $F_\alpha(u)=\int_{-\sigma}^{\sigma}f(t)K_\alpha(u,t)\,dt$, and by duality and the symmetry of the FRFT and its inversion formula, the sampling expansion (8) can be obtained in the same way as (7). □

It should be noted that formulae (7) and (8) are generalization of Goldman's formula (see [5; 15, p. 67]) and they reduce to it when $\alpha = \pi/2$.

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