A note on continuous stable sampling*

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Abstract

As a starting point we assume to have a continuous frame in a Hilbert space with respect to a measure space. This frame inherits a unitary structure from a unitary representation of a locally compact abelian group in the Hilbert space. In this setting we state a continuous sampling result for the range space of the associated analysis frame operator. The data sampling are functions also defined by using the underlying unitary structure. The result is illustrated by using continuous frames in Paley-Wiener and shift-invariant spaces generated by translates of fixed functions. A sampling strategy working only for discrete abelian groups is also discussed.

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1 Introduction

The aim of this note is to state some basic ideas on continuous stable sampling by using continuous frames in a Hilbert space with respect to a measure space. Here, the data sampling are functions instead of sequences obtained from the function to be recovered. Thus, in this general setting, we could include the tomography related with the Radon transform, the continuous wavelet or Gabor transforms, etc. We restrict ourselves to the case where the continuous frame has a unitary structure given by the action of a locally compact abelian (LCA) group G on a Hilbert space $\mathcal{H}$ by means of a unitary representation $g \mapsto U(g)$ of the group G on $\mathcal{H}$. The sampling will be carried out in the range space of the analysis operator associated with the continuous frame, a closed

*The authors are very pleased to dedicate this paper to Professor F. H. Szafraniec on the occasion of his 80th birthday. Professor Szafraniec's research and mentorship have, over the years, inspired and influenced many mathematicians throughout the world; we are fortunate to be two of these mathematicians.

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subspace of $L^2(G)$. Recall that a unitary representation of G on $\mathcal{H}$ is a continuous homomorphism of G into the group of unitary operators in $\mathcal{H}$. The main features on continuous frames can be found in a brief included in Section 2. In particular, the definition of a continuous frame with respect to a measure space is given in Eq. (1) below.

More specifically, we assume that for a fixed $a \in \mathcal{H}$ the family $\{U(t)a\}_{t\in G}$ is a continuous frame for the Hilbert space $\mathcal{H}_a := \overline{\text{span}}_{\mathcal{H}}\big\{U(t)a\big\}_{t\in G}$ with respect to (G,μ) where μ denotes the Haar measure associated to the group G. The idea is to recover is a stable way any function F_x, defined for $x \in \mathcal{H}_a$, as $F_x(t) = \big\langle x, U(t)a\big\rangle_{\mathcal{H}}$, $t \in G$, from a finite set of sampling functions $\mathcal{L}_m F_x(t) := \big\langle x, U(t)b_m\big\rangle_{\mathcal{H}}$, $t \in G$, where $b_m \in \mathcal{H}_a$ for $m = 1, 2, \dots, M$.

In a wide sense the *sampling and reconstruction problem* consists of the stable recovering of any function $f \in \mathcal{V}_{samp}$, usually a closed subspace of a Hilbert space $\mathcal{H}$, from some available data $\mathcal{L}f = (\mathcal{L}_1 f, \mathcal{L}_2 f, \dots, \mathcal{L}_M f)$ associated with the function f. This available data $\mathcal{L}f$ could be a sequence of its samples $\{f(t_n)\}$, a vector sequence of averages of the function $\big(\big\{\langle f, \psi_m(\cdot - t_n)\rangle\big\}\big)_{m=1}^{M}$, a vector sequence of filtered versions of a related function $\widetilde{f}$, i.e., $\big(\big\{(\widetilde{f} * \psi_m)(t_n)\big\}\big)_{m=1}^{M}$, or whatever data information providing a stable reconstruction in $\mathcal{V}_{samp}$. The last means that the available data provides an equivalent norm to that in the space $\mathcal{V}_{samp}$, i.e., there two exist positive constants $0 < A \le B$ such that $A\|f\|^2 \le \|\mathcal{L}f\|^2 \le B\|f\|^2$ for any $f \in \mathcal{V}_{samp}$.

In general, this is done by using a suitable representation of the data $\mathcal{L}f$ in an auxiliary space $\mathcal{V}_{aux}$ where $\|\mathcal{L}f\|$ is just the norm of $\mathcal{L}f$ in this space $\mathcal{V}_{aux}$. For instance, consider the classical Paley-Wiener space PW_π of bandlimited functions in $L^2(\mathbb{R})$ to the interval $[-\pi, \pi]$, i.e.,

$$PW_\pi = \Big\{ f \in L^2(\mathbb{R}) \,:\, \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi] \Big\},$$

where $\widehat{f}$ denotes the Fourier transform of f. For each $f \in PW_\pi$, we have the expression for the samples $f(n) = \frac{1}{\sqrt{2\pi}}\langle \widehat{f}, \mathrm{e}^{-inw}\rangle_{L^2[-\pi,\pi]}$, $n \in \mathbb{Z}$. Since $\{\frac{\mathrm{e}^{-inw}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[-\pi, \pi]$, by using Parseval equality, we get

$$\sum_{n\in\mathbb{Z}} |f(n)|^2 = \|\widehat{f}\|^2 = \|f\|^2 \quad \text{for any } f \in PW_\pi$$

Finally, the famous WSK sampling formula

$$f(t) = \sum_{n\in\mathbb{Z}} f(n) \operatorname{sinc}(t - n)\,, \quad t \in \mathbb{R}\,,$$

where sinc denotes the sine cardinal function $\frac{\sin \pi t}{\pi t}$, is obtained by applying the inverse Fourier transform $\mathcal{F}^{-1}$ in the expansion of $\widehat{f} \in L^2[-\pi, \pi]$ with respect to the above orthonormal basis. Besides, PW_π is a reproducing kernel Hilbert space (RKHS hereafter) where convergence in norm implies pointwise convergence. Recall that a RKHS is a Hilbert space of functions in the same domain where any evaluation functional $E_t : f \mapsto f(t)$ is bounded. Usually, this auxiliary space is an $L^2[a, b]$ or an $\ell^2(\mathbb{Z})$ space (see, for instance, Refs. [13, 14] and references therein).

The above consideration is a general pattern in dealing with sampling and reconstruction problems. The available data $\mathcal{L}f$ will be expressed in terms of a *frame* for the auxiliary space $\mathcal{V}_{aux}$. For the definition of a (discrete) *frame* $\{u_n\}$ for a separable Hilbert space $\mathcal{H}$ see Eq. (2) below. Given a frame $\{u_n\}$ for $\mathcal{H}$ the representation property of any vector $u \in \mathcal{H}$ as a series $u = \sum_n c_n u_n$ is retained, but, unlike the case of Riesz (orthonormal) bases, the uniqueness of this representation is sacrificed. Suitable frame coefficients c_n which depend continuously and linearly on u are obtained by using the dual frames $\{v_n\}$ of $\{u_n\}$, i.e., $\{v_n\}_{n\in\mathbb{Z}}$ is another frame for $\mathcal{H}$ such that

$$u = \sum_n \langle u, v_n\rangle u_n = \sum_n \langle u, u_n\rangle v_n \quad \text{for each } u \in \mathcal{H}\,.$$

In particular, frames include orthonormal and Riesz bases for $\mathcal{H}$. For more details and proofs see Ref. [7].

In this paper the vector sampling function $\mathcal{L}f$ will be expressed in terms of a continuous frame in an auxiliary space. Thus we will deal with continuous frames in a Hilbert space with respect to a measure space; we include a summary of the main results along with a set of references on this topic and their most important applications in Section 2. Continuous and discrete frames share the main needed properties which are included in the brief. The rest of the paper is organized as follows: Section 3 is devoted to show as convolution systems is a good strategy in sampling theory which only works for discrete groups G. In Section 4 we give a continuous sampling result valid for the range space of the analysis operator of U-structured continuous frames. Finally, this result is illustrated in the case of continuous frames obtained from translates of fixed functions in the Paley-Wiener PW_π, or in a principal shift-invariant subspace of $L^2(\mathbb{R})$.

2 A brief on continuous frames

Let $\mathcal{H}$ be a Hilbert space and let (Ω, μ) be a measure space. A mapping $F : \Omega \to \mathcal{H}$ is a *continuous frame* for $\mathcal{H}$ with respect to (Ω, μ) if F is weakly measurable, i.e., for each $f \in \mathcal{H}$ the function $w \mapsto \langle f, F(w)\rangle$ is measurable, and there exist constants $0 < A \le B$ such that

$$A\|f\|^2 \le \int_\Omega \big|\langle f, F(w)\rangle\big|^2 d\mu(w) \le B\|f\|^2 \quad \text{for each } f \in \mathcal{H}\,. \tag{1}$$

The constants A and B are the lower and upper continuous frame bounds respectively. The mapping F is a *tight continuous frame* if $A = B$; a *Parseval continuous frame* if $A = B = 1$. The mapping F is called *Bessel* if the right hand inequality holds. Along this paper we refer a continuous frame as the mapping $F : \Omega \to \mathcal{H}$, or as the family $\{F(w)\}_{w\in\Omega}$ or $\{F_w\}_{w\in\Omega}$ in the Hilbert space $\mathcal{H}$. The counting measure μ on $\Omega = \mathbb{N}$ gives the classical definition of (discrete) frame $\{f_n\}_{n=1}^\infty$:

$$A\|f\|^2 \le \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \le B\|f\|^2 \quad \text{for each } f \in \mathcal{H}\,. \tag{2}$$

There are a lot of examples of continuous frames in the mathematical literature; among them we encounter:

- Let $\mathcal{H} \subset L^2(\Omega,\mu)$ be a RKHS (reproducing kernel Hilbert space) of functions defined on Ω with reproducing kernel $k_x(t)$, i.e., $f(x) = \langle f, k_x\rangle$, $x \in \Omega$, for each $f \in \mathcal{H}$. Then, the family $\{k_x\}_{x\in\Omega}$ is a Parseval continuous frame for $\mathcal{H}$ with respect to (Ω,μ). Indeed, for each $f \in \mathcal{H}$ we have

$$\int_\Omega |\langle f, k_x\rangle|^2 d\mu(x) = \int_\Omega |f(x)|^2 d\mu(x) = \|f\|^2 .$$

- For a fixed non zero function $g \in L^2(\mathbb{R}^d)$, the *Gabor system* defined as $\big\{E_\xi T_x g : (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d\big\}$ is a tight continuous frame for $L^2(\mathbb{R}^d)$ with respect to $\big(\mathbb{R}^d \times \mathbb{R}^d, dx\, d\xi\big)$, where E_ξ and T_x denote the modulation and translation operators in $L^2(\mathbb{R}^d)$ respectively (see, for instance, Ref. [7]).

- For an admissible function $\psi \in L^2(\mathbb{R})$, i.e., a function for which the constant $C_\psi := \int_\mathbb{R} \frac{|\widehat{\psi}(w)|^2}{|w|} dw < +\infty$, the *wavelet system* defined by $\big\{\psi^{a,b} := T_b D_a \psi : (a,b) \in (\mathbb{R}\setminus\{0\}) \times \mathbb{R}\big\}$ is a tight continuous frame for $L^2(\mathbb{R})$ with respect to $\big((\mathbb{R}\setminus\{0\}) \times \mathbb{R}, \dfrac{dadb}{a^2}\big)$, where T_b and D_a denote the translation and dilation operators in $L^2(\mathbb{R})$ respectively (see, for instance, Ref. [7]).

- Other examples involve *coherent states* in physics (Refs. [3, 4]), *square-integrable group representations* (Refs. [1, 2]), *Gabor/wavelet frames on the sphere* (Refs. [6, 20, 21]), or *mixed Gabor/wavelet transform* (Ref. [10]).

Associated to a continuous frame there exists a unique operator $S_F : \mathcal{H} \to \mathcal{H}$ such that

$$\langle S_F f, g\rangle = \int_\Omega \langle f, F(w)\rangle\, \langle F(w), g\rangle d\mu(w)\,, \quad f, g \in \mathcal{H}\,.$$

This operator S_F is bounded, self-adjoint, positive and invertible and it is called the *continuous frame operator* of F. We use the notation

$$S_F f = \int_\Omega \langle f, F(w)\rangle\, F(w)\, d\mu(w)\,.$$

For any $f \in \mathcal{H}$ we have the weakly representations

$$f = S_F^{-1} S_F f = \int_\Omega \langle f, F(w)\rangle S_F^{-1} F(w) d\mu(w)\,,$$
$$f = S_F S_F^{-1} f = \int_\Omega \langle f, S_F^{-1} F(w)\rangle F(w) d\mu(w)\,.$$

The operator $T_F : L^2(\Omega,\mu) \to \mathcal{H}$ weakly defined by

$$\langle T_F \varphi, h\rangle = \int_\Omega \varphi(w) \langle F(w), h\rangle d\mu(w)\,, \quad h \in \mathcal{H}\,,$$

is linear and bounded; it is called the *synthesis operator* of F. Its adjoint operator $T_F^* : \mathcal{H} \to L^2(\Omega,\mu)$ is given by $(T_F^* h)(w) = \langle h, F(w)\rangle$, $w \in \Omega$, and it is called the *analysis operator* of F. Moreover, $S_F = T_F T_F^*$. In case (Ω,μ) is a σ-finite measure space, the mapping F is a continuous frame with respect (Ω, μ) for $\mathcal{H}$ if and only if the operator S_F is a bounded invertible operator.

The analysis operator $f \mapsto \langle f, F(w)\rangle$ defines a linear transform in $\mathcal{H}$ which is bounded and boundedly invertible on its range. In the case of a Gabor system is the so called *short-time Fourier transform*, and for a wavelet system is the *continuous wavelet transform*.

Let F and G be continuous frames for $\mathcal{H}$ with respect to (Ω, μ). We call G a *dual* of F if

$$\langle f, g\rangle = \int_\Omega \langle f, F(w)\rangle \, \langle G(w), g\rangle \, d\mu(w)\,, \quad f, g \in \mathcal{H}$$

holds. Thus for any $f \in \mathcal{H}$ we have the weak representation

$$f = \int_\Omega \langle f, F(w)\rangle \, G(w)\, d\mu(w)\,.$$

This is equivalent to $T_G T_F^* = I_{\mathcal{H}}$. In particular, $S_F^{-1}F$ is always a dual of F called the *standard dual frame* for F.

A continuous frame F is a *Riesz-type frame* if F has only one dual. The following characterization of Riesz-type frame holds, the mapping F is a Riesz-type frame if and only if $\operatorname{Range} T_F^* = L^2(\Omega, \mu)$, i.e, T_F^* is an isomorphism between $\mathcal{H}$ and $L^2(\Omega, \mu)$.

For more details on continuous frames see, for instance, Refs. [3, 5, 9, 12, 16, 19, 22].

3 Convolution systems on locally compact abelian groups and sampling

Let $\mathcal{H}$ be a separable Hilbert space, and let $t \in G \mapsto U(t) \in \mathcal{U}(\mathcal{H})$ be a unitary representation of a LCA group $(G, +)$ on $\mathcal{H}$, i.e., it satisfies that $U(t+s) = U(t)U(s)$, $U(-t) = U^{-1}(t) = U^*(t)$ for $t, s \in G$, and the map $t \mapsto U(t)$ is strongly continuous for each $t \in G$.

For a fixed $\varphi \in \mathcal{H}$, assume that the family $\{U(t)\varphi\}_{t\in G}$ is Bessel in $\mathcal{H}$ with respect to (G, μ), where $\mu = \mu_G$ denotes the Haar measure in G. We can define elements in $\mathcal{H}$, in the weak sense, as follows: Consider the sesquilinear form

$$\begin{array}{rccl} \Psi: & L^2(G) \times \mathcal{H} & \longrightarrow & \mathbb{C} \\ & (x, f) & \longmapsto & \displaystyle\int_G x(t)\langle U(t)\varphi, f\rangle \, d\mu(t)\,. \end{array}$$

Cauchy-Schwarz's inequality and the Bessel character shows that Ψ is bounded. As a consequence of Theorem 2.3.6 in Ref. [18] there exists a unique operator $u : L^2(G) \to \mathcal{H}$ such that $\Psi(x, f) = \langle u(x), f\rangle$, for all $x \in L^2(G)$ and $f \in \mathcal{H}$. Moreover, $\|u\| = \|\Psi\|$. As usual, we will use the notation $u(x) = \int_G x(t)\, U(t)\varphi \, d\mu(t)$.

Now, we consider N generators $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ in $\mathcal{H}$, and we assume in addition that the family $\big\{U(t)\varphi_n\big\}_{t\in G;\, n=1,2,\dots,N}$ is a continuous frame for the closed subspace of $\mathcal{H}$ defined by $\mathcal{V}_\Phi := \overline{\operatorname{span}}_{\mathcal{H}}\{U(t)\varphi_n\}_{t\in G;\, n=1,2,\dots,N}$ with respect to (G,μ). Hence, the closed subspace $\mathcal{V}_\Phi$ of $\mathcal{H}$ can be described (in the weak sense) as

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \int_G x_n(t)U(t)\varphi_n \, d\mu(t) \;:\; x_n \in L^2(G),\ n = 1,2,\dots,N \Big\}. \tag{3}$$

Besides, the mapping defined by

$$\begin{array}{rccl} \mathcal{T}_{U,\Phi}: & L^2_N(G) & \longrightarrow & \mathcal{V}_\Phi \\ & \mathbf{x} = (x_1,x_2,\dots,x_N)^\top & \longmapsto & u = \displaystyle\sum_{n=1}^{N} \int_G x_n(t)U(t)\varphi_n \, d\mu(t)\,, \end{array}$$

is a bounded operator from the Hilbert space $L^2_N(G) := L^2(G)\times\cdots\times L^2(G)$ (N times) onto $\mathcal{V}_\Phi$. Indeed, it is bounded since $\big\{U(t)\varphi_n\big\}_{t\in G;\, n=1,2,\dots,N}$ is a Bessel family and onto since it is a continuous frame for $\mathcal{V}_\Phi$. Moreover, due to the invariance of the Haar measure μ on G, the operator $\mathcal{T}_{U,\Phi}$ satisfies the *shifting property*

$$\mathcal{T}_{U,\Phi}\big(T_t\,\mathbf{x}\big) = U(t)\mathcal{T}_{U,\Phi}\mathbf{x}\,, \quad \mathbf{x} \in L^2_N(G)\,, \tag{4}$$

where $T_t\,\mathbf{x}(s) = \mathbf{x}(s-t)$, $\;t,s\in G$.

Given M elements ψ_m in $\mathcal{H}$, $m = 1,2,\dots,M$, non necessarily in $\mathcal{V}_\Phi$, for each $u \in \mathcal{V}_\Phi$ we define, for $m = 1,2,\dots,M$, the *generalized average sampling function* as

$$\mathcal{L}_m u(t) := \big\langle u, U(t)\psi_m \big\rangle_{\mathcal{H}}, \quad t \in G\,. \tag{5}$$

This definition is motivated by the average sampling in classical shift-invariant subspaces of $L^2(\mathbb{R}^d)$. The sampling function in (5) can be expressed as the output of a convolution system. Indeed, for any $u = \sum_{n=1}^{N} \int_G x_n(t)U(t)\varphi_n d\mu(t)$ in $\mathcal{V}_\Phi$, and $m = 1,2,\dots,M$ one gets

$$\begin{aligned} \mathcal{L}_m u(t) &= \big\langle \sum_{n=1}^{N} \int_G x_n(s)U(s)\varphi_n d\mu(s), U(t)\psi_m \big\rangle_{\mathcal{H}} \\ &= \sum_{n=1}^{N} \int_G x_n(s)\langle U(s)\varphi_n, U(t)\psi_m\rangle_{\mathcal{H}} = \sum_{n=1}^{N} \int_G x_n(s)\langle \varphi_n, U(t-s)\psi_m\rangle_{\mathcal{H}} \\ &= \sum_{n=1}^{N} \int_G x_n(s)\, a_{m,n}(t-s) d\mu(s) = \sum_{n=1}^{N} (a_{m,n} * x_n)(t), \;\; t \in G\,, \end{aligned}$$

where $a_{m,n}(t) := \langle \varphi_n, U(t)\psi_m\rangle_{\mathcal{H}}$, $t \in G$. Notice that each $a_{m,n}$ belongs to $L^2(G)$ since the sequence $\{U(t)\varphi_n\}_{t\in G;\, n=1,2,\dots,N}$ is, in particular, a Bessel family in $\mathcal{H}$ with respect to (G,μ).

This particular example leads us to introduce, in general, a *convolution sampling procedure* in $\mathcal{V}_\Phi$ as follows: Given a matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}(L^2(G))$, i.e., a $M\times N$ matrix with entries in $L^2(G)$, we consider the convolution system defined in $L^2_N(G)$ as

$$\mathcal{A}(\mathbf{x}) = A * \mathbf{x}\,, \quad \mathbf{x} = (x_1,x_2,\dots,x_N)^\top \in L^2_N(G)\,,$$

where $A * \mathbf{x}$ denotes the (matrix) convolution

$$(A * \mathbf{x})(t) = \sum_{s \in G} A(t-s)\, \mathbf{x}(s), \quad t \in G\,.$$

Note that the m-th entry of $A * \mathbf{x}$ is $\sum_{n=1}^{N}(a_{m,n} * x_n)$, where x_n denotes the n-th entry of $\mathbf{x} \in L^2_{_N}(G)$. For $u = \sum_{n=1}^{N} \int_G x_n(t) U(t)\varphi_n \, d\mu(t)$ in $\mathcal{V}_\Phi$ we define the *sampling function* $\boldsymbol{\mathcal{L}} f$ as

$$\boldsymbol{\mathcal{L}}u(t) := \big(\mathcal{L}_1 u(t), \mathcal{L}_2 u(t), \dots, \mathcal{L}_{_M} u(t)\big)^\top = (A * \mathbf{x})(t) = \big[\mathcal{A}(\mathbf{x})\big](t)\,, \quad t \in G\,.$$

where $\mathbf{x} = (x_1, x_2, \dots, x_N)^\top \in L^2_N(G)$.

We say that the *convolution sampling procedure* $\boldsymbol{\mathcal{L}}u(t) = \big(A * \mathbf{x})(t)$, $t \in G$, defined in $\mathcal{V}_\Phi$ by means of the convolution system $\mathcal{A} : L^2_N(G) \to L^2_M(G)$ is *stable* if there exist two positive constants $0 < \alpha \le \beta$ such that $\alpha\|u\|^2 \le \|\boldsymbol{\mathcal{L}}u\|^2 \le \beta\|u\|^2$ for all $u \in \mathcal{V}_\Phi$.

The inequality in the right side is related to the boundedness of the convolution system $\mathcal{A} : L^2_N(G) \to L^2_M(G)$, meanwhile the left one is related to the existence of a bounded convolution operator $\mathcal{B} : L^2_M(G) \to L^2_N(G)$ such that $\mathcal{B}\mathcal{A} = \mathcal{I}_{L^2_N(G)}$, i.e., a left-inverse bounded convolution system of $\mathcal{A}$. Thus, roughly speaking, we could recover any function $u \in \mathcal{V}_\Phi$ from its sampling function $\boldsymbol{\mathcal{L}}u \in L^2_M(G)$ in a stable way.

In next section we present in short the results obtained in the case of a countable discrete abelian group G; more details and examples can be found in Ref. [14].

3.1 The case where G is a countable discrete group

Let $(G, +)$ be a countable discrete abelian group and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unidimensional torus. We say that $\xi : G \mapsto \mathbb{T}$ is a character of G if $\xi(g + g') = \xi(g)\xi(g')$ for all $g, g' \in G$. We denote $\xi(g) = (g, \xi)$. By defining $(\xi + \xi')(g) = \xi(g)\xi'(g)$, the set of characters $\widehat{G}$ is a group, called the dual group of G. In case G is a countable discrete group, its dual group $\widehat{G}$ is compact (see Ref. [8]). There exists a unique measure, the Haar measure $\mu = \mu_{\widehat{G}}$ on $\widehat{G}$ satisfying $\mu(\xi + E) = \mu(E)$, for every Borel set $E \subset \widehat{G}$, and $\mu(\widehat{G}) = 1$.

Recall that for $x \in \ell^1(G)$ its *Fourier transform* is defined as

$$\widehat{x}(\xi) := \sum_{g \in G} x(g)\overline{(g, \xi)} = \sum_{g \in G} x(g)(-g, \xi)\,, \quad \xi \in \widehat{G}\,.$$

The Plancherel theorem extends uniquely the Fourier transform on $\ell^1(G) \cap \ell^2(G)$ to a unitary isomorphism from $\ell^2(G)$ to $L^2(\widehat{G})$. For the details see, for instance, Ref. [8]. We will denote the involved L^p spaces as $\ell^2(G)$, $\ell^2_{_N}(G)$, or $\ell^2_{_M}(G)$ respectively, and $L^\infty(\widehat{G}) \subset L^2(\widehat{G})$. The following result on the convolution in $\ell^2(G)$ holds [11]: Assume that $a, b \in \ell^2(G)$ and $\widehat{a}(\xi)\,\widehat{b}(\xi) \in L^2(\widehat{G})$. Then the convolution $a * b$ belongs to $\ell^2(G)$ and $\widehat{a * b}(\xi) = \widehat{a}(\xi)\,\widehat{b}(\xi)$, a.e. $\xi \in \widehat{G}$.

Whenever G is a countable discrete abelian group, the closed subspace $\mathcal{V}_\Phi$ of $\mathcal{H}$ in (3) can be described as

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{t \in G} x_n(t) U(t)\varphi_n \,:\, x_n \in \ell^2(G),\ n = 1, 2, \dots, N \Big\}.$$

Consider $A = [a_{m,n}] \in \mathcal{M}_{M\times N}(\ell^2(G))$ such that its *transfer matrix* defined by $\widehat{A} := [\widehat{a}_{m,n}]$ has entries in $L^\infty(\widehat{G})$. Thus, the convolution system $\mathcal{A}(\mathbf{x}) := A * \mathbf{x}$ defines a bounded operator $\mathcal{A} : \ell^2_N(G) \to \ell^2_M(G)$.

It is known that its adjoint operator $\mathcal{A}^* : \ell^2_M(G) \to \ell^2_N(G)$ is also a bounded convolution system with associated matrix $A^* = [a^*_{m,n}]^\top \in \mathcal{M}_{N\times M}(\ell^2(G))$, where its entries are given by the involution $a^*_{m,n}(t) = \overline{a_{m,n}(-t)}$, $t \in G$. By using properties of the Fourier transform in $\ell^2(G)$, the transfer matrix of $\mathcal{A}^*$ is $\widehat{A^*}(\xi) = \widehat{A}(\xi)^*$, a.e. $\xi \in \widehat{G}$, i.e., the transpose conjugate of matrix $\widehat{A}(\xi)$.

The bounded operator $\mathcal{A}^*\mathcal{A}$ is invertible if and only if operator $\mathcal{A}$ is injective with a closed range which happens if and only if the constant

$$\delta_A := \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)^*\widehat{A}(\xi)] > 0\,. \tag{6}$$

Therefore, choosing the operator $\mathcal{B} = (\mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^*$ we get $\mathcal{B}\mathcal{A} = \mathcal{I}_{\ell^2_N(G)}$. The transfer matrix of $\mathcal{B}$ is given by the Moore-Penrose pseudo-inverse

$$\widehat{A}(\xi)^\dagger = \big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$$

which has entries in $L^\infty(\widehat{G})$. Moreover, notice that any $N \times M$ matrix $\widehat{B}(\xi)$ solution of $\widehat{B}(\xi)\widehat{A}(\xi) = I_N$ a.e. $\xi \in \widehat{G}$ with entries in $L^\infty(\widehat{G})$ can be expressed in terms of $\widehat{A}(\xi)^\dagger$ as

$$\widehat{B}(\xi) := \widehat{A}(\xi)^\dagger + C(\xi)\big[I_M - \widehat{A}(\xi)\widehat{A}(\xi)^\dagger\big]\,,$$

where $C(\xi)$ denotes any $N \times M$ matrix with entries in $L^\infty(\widehat{G})$.

The above considerations have an easy interpretation in terms of discrete frames of translates in $\ell^2_N(G)$ as follows. For $u = \mathcal{T}_{U,\Phi}\mathbf{x} \in \mathcal{V}_\Phi$ consider the sampling function $\boldsymbol{\mathcal{L}}u(t) = (A * \mathbf{x})(t) = \big[\mathcal{A}(\mathbf{x})\big](t)$, $t \in G$. The m-th component of $\boldsymbol{\mathcal{L}}u(t)$ satisfies

$$\mathcal{L}_m u(t) = [\mathcal{A}(\mathbf{x})]_m(t) = [A * \mathbf{x}]_m(t) = \big\langle \mathbf{x}, T_t\,\mathbf{a}^*_m \big\rangle_{\ell^2_N(G)}\,, \tag{7}$$

where $\mathbf{a}^*_m = (a_{m,1}, a_{m,2}, \ldots, a_{m,N})^\top \in \ell^2_N(G)$ denotes the m-th column of the matrix A^*, the associated matrix of the adjoint operator $\mathcal{A}^*$ of the convolution system $\mathcal{A}$, and $T_t\,\mathbf{a}^*_m = \mathbf{a}^*_m(\cdot - t)$.

As a consequence, the sampling procedure $\boldsymbol{\mathcal{L}}$ is stable in $\mathcal{V}_\Phi$ if and only if the sequence $\big\{T_t\,\mathbf{a}^*_m\big\}_{t\in G;\, m=1,2,\ldots,M}$ is a (discrete) frame for $\ell^2_N(G)$. Moreover, the operator $\mathcal{A}$ is its analysis operator.

Therefore, its frame operator $\mathcal{A}^*\mathcal{A}$ must be a bounded invertible operator. In case, the operator $\mathcal{A}$ is bounded, it will be invertible if and only if $\delta_A > 0$, where δ_A is the constant introduced in Eq. (6).

Now, let $\widehat{B}$ be a matrix in $\mathcal{M}_{N\times M}\big(L^\infty(\widehat{G})\big)$ such that $\widehat{B}(\xi)\,\widehat{A}(\xi) = I_N$, a.e. $\xi \in \widehat{G}$. Consider the m-th column $\mathbf{b}_m = (b_{1,m}, b_{2,m}, \ldots, b_{N,m})^\top \in \ell^2_N(G)$ of the matrix B in $\mathcal{M}_{N\times M}\big(L^2(G)\big)$ whose transfer matrix is $\widehat{B}$. Note that $\mathcal{B}$ is the synthesis operator of the frame $\big\{T_t\,\mathbf{b}_m\big\}_{t\in G;\, m=1,2,\ldots,M}$, since it can be written as

$$\mathcal{B}(\mathbf{x}) = B * \mathbf{x} = \sum_{m=1}^{M}\sum_{t\in G} x_m(t)\, T_t\mathbf{b}_m\,, \quad \mathbf{x} \in \ell^2_M(G)\,.$$

As a consequence, from $\mathcal{B}\,\mathcal{A} = \mathcal{I}_{\ell^2_N(G)}$ and Eq. (7), we obtain that the (discrete) frames $\big\{T_t\,\mathbf{b}_m\big\}_{t\in G;\, m=1,2,\dots,M}$ and $\big\{T_t\,\mathbf{a}^*_m\big\}_{t\in G;\, m=1,2,\dots,M}$ for $\in \ell^2_N(G)$ form a dual pair.

Hence we have $\mathbf{x} = \sum_{m=1}^M \sum_{t\in G} \big\langle \mathbf{x}, T_t\,\mathbf{a}^*_m \big\rangle_{\ell^2_N(G)} T_t\,\mathbf{b}_m$ for each $\mathbf{x} \in \ell^2_N(G)$. Applying the operator $\mathcal{T}_{U,\Phi}$ and the shifting property (4) we get a reconstruction formula for any $u \in \mathcal{V}_\Phi$ from its samples $\big\{\mathcal{L}_m u(t)\big\}_{t\in G;\, m=1,2,\dots,M}$ in (7) as

$$
\begin{aligned}
u &= \sum_{m=1}^M \sum_{t\in G} \mathcal{L}_m u(t)\, \mathcal{T}_{U,\Phi}\big(T_t\,\mathbf{b}_m\big) = \sum_{m=1}^M \sum_{t\in G} \mathcal{L}_m u(t)\, U(t) \mathcal{T}_{U,\Phi}\mathbf{b}_m \\
&= \sum_{m=1}^M \sum_{t\in G} \mathcal{L}_m u(t)\, U(t) S_m \quad \text{in } \mathcal{H}\,,
\end{aligned}
\tag{8}
$$

where the reconstruction functions $S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m$, $m = 1, 2, \dots, M$, belong to $\mathcal{V}_\Phi$. Moreover, the sequence $\{U(t)S_m\}_{t\in G;\, m=1,2,\dots,M}$ is a (discrete) frame for $\mathcal{V}_\Phi$.

Notice that the above convolution sampling procedure is stable in $\mathcal{V}_\Phi$ if and only if the mapping $\mathcal{T}_{U,\Phi}$ is an isomorphism between the Hilbert spaces $\ell^2_N(G)$ and $\mathcal{V}_\Phi$, or equivalently, the sequence $\big\{U(t)\varphi_n\big\}_{t\in G;\, n=1,2,\dots,N}$ is a Riesz basis for $\mathcal{V}_\Phi$. Furthermore, under the hypothesis $\widehat{A} \in \mathcal{M}_{M\times N}(L^\infty(\widehat{G}))$, the existence of a sampling formula like those in Eq. (8) for $\mathcal{V}_\Phi$ is equivalent to condition $\delta_A > 0$ in Eq. (6) (see Ref. [14]).

The above study fails whenever the group $\widehat{G}$ is not compact or, equivalently, the group G is not discrete. In this case, under the hypothesis $\widehat{A} \in \mathcal{M}_{M\times N}(L^2(\widehat{G}) \cap L^\infty(\widehat{G}))$ condition (6) cannot occur since $\det[\widehat{A}(\xi)^* \widehat{A}(\xi)]$ should be a positive function in $L^2(\widehat{G})$ bounded away from zero in $\widehat{G}$ non compact!

In next section we will study continuous stable sampling in the range space of the analysis operator associated with a continuous frame $\{\psi(w)\}_{w\in\Omega}$ for a Hilbert space $\mathcal{H}$ with respect to (Ω, μ). This range space is a reproducing kernel Hilbert space (RKHS) included in $L^2(\Omega, \mu)$ (see, for instance, Ref. [9]).

4 Continuous frames and sampling: a case study

Assume that $\{\psi(w)\}_{w\in\Omega}$ is a continuous frame for a Hilbert space $\mathcal{H}$ with respect to (Ω, μ) such that the mapping $w \mapsto \psi(w)$ is weakly continuous. Its analysis operator $T^*_\psi : \mathcal{H} \to L^2(\Omega, \mu)$ is a bounded and boundedly invertible operator on its range denoted as $\mathcal{H}_\psi :=$ Range T^*_ψ. This is a closed subspace of $L^2(\Omega, \mu)$ described as the functions F such that

$$
\begin{array}{ccccl}
\mathcal{H} & \longrightarrow & \mathcal{H}_\psi & & \\
f & \longmapsto & F_f & : & F_f(w) = \langle f, \psi(w) \rangle_{\mathcal{H}}\,, \quad w \in \Omega\,.
\end{array}
$$

Besides $\mathcal{H}_\psi$ is a RKHS (of continuous functions in Ω) whose reproducing kernel is given by

$$
k_\psi(u, v) = \langle \psi(v), S^{-1}_\psi \psi(u) \rangle_{\mathcal{H}}\,, \quad u, v \in \Omega\,,
$$

where S_ψ^{-1} denotes the inverse of the frame operator S_ψ associated to $\{\psi(w)\}_{w\in\Omega}$. That is, for any $F_f \in \mathcal{H}_\psi$ we have

$$F_f(u) = \int_\Omega F_f(v)\, k_\psi(u,v)\, d\mu(v) = \langle F_f, k_\psi(\cdot,u)\rangle_{L^2(\Omega,\mu)}\,, \quad u \in \Omega\,.$$

The aim in this section is to obtain a sort of continuous stable sampling theory in case the continuous frame has a unitary structure.

4.1 Continuous frames with a unitary structure

Let $t \in G \mapsto U(t) \in \mathcal{U}(\mathcal{H})$ be a unitary representation of a LCA group $(G,+)$ on a separable Hilbert space $\mathcal{H}$. Assume that for a fixed $a \in \mathcal{H}$ the family $\{U(t)a\}_{t\in G}$ is a continuous frame for the Hilbert space $\mathcal{H}_a := \overline{\operatorname{span}}_{\mathcal{H}}\big\{U(t)a\big\}_{t\in G}$ with respect to (G,μ) where μ denotes the Haar measure associated to G. In order to avoid some technical problems we will assume that G is a σ-compact group; thus its Haar measure will be σ-finite. For the details see, for instance, Refs. [8, 11].

In this section we consider the functions F in a closed subspace $\mathcal{H}_U \subset L^2(G)$ defined as

$$\begin{array}{ccl} \mathcal{H}_a \subset \mathcal{H} & \longrightarrow & \mathcal{H}_U \subset L^2(G) \\ x & \longmapsto & F_x \,:\, F_x(t) = \big\langle x, U(t)a\big\rangle_{\mathcal{H}}\,, \quad t \in G\,. \end{array}$$

Given M elements $b_m \in \mathcal{H}_a$, $m = 1,2,\ldots,M$, for each $F \in \mathcal{H}_U$ (in the sequel we omit the subscript x) we define the generalized sampling functions as

$$\mathcal{L}_m F(t) := \big\langle x, U(t)b_m\big\rangle_{\mathcal{H}}, \quad t \in G\,, \quad m = 1,2,\ldots,M\,. \tag{9}$$

The stable sampling condition reads as: there exist two positive constants $0 < A \le B$ such that

$$A\|F\|^2 \le \sum_{m=1}^{M} \int_G |\mathcal{L}_m F(t)|^2 d\mu(t) \le B\|F\|^2\,, \quad \text{for all } F \in \mathcal{H}_U\,.$$

Equivalently, the family $\{U(t)b_m\}_{t\in G;\, m=1,2,\ldots,M}$ is a continuous frame for the Hilbert space $\mathcal{H}_a$ with respect to (G,μ). In order to obtain a structured sampling (reconstruction) formula for any $F \in \mathcal{H}_U$ we need a dual of the above continuous frame having the same structure.

Assume that there exist M elements $c_1, c_2, \ldots, c_M$ in $\mathcal{H}_a$ such that the family $\{U(t)c_m\}_{t\in G;\, m=1,2,\ldots,M}$ is a dual of $\{U(t)b_m\}_{t\in G;\, m=1,2,\ldots,M}$ in $\mathcal{H}_a$ with respect to (G,μ). Then, a sampling formula in $\mathcal{H}_U$ is easily obtained. Indeed, for each $x \in \mathcal{H}_a$ we have, in the weak sense

$$x = \sum_{m=1}^{M} \int_G \big\langle x, U(s)b_m\big\rangle_{\mathcal{H}}\, U(s)c_m\, d\mu(s)\,.$$

Therefore, for $F(t) = \langle x, U(t)a\rangle_{\mathcal{H}}$, $t \in G$, we have

$$\begin{aligned} F(t) &= \sum_{m=1}^{M} \int_G \mathcal{L}_m F(s)\, \big\langle U(s)c_m, U(t)a\big\rangle_{\mathcal{H}}\, d\mu(s) \\ &= \sum_{m=1}^{M} \int_G \mathcal{L}_m F(s)\, S_m(t-s)\, d\mu(s)\,, \quad t \in G\,, \end{aligned}$$

where $S_m(u) = \langle c_m, U(u)a \rangle_{\mathcal{H}}$, $u \in G$, $m = 1, 2, \dots, M$. Hence we have obtained the following continuous sampling result in $\mathcal{H}_U$:

Theorem 1. *For each $F \in \mathcal{H}_U$ consider its function samples $\{\mathcal{L}_m F\}_{m=1,2,\dots,M}$ defined in Eq. (9). If the continuous frame $\{U(t)b_m\}_{t\in G;\, m=1,2,\dots,M}$ for $\mathcal{H}_a$ with respect to (G, μ) has a dual of the form $\{U(t)c_m\}_{t\in G;\, m=1,2,\dots,M}$, then the sampling formula in $\mathcal{H}_U$*

$$F(t) = \sum_{m=1}^{M} \int_G \mathcal{L}_m F(s)\, S_m(t-s)\, d\mu(s)\,, \quad t \in G\,. \tag{10}$$

holds, where the functions $S_m(u) = \langle c_m, U(u)a \rangle_{\mathcal{H}} \in \mathcal{H}_U$, $m = 1, 2, \dots, M$. Moreover, the family $\{S_m(\cdot - t)\}_{t\in G}$ forms a continuous frame for $\mathcal{H}_U$ with respect to (G, μ).

Notice that the above theorem is also valid in case the group G is not abelian; it is enough to replace $S_m(t-s)$ by $S_m(s^{-1}t)$ in formula (10).

In case $\{U(t)a\}_{t\in G}$ is a tight continuous frame for $\mathcal{H}_a$ with respect to (G, μ) with constant $A > 0$, formula (10) applied to $F(t) = \langle x, U(t)a \rangle_{\mathcal{H}}$, $t \in G$, gives the reproducing formula in $\mathcal{H}_U$

$$F(t) = \frac{1}{A} \int_G F(s)\, k_a(s-t)\, d\mu(s)\,, \quad s \in G\,,$$

where $k_a(t) = \langle U(t)a, a \rangle_{\mathcal{H}}$, $t \in G$.

Next we consider two important toy examples in $\mathcal{H} = L^2(\mathbb{R})$ involving the unitary representation $[U(t)f](s) = f(s-t)$ of the group $(\mathbb{R}, +)$ in $L^2(\mathbb{R})$. In the first one, $\mathcal{H}_{\text{sinc}} = PW_\pi$ is the Paley-Wiener space of bandlimited functions to the interval $[-\pi, \pi]$ and, in the second one, $\mathcal{H}_\varphi = V_\varphi^2$ is a principal shift-invariant subspace generated by $\varphi \in L^2(\mathbb{R})$.

4.2 The case of Paley-Wiener spaces

Here, the family $\{\operatorname{sinc}(\cdot - x)\}_{x\in\mathbb{R}}$ is a Parseval continuous frame for the Paley-Wiener space $PW_\pi = \mathcal{H}_{\text{sinc}} \subset L^2(\mathbb{R})$ with respect to $(\mathbb{R}, dx)$. Due to the reproducing property, the Hilbert space $\mathcal{H}_U$ coincides also with PW_π. Given M functions $\varphi_m \in PW_\pi$, $m = 1, 2, \dots, M$, the goal is to characterize the family $\{\varphi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ as a continuous frame for PW_π, and to find its duals with the same structure, i.e., having the form $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ for some $\psi_m \in PW_\pi$, $m = 1, 2, \dots, M$. For the first question, by using the Parseval formula in $L^2(\mathbb{R})$, we have that

$$\begin{aligned}\langle f, \varphi_m(\cdot - x)\rangle &= \langle \widehat{f}(\xi), \mathrm{e}^{-ix\xi}\widehat{\varphi}_m(\xi)\rangle = \int_{-\pi}^{\pi} \widehat{f}(\xi)\, \overline{\widehat{\varphi}_m(\xi)}\, \mathrm{e}^{ix\xi}\, d\xi \\ &= \sqrt{2\pi}\mathcal{F}^{-1}(\widehat{f}\, \overline{\widehat{\varphi}_m})(x),\end{aligned}$$

where the Fourier transform, defined as $\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\mathrm{e}^{-i\xi t} dt$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, is extended to a unitary operator on $L^2(\mathbb{R})$ by Plancherel theorem. Hence,

$$\begin{aligned}\int_{\mathbb{R}} |\langle f, \varphi_m(\cdot - x)\rangle|^2 dx &= 2\pi \|\mathcal{F}^{-1}(\widehat{f}\, \overline{\widehat{\varphi}_m})\|^2 = 2\pi \|\widehat{f}\, \overline{\widehat{\varphi}_m}\|^2 \\ &= 2\pi \int_{-\pi}^{\pi} |\widehat{f}(\xi)\, \widehat{\varphi}_m(\xi)|^2 d\xi\,.\end{aligned}$$

Assume that

$$0 \le \frac{A_m}{2\pi} := \operatorname*{ess\,inf}_{\xi\in[-\pi,\pi]} |\widehat{\varphi}_m(\xi)|^2 \le \operatorname*{ess\,sup}_{\xi\in[-\pi,\pi]} |\widehat{\varphi}_m(\xi)|^2 := \frac{B_m}{2\pi} < \infty \,.$$

Then, we have

Proposition 1. *The family* $\{\varphi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *as a continuous frame for* PW_π *with respect to* $(\mathbb{R}, dx)$ *if and only if the constants* $A_M := \sum_{m=1}^M A_m > 0$ *and* $B_M := \sum_{m=1}^M B_m < \infty$. *Moreover, the bounds* A_M *and* B_M *are the optimal ones.*

In particular, the family $\{\varphi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ is a Parseval continuous frame for PW_π with respect to $(\mathbb{R}, dx)$ if and only if $\sum_{m=1}^M |\widehat{\varphi}_m(\xi)|^2 = \frac{1}{2\pi}$ a.e. $\xi \in [-\pi, \pi]$.

Concerning the second question, we look for a dual of $\{\varphi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ having the form $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$. For any $f, g \in PW_\pi$ it should satisfy

$$\begin{aligned}
\langle f, g\rangle &= \sum_{m=1}^M \int_{\mathbb{R}} \langle f, \varphi_m(\cdot - x)\rangle\, \overline{\langle g, \psi_m(\cdot - x)\rangle}\, dx \\
&= 2\pi \sum_{m=1}^M \int_{\mathbb{R}} \mathcal{F}^{-1}(\widehat{f}\,\overline{\widehat{\varphi}}_m)(x)\overline{\mathcal{F}^{-1}(\widehat{g}\,\overline{\widehat{\psi}}_m)(x)}\, dx = 2\pi \sum_{m=1}^M \big\langle \widehat{f}\,\overline{\widehat{\varphi}}_m, \widehat{g}\,\overline{\widehat{\psi}}_m\big\rangle \\
&= 2\pi\Big\langle \widehat{f}, \widehat{g}\,\big(\sum_{m=1}^M \widehat{\varphi}_m\,\overline{\widehat{\psi}}_m\big)\Big\rangle = \langle \widehat{f}, \widehat{g}\,\rangle \,.
\end{aligned}$$

Hence we derive the following result

Proposition 2. *The family* $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *is a dual of the continuous frame* $\{\varphi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *if and only if* $\displaystyle\sum_{m=1}^M \widehat{\varphi}_m(\xi)\,\overline{\widehat{\psi}}_m(\xi) = \frac{1}{2\pi}$ *a.e.* $\xi \in [-\pi, \pi]$.

A dual having the same form always exists; for instance the functions ψ_m whose Fourier transforms are $\widehat{\psi}_m(\xi) = \dfrac{\widehat{\varphi}_m(\xi)}{2\pi\sum_{n=1}^M |\widehat{\varphi}_n(\xi)|^2}$, a.e. $\xi \in [-\pi, \pi]$, and $m = 1, 2, \dots, M$.

Finally, having in mind Propositions 1–2, from Theorem 1 we get the following sampling result in PW_π

Corollary 1. *Given* M *fixed functions* $\varphi_m \in L^2(\mathbb{R})$, *for each* $f \in PW_\pi$ *consider the sampling functions* $\mathcal{L}_m f(x) := \langle f, \varphi_m(\cdot - x)\rangle$, $x \in \mathbb{R}$, *and* $m = 1, 2, \dots, M$. *Assume that condition*

$$0 < \operatorname*{ess\,inf}_{\xi\in[-\pi,\pi]} \sum_{m=1}^M |\widehat{\varphi}_m \chi_{[-\pi,\pi]}(\xi)|^2 \le \operatorname*{ess\,sup}_{\xi\in[-\pi,\pi]} \sum_{m=1}^M |\widehat{\varphi}_m \chi_{[-\pi,\pi]}(\xi)|^2 < \infty$$

holds. Then, there exist M *functions* $\psi_m \in PW_\pi$, $m = 1, 2, \dots, M$ *such that for any* $f \in PW_\pi$ *we have the sampling formula*

$$f(t) = \sum_{m=1}^M \int_{\mathbb{R}} \mathcal{L}_m f(x)\, \psi_m(t - x)\, dx\,, \quad t \in \mathbb{R}\,.$$

Proof. Note that the functions φ_m, $m = 1, 2, \ldots, M$, do not need to belong necessarily to PW_π. However, we could consider the functions $\widetilde{\varphi}_m = \mathcal{F}^{-1}(\widehat{\varphi}_m \chi_{[-\pi,\pi]}) \in PW_\pi$, $m = 1, 2, \ldots, M$; the family $\{\widetilde{\varphi}_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\ldots,M}$ is a continuous frame in PW_π and proceeding as in Proposition 2 we obtain a dual $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\ldots,M}$. Notice that, for any $f \in PW_\pi$, we have $\langle f, \widetilde{\varphi}_m(\cdot - x)\rangle = \langle f, \varphi_m(\cdot - x)\rangle$, $x \in \mathbb{R}$ and $m = 1, 2, \ldots, M$. □

4.3 The case of a shift-invariant subspace in $L^2(\mathbb{R})$

Here, we consider $\mathcal{H}_\varphi := V_\varphi^2 \subset L^2(\mathbb{R})$ the principal shift-invariant subspace of $L^2(\mathbb{R})$ generated by $\varphi \in L^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. We assume that the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(\mathbb{R})$, i.e., a Riesz basis for V_φ^2. Equivalently,

$$0 < \frac{A}{2\pi} := \operatorname*{ess\,inf}_{\xi\in[0,2\pi]} \Phi(\xi) \le \frac{B}{2\pi} := \operatorname*{ess\,sup}_{\xi\in[0,2\pi]} \Phi(\xi) < \infty\,,$$

where $\Phi(\xi) := \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi m)|^2$. Thus, the shift-invariant subspace V_φ^2 can be described as

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} c_n\, \varphi(t-n) \ : \ \{c_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

Moreover, the Fourier transform of any $f \in V_\varphi^2$ can be characterized as $\widehat{f}(\xi) = c_f(\xi)\, \widehat{\varphi}(\xi)$ where c_f is the 2π-periodic function in $L^2[0, 2\pi]$ given by $c_f(\xi) = \sum_{n\in\mathbb{Z}} c_n\, \mathrm{e}^{-in\xi}$.

Our first task is to characterize the family $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$ as a continuous frame in V_φ^2 with respect to $(\mathbb{R}, dx)$. Indeed, by using the Parseval formula in $L^2(\mathbb{R})$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |\langle f, \phi(\cdot - x)\rangle|^2 dx &= 2\pi \|\widehat{f}\,\overline{\widehat{\phi}}\|^2 = 2\pi \int_{\mathbb{R}} |c_f(\xi)\widehat{\varphi}(\xi)\overline{c_\phi(\xi)\widehat{\varphi}(\xi)}|^2 d\xi \\
&= 2\pi \sum_{m\in\mathbb{Z}} \int_{2\pi m}^{2\pi(m+1)} |c_f(\xi)c_\phi(\xi)|^2 |\widehat{\varphi}(\xi)|^4 d\xi \\
&= 2\pi \int_0^{2\pi} |c_f(\xi)c_\phi(\xi)|^2 \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi m)|^4 d\xi \\
&= 2\pi \int_0^{2\pi} |c_f(\xi)c_\phi(\xi)|^2 \widetilde{\Phi}(\xi) d\xi\,,
\end{aligned}$$

where the change of variable $\xi \mapsto \xi + 2\pi m$ has been done, and $\widetilde{\Phi}$ is the 2π-periodic function in $L^2[0, 2\pi]$ defined by $\widetilde{\Phi}(\xi) := \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi m)|^4$. Having in mind that $2\pi \sum_n |c_n|^2 = \int_0^{2\pi} |c_f(\xi)|^2 d\xi$, and assuming that

$$0 < \frac{\widetilde{A}}{2\pi} := \operatorname*{ess\,inf}_{\xi\in[0,2\pi]} \widetilde{\Phi}(\xi) \le \frac{\widetilde{B}}{2\pi} := \operatorname*{ess\,sup}_{\xi\in[0,2\pi]} \widetilde{\Phi}(\xi) < \infty\,, \tag{11}$$

we obtain that

Proposition 3. *The family $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$ is a continuous frame in V_φ^2 with respect to $(\mathbb{R}, dx)$ if and only if $0 < \frac{C}{2\pi} := \operatorname*{ess\,inf}_{\xi\in[0,2\pi]} |c_\phi(\xi)|^2 \le \frac{D}{2\pi} := \operatorname*{ess\,sup}_{\xi\in[0,2\pi]} |c_\phi(\xi)|^2 < \infty$. In this case, the optimal frame bounds are $\frac{C\widetilde{A}}{B}$ and $\frac{D\widetilde{B}}{A}$, respectively.*

In particular, we deduce

Corollary 2. *The family* $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$ *is a continuous frame in* V_φ^2 *with respect to* $(\mathbb{R}, dx)$ *if and only if the sequence* $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$ *is a Riesz basis for* V_φ^2.

Proof. Notice that $\sum_{m\in\mathbb{Z}} |\widehat{\phi}(\xi + 2\pi m)|^2 = |c_\phi(\xi)|^2 \sum_{m\in\mathbb{Z}} |\widehat{\varphi}(\xi + 2\pi m)|^2$, where $\widehat{\phi}(\xi) = c_\phi(\xi)\widehat{\varphi}(\xi)$ a.e. $\xi \in [0, 2\pi]$. □

We are interested in finding the duals of the continuous frame $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$ with the same structure, i.e., having the form $\{\psi(\cdot - x)\}_{x\in\mathbb{R}}$. To this end, for any $f, g \in V_\varphi^2$, by using Parseval's formula in $L^2(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}} \langle f, \phi(\cdot - x)\rangle \overline{\langle g, \psi(\cdot - x)\rangle}\, dx &= 2\pi \big\langle \widehat{f}\, \overline{\widehat{\phi}}, \widehat{g}\, \overline{\widehat{\psi}}\, \big\rangle \\ &= 2\pi \int_{\mathbb{R}} c_f(\xi)\, \overline{c_g(\xi)}\, \overline{c_\phi(\xi)} c_\psi(\xi)\, |\widehat{\varphi}(\xi)|^4 d\xi \\ &= 2\pi \int_0^{2\pi} c_f(\xi)\, \overline{c_g(\xi)}\, \overline{c_\phi(\xi)}\, c_\psi(\xi)\, \widetilde{\Phi}(\xi)\, d\xi \\ &= 2\pi \int_0^{2\pi} c_f(\xi)\, \overline{c_g(\xi)}\, \Phi(\xi) \Big[c_\psi(\xi)\, \overline{c_\phi(\xi)}\, \frac{\widetilde{\Phi}(\xi)}{\Phi(\xi)}\Big]\, d\xi\,.\end{aligned}$$

This should equal to $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\,\rangle = \int_0^{2\pi} c_f(\xi)\, \overline{c_g(\xi)}\, \Phi(\xi)\, d\xi$, which happens if and only if $c_\psi(\xi)\, \overline{c_\phi(\xi)}\, \dfrac{\widetilde{\Phi}(\xi)}{\Phi(\xi)} = \dfrac{1}{2\pi}$ a.e. $\xi \in [0, 2\pi]$. Thus we have

Proposition 4. *The family* $\{\psi(\cdot - x)\}_{x\in\mathbb{R}}$ *is a dual in* V_φ^2 *of the continuous frame* $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$ *with respect to* $(\mathbb{R}, dx)$ *if and only if* $c_\psi(\xi) = \dfrac{\Phi(\xi)}{2\pi\, \overline{c_\phi(\xi)}\, \widetilde{\Phi}(\xi)}$ *a.e.* $\xi \in [0, 2\pi]$.

Having in mind Corollary 2, at this stage a question arises concerning the function $\psi \in V_\varphi^2$ in the above proposition: When will be the sequence $\{\psi(\cdot - n)\}_{n\in\mathbb{Z}}$ the dual Riesz basis of $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$? To answer that, a simple calculation gives

$$\begin{aligned}\big\langle \phi(\cdot - n), \psi(\cdot - m)\big\rangle &= \big\langle e^{-in\xi}\, \widehat{\phi}(\xi), e^{-im\xi}\, \widehat{\psi}(\xi)\big\rangle \\ &= \big\langle e^{-in\xi}\, c_\phi(\xi)\widehat{\varphi}(\xi), e^{-im\xi}\, c_\psi(\xi)\widehat{\varphi}(\xi)\big\rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n)\xi}\, \Phi(\xi)\, d\xi = \delta_{n,m} \\ &\iff \Phi(\xi) = 1 \text{ a.e. } \xi \in [0, 2\pi]\,,\end{aligned}$$

that is, if and only if the sequence $\{\frac{1}{\sqrt{2\pi}}\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_φ^2. In this case, as a byproduct, we derive that the equality (assuming that V_φ^2 is a RKHS)

$$\int_{\mathbb{R}} \langle f, \phi(\cdot - x)\rangle\, \psi(t - x)\, dx = \sum_{n\in\mathbb{Z}} \langle f, \phi(\cdot - n)\rangle\, \psi(t - n)\,, \quad t \in \mathbb{R}\,,$$

holds for any $f \in V_\varphi^2$. In general, the equality above holds with different functions ψ and $\widetilde{\psi}$, where $\{\psi(\cdot - x)\}_{x\in\mathbb{R}}$ is a continuous dual of $\{\phi(\cdot - x)\}_{x\in\mathbb{R}}$, and $\{\widetilde{\psi}(\cdot - n)\}_{n\in\mathbb{Z}}$ is the dual Riesz basis of $\{\phi(\cdot - n)\}_{n\in\mathbb{Z}}$.

Proceeding as before, it is straightforward to derive the general case with M functions $\phi_m \in V_\varphi^2,\ \ m = 1, 2, \dots, M$.

Proposition 5. *Assume that hypothesis* (11) *on* $\widetilde{\Phi}$ *holds in* V_φ^2*. Then:*

1. *The family* $\{\phi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *is a continuous frame for* V_φ^2 *with respect to* $(\mathbb{R}, dx)$ *if and only if*

$$0 < \operatorname*{ess\,inf}_{\xi\in[0,2\pi]} \sum_{m=1}^{M} |c_{\phi_m}(\xi)|^2 \le \operatorname*{ess\,sup}_{\xi\in[0,2\pi]} \sum_{m=1}^{M} |c_{\phi_m}(\xi)|^2 < \infty\,. \tag{12}$$

2. *The family* $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *is a dual of* $\{\phi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ *if and only if*

$$\Big(\sum_{m=1}^{M} c_{\psi_m}(\xi)\,\overline{c_{\phi_m}(\xi)}\Big)\,\frac{\widetilde{\Phi}(\xi)}{\Phi(\xi)} = \frac{1}{2\pi} \ \ \textit{a.e.}\,\xi \in [0, 2\pi]\,.$$

Notice that a dual of $\{\phi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ with the same form always exists; a solution for the functions ψ_m, $m = 1, 2, \dots, M$, is given by

$$c_{\psi_m}(\xi) = \frac{c_{\phi_m}(\xi)}{2\pi \sum_{n=1}^{M} |c_{\phi_n}(\xi)|^2}\,\frac{\widetilde{\Phi}(\xi)}{\Phi(\xi)}\,,\ \text{a.e. } \xi \in [0, 2\pi]\,, \tag{13}$$

that is, $\widehat{\psi_m}(\xi) = c_{\psi_m}(\xi)\,\widehat{\varphi}(\xi),\ m = 1, 2, \dots, M$.

Concerning the hypothesis (11), notice firstly that $\Phi(\xi) \le \frac{B}{2\pi}$, a.e. $\xi \in [0, 2\pi]$ implies that $\widetilde{\Phi}(\xi) \le \big(\frac{B}{2\pi}\big)^2$, a.e. $\xi \in [0, 2\pi]$. The other inequality holds if, for instance, the condition $\operatorname{ess\,inf}_{\xi\in[-\pi,\pi]} |\widehat{\varphi}(\xi)| > 0$ is satisfied. This condition is satisfied, for instance, by classical B-splines N_p defined by $N_p := \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ (p times) since $\widehat{N}_p(\xi) = \frac{\mathrm{e}^{-ip\xi/2}}{\sqrt{2\pi}}\big(\frac{\sin\xi/2}{\xi/2}\big)^p$ and $\operatorname{ess\,inf}_{\xi\in[-\pi,\pi]} |\widehat{N}_p(\xi)| = \frac{1}{\sqrt{2\pi}}\big(\frac{2}{\pi}\big)^p > 0$.

Finally, by using Proposition 5, from Theorem 1 we derive a continuous sampling result for the corresponding RKHS $\mathcal{H}_U$ obtained from V_φ^2. Since the family $\{\varphi(\cdot - x)\}_{x\in\mathbb{R}}$ is a continuous frame for V_φ^2 we have here that

$$\mathcal{H}_U = \Big\{F(x) = \langle f, \varphi(\cdot - x)\rangle,\, x \in \mathbb{R},\ :\ f \in V_\varphi^2\Big\} = \Big\{f * \varphi^*\ :\ f \in V_\varphi^2\Big\}\,,$$

where φ^* denotes the involution of φ given by $\varphi^*(u) := \overline{\varphi(-u)}$, $u \in \mathbb{R}$.

Corollary 3. *Given* M *fixed functions* $\phi_m \in V_\varphi^2$*, for each* $F(x) = \langle f, \varphi(\cdot - x)\rangle,\ x \in \mathbb{R}$*, in* $\mathcal{H}_U$ *consider, for* $m = 1, 2, \dots, M$*, its sampling functions* $\mathcal{L}_m F(x) := \langle f, \phi_m(\cdot - x)\rangle$*,* $x \in \mathbb{R}$*. Assume that condition* (12) *for the functions* ϕ_m *holds. Then, there exist* M *functions* $S_m \in \mathcal{H}_U$*,* $m = 1, 2, \dots, M$*, such that for any* $f \in \mathcal{H}_U$ *we have the sampling formula*

$$F(t) = \sum_{m=1}^{M} \int_{\mathbb{R}} \mathcal{L}_m F(x)\, S_m(t - x)\, dx\,, \quad t \in \mathbb{R}\,.$$

Proof. For $F \in \mathcal{H}_U$ let f be in V_φ^2 be such that $F(x) = \langle f, \varphi(\cdot - x)\rangle$, $x \in \mathbb{R}$. According to Proposition 5(2) let the family $\{\psi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ be a dual of the continuous frame $\{\phi_m(\cdot - x)\}_{x\in\mathbb{R};\, m=1,2,\dots,M}$ in V_φ^2 with respect to $(\mathbb{R}, dt)$ (such as the one given in Eq. (13)). Thus, in the weak sense, we have $f = \sum_{m=1}^M \int_\mathbb{R} \langle f, \phi_m(\cdot - t)\rangle\, \psi_m(\cdot - t)\, dt$. Therefore,

$$
\begin{aligned}
F(x) &= \sum_{m=1}^{M} \int_\mathbb{R} \mathcal{L}_m F(t)\, \big\langle \psi_m(\cdot - t), \varphi(\cdot - x)\big\rangle\, dt \\
&= \sum_{m=1}^{M} \int_\mathbb{R} \mathcal{L}_m F(t)\, S_m(x - t)\, dt,\ \ x \in \mathbb{R}\,,
\end{aligned}
$$

where the function $S_m(u) = \langle \psi_m, \varphi(\cdot - u)\rangle$, $u \in \mathbb{R}$, belongs to $\mathcal{H}_U$ for $m = 1, 2, \dots, M$. Moreover, the family $\{S_m(\cdot - t)\}_{t\in\mathbb{R}}$ is a continuous frame for $\mathcal{H}_U$ with respect to $(\mathbb{R}, dt)$. □

4.3.1 Final comments

Closing the paper the following comments on future work are pertinent:

1. For each $m = 1, 2, \dots, M$ the sampling function $\mathcal{L}_m F(x) := \langle f, \phi_m(\cdot - x)\rangle$, $x \in \mathbb{R}$, for $F \in \mathcal{H}_U$ can be expressed as a semi-discrete convolution. Indeed, for $f(t) = \sum_{n\in\mathbb{Z}} c_n\, \varphi(t - n)$ in V_φ^2 it is straightforward to derive that

$$
\mathcal{L}_m F(x) = \big\langle f, \phi_m(\cdot - x)\big\rangle = \sum_{n\in\mathbb{Z}} c_n\, \psi_m(x - n) = \big(\psi_m *' c\big)(x)\,, \quad x \in \mathbb{R}\,,
$$

   where $c = \{c_n\}$ and $\psi_m(t) := \langle \varphi, \phi_m(\cdot - t)\rangle$, $t \in \mathbb{R}$; as usual, the symbol $*'$ denotes the semi-discrete convolution. The underlying idea would be to define generalized sampling functions for $F \in \mathcal{H}_U$ by means of semi-discrete convolutions as above, and searching for necessary and sufficient conditions on the involved functions ψ_m, $m = 1, 2, \dots, M$, in order to obtain a sampling result, as in Corollary 3, in the light of the theory considered in Section 3.

2. Similar results can be obtained in V_Φ^2, a shift-invariant subspace in $L^2(\mathbb{R}^d)$ with a set of N generators $\Phi := \{\varphi_1, \varphi_2, \dots, \varphi_N\}$.

3. The results in this section could be extended to shift-invariant subspaces of $L^2(G)$ where G is a locally compact abelian group by using, for instance, the mathematical techniques in Refs. [15, 17].

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