Convolution systems on discrete abelian groups as a unifying strategy in sampling theory

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Abstract

A regular sampling theory in a multiply generated unitary invariant subspace of a separable Hilbert space $\mathcal{H}$ is proposed. This subspace is associated to a unitary representation of a countable discrete abelian group G on $\mathcal{H}$. The samples are defined by means of a filtering process which generalizes the usual sampling settings. The multiply generated setting allows to consider some examples where the group G is non-abelian as, for instance, crystallographic groups. Finally, it is worth to mention that classical average or pointwise sampling in shift-invariant subspaces are particular examples included in the followed approach.

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1 Introduction

In this paper we propose a regular sampling theory for a multiply generated U-invariant subspace of a separable Hilbert space $\mathcal{H}$. By regular sampling we mean that the samples are taken following the pattern given by the action of a discrete abelian group G on $\mathcal{H}$ by means of a unitary representation $g \mapsto U(g)$ of the group G on $\mathcal{H}$ which also defines the U-invariant subspace where the sampling will be carried out. Recall that a unitary representation of G on $\mathcal{H}$ is a homomorphism of G into the group of unitary operators in $\mathcal{H}$.

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In classical shift-invariant subspaces of $L^2(\mathbb{R}^d)$ this group is $\mathbb{Z}^d$ or a subgroup of it, and the unitary representation is given by the integer shifts. In general, the U-invariant subspace in $\mathcal{H}$ looks like

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n \ : \ x_n \in \ell^2(G), \ n = 1, 2, \dots, N \Big\},$$

where $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ denotes a fixed set of generators in $\mathcal{H}$.

For each $f \in \mathcal{V}_\Phi$ we consider two sets of samples $\{\mathcal{L}_m f(g)\}_{g\in G;\, n=1,2,\dots,M}$ defined as

$$\mathcal{L}_m f(g) := \big\langle f, U(g)\psi_m \big\rangle_{\mathcal{H}} \quad \text{or} \quad \mathcal{L}_m f(g) := \big[U(-g)f\big](t_m), \quad g \in G\,,$$

where, in the first case $\psi_1, \psi_2, \dots, \psi_M$ denote M elements in $\mathcal{H}$, which do not belong necessarily to $\mathcal{V}_\Phi$, and, in the second case, we take $\mathcal{H} := L^2(\mathbb{R}^d)$ and $t_1, t_2, \dots, t_M$ are M fixed points in $\mathbb{R}^d$. In the special case where $\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big[U(p)f\big](t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$, the above samples correspond to average or pointwise sampling, respectively, in the corresponding shift-invariant subspace V^2_Φ of $L^2(\mathbb{R}^d)$.

These data samples have in common that can be expressed as a convolution system in the product Hilbert space $\ell^2_N(G) := \ell^2(G) \times \cdots \times \ell^2(G)$ (N times); namely, for $f = \sum_{n=1}^N \sum_{g\in G} x_n(g)U(g)\varphi_n$ in $\mathcal{V}_\Phi$ and $m = 1, 2, \dots, M$ we have

$$\mathcal{L}_m f(g) = \sum_{n=1}^{N} (a_{m,n} * x_n)(g), \ \ g \in G\,,$$

for some MN sequences $a_{m,n} \in \ell^2(G)$ (see Section 2 below). Thus, in general, the acquisition of samples can be modeled as a filtering process in $\ell^2_N(G)$. This is very usual situation: the regular samples $\{f(n)\}_{n\in\mathbb{Z}}$ of any bandlimited function f in the Paley-Wiener space PW_π are given as $f(n) = (f * \text{sinc})(n), \ \ n \in \mathbb{Z}$, where sinc denotes the cardinal sine function. In the case of average sampling we have that $\langle f, \psi(\cdot - n)\rangle_{L^2(\mathbb{R})} = (f * \widetilde{\psi})(n), \ \ n \in \mathbb{Z}$, where $\widetilde{\psi}(t) = \overline{\psi(-t)}$ is the average function.

Under appropriate hypotheses on the Fourier transforms $\widehat{a}_{m,n} \in L^2(\widehat{G})$ of $a_{m,n} \in \ell^2(G)$ we obtain (see Thm. 1 in Section 4) necessary and sufficient conditions for the existence of stable reconstruction formulas in $\mathcal{V}_\Phi$ having the form

$$f = \sum_{m=1}^{M} \sum_{g\in G} \mathcal{L}_m f(g)\, U(g) S_m\,,$$

for some sampling functions $S_m \in \mathcal{V}_\Phi, \ m = 1, 2, \dots, M$, where the corresponding sequence $\{U(g)S_m\}_{g\in G;\, m=1,2,\dots,M}$ forms a frame for $\mathcal{V}_\Phi$. The use of the Fourier transform in $\ell^2(G)$, and the discrete nature of the sampling problem treated here impose that G will be a countable discrete abelian group. However, as we will see in Section 4.2, some cases involving non-abelian groups expressed as semi-direct or direct product of groups can be considered inside our study; this is the case of crystallographic groups. Notice that working in locally compact abelian groups is not just a unified way of dealing with the classical groups $\mathbb{R}^d, \mathbb{Z}^d, \mathbb{T}^d, \mathbb{Z}^d_s$: signal processing often involves products of these groups which are also locally compact abelian groups. For example, multichannel video signal involves the group $\mathbb{Z}^d \times \mathbb{Z}_s$, where d is the number of channels and s the number of pixels of each image.

The used mathematical technique is that of frame theory (see, for instance, Ref. [7]). The existence of the above sampling formula relies on the existence of dual frames for the Hilbert space product $\ell^2_{_N}(G)$ having the form $\{T_g\,\mathbf{b}_m\}_{g\in G;\, m=1,2,\dots,M}$, where $\mathbf{b}_m \in \ell^2_{_N}(G)$ and $T_g\mathbf{b}_m = \mathbf{b}_m(\cdot - g)$ denotes the translation operator in $\ell^2_{_N}(G)$. This can be reformulated as follows: given an analysis convolution system $\mathcal{A} : \ell^2_{_N}(G) \to \ell^2_{_M}(G)$ associated to data sampling, there exists another synthesis convolution system $\mathcal{B} : \ell^2_{_M}(G) \to \ell^2_{_N}(G)$ such that $\mathcal{B}\,\mathcal{A} = \mathcal{I}_{\ell^2_{_N}(G)}$. In other words, working in the Fourier domain $L^2(\widehat{G})$, we exploit the relationship between bounded convolution systems and frame theory in the product Hilbert space $\ell^2_{_N}(G)$. All needed results on this relationship are included in Section 3.

Finally, it is worth to mention that most of the well known sampling results can be considered as particular examples of this approach; see Sections 4.2–4.4 for the details. A comparison with some previous similar sampling results is presented in Section 4.5, where some affinities and differences are exhibited.

2 Data samples as a filtering process

Let $\mathcal{H}$ be a separable Hilbert space, and let $G \ni g \mapsto U(g) \in \mathcal{U}(\mathcal{H})$ be a unitary representation of a countable discrete abelian group $(G,+)$ on $\mathcal{H}$, i.e., it satisfies $U(g+g') = U(g)U(g')$, $U(-g) = U^{-1}(g) = U^*(g)$ for $g, g' \in G$. Given a set $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ of generators in $\mathcal{H}$ we consider the subspace $\mathcal{H}$ defined as $\mathcal{V}_\Phi := \overline{\operatorname{span}}_{\mathcal{H}}\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$. Assuming that $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ is a Riesz sequence in $\mathcal{H}$, i.e., a Riesz basis for $\mathcal{V}_\Phi$, this subspace can be expressed as

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n \;:\; x_n \in \ell^2(G),\ n = 1,2,\dots,N \Big\}.$$

Let us motivate the sampling approach followed in this work by means of a couple of examples:
• Given M elements $\psi_m \in \mathcal{H}$, $m = 1,2,\dots,M$, which do not belong necessarily to $\mathcal{V}_\Phi$, for any $f \in \mathcal{V}_\Phi$ we define for $m = 1,2,\dots,M$ its (generalized) average samples as

$$\mathcal{L}_m f(g) := \big\langle f, U(g)\psi_m \big\rangle_{\mathcal{H}}, \quad g \in G\,. \tag{1}$$

These samples can be expressed as the output of a convolution system. Indeed, for any $f = \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n$ in $\mathcal{V}_\Phi$, for each $m = 1,2,\dots,M$ one immediately gets

$$\mathcal{L}_m f(g) = \sum_{n=1}^{N} (a_{m,n} * x_n)(g), \ \ g \in G\,, \tag{2}$$

with $a_{m,n}(g) = \langle \varphi_n, U(g)\psi_m \rangle_{\mathcal{H}}$, $g \in G$. Notice that each $a_{m,n}$ belongs to $\ell^2(G)$ since the sequence $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ is, in particular, a Bessel sequence in $\mathcal{H}$.

• Suppose now that $\mathcal{H} = L^2(\mathbb{R}^d)$ and consider M fixed points $t_m \in \mathbb{R}^d$, $m = 1, 2, \dots, M$. For each $f \in \mathcal{V}_\Phi$ we define formally its samples, for $m = 1, 2, \dots, M$, as

$$\mathcal{L}_m f(g) := \big[U(-g)f\big](t_m), \quad g \in G\,. \tag{3}$$

It is straightforward to check that, for $f = \sum_{n=1}^{N}\sum_{g\in G} x_n(g)U(g)\varphi_n$ in $\mathcal{V}_\Phi$, expression 2 holds for $a_{m,n}(g) = \big[U(-g)\varphi_n\big](t_m)$, $g \in G$. Under mild hypotheses (see Section 4.3) one can obtain that $\mathcal{V}_\Phi$ is a reproducing kernel Hilbert space of continuous functions in $L^2(\mathbb{R}^d)$ where samples (3) are well defined with corresponding $a_{m,n} \in \ell^2(G)$, and yielding pointwise sampling in $\mathcal{V}_\Phi$.

The above two situations englobe most of the regular (average or pointwise) sampling appearing in mathematical or engineering literature as we will see in Section 4.

Consequently, one can think of a sampling process in subspace $\mathcal{V}_\Phi$ as M expressions like (2), i.e., a convolution system $\mathcal{A}$ defined in the product Hilbert space $\ell^2_N(G) := \ell^2(G) \times \cdots \times \ell^2(G)$ (N times) by means of a matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}(\ell^2(G))$, i.e., a $M \times N$ matrix with entries in $\ell^2(G)$, as

$$\mathcal{A}(\mathbf{x}) = A * \mathbf{x}\,, \quad \mathbf{x} = (x_1, x_2, \dots, x_N)^\top \in \ell^2_N(G)\,,$$

where $A * \mathbf{x}$ denotes the (matrix) convolution

$$(A * \mathbf{x})(g) = \sum_{g'\in G} A(g - g')\,\mathbf{x}(g'), \quad g \in G\,.$$

Note that the m-th entry of $A * \mathbf{x}$ is $\sum_{n=1}^{N}(a_{m,n} * x_n)$, where x_n denotes the n-th entry of $\mathbf{x} \in \ell^2_N(G)$.

The main aim in this paper is to recover, in a stable way, any $\mathbf{x} \in \ell^2_N(G)$, or equivalently, the corresponding $f = \sum_{n=1}^{N}\sum_{g\in G} x_n(g)U(g)\varphi_n \in \mathcal{V}_\Phi$, from the vector data

$$\boldsymbol{\mathcal{L}} f(g) := \big(\mathcal{L}_1 f(g), \mathcal{L}_2 f(g), \dots, \mathcal{L}_M f(g)\big)^\top = \big[\mathcal{A}(\mathbf{x})\big](g)\,, \quad g \in G\,,$$

i.e., from the output $\mathcal{A}(\mathbf{x})$ of the convolution system $\mathcal{A}$ with associated matrix A, in case the vector sampling $\boldsymbol{\mathcal{L}} f \in \ell^2_M(G)$.

2.1 A brief on harmonic analysis on discrete abelian groups

Let $(G, +)$ be a countable discrete abelian group and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unidimensional torus. We say that $\xi : G \mapsto \mathbb{T}$ is a character of G if $\xi(g + g') = \xi(g)\xi(g')$ for all $g, g' \in G$. We denote $\xi(g) = \langle g, \xi\rangle$. By defining $(\xi+\xi')(g) = \xi(g)\xi'(g)$, the set of characters $\widehat{G}$ is a group, called the dual group of G; since G is discrete, the group $\widehat{G}$ is compact [9, Prop. 4.4]. In particular, it is known that $\widehat{\mathbb{Z}} \cong \mathbb{T}$, with $\langle n, z\rangle = z^n$, and $\widehat{\mathbb{Z}}_s \cong \mathbb{Z}_s := \mathbb{Z}/s\mathbb{Z}$, with $\langle n, m\rangle = W_s^{nm}$, where $W_s = e^{2\pi i/s}$.

There exists a unique measure, the Haar measure μ on $\widehat{G}$ satisfying $\mu(\xi+E)=\mu(E)$, for every Borel set $E\subset\widehat{G}$, and $\mu(\widehat{G})=1$. We denote $\int_{\widehat{G}} X(\xi)d\xi=\int_{\widehat{G}} X(\xi)d\mu(\xi)$.
If $G=\mathbb{Z}$,

$$\int_{\widehat{G}} X(\xi)d\xi=\int_{\mathbb{T}} X(z)dz=\frac{1}{2\pi}\int_0^{2\pi} X(e^{iw})dw\,,$$

and if $G=\mathbb{Z}_s$,

$$\int_{\widehat{G}} X(\xi)d\xi=\int_{\mathbb{Z}_s} X(n)dn=\frac{1}{s}\sum_{n\in\mathbb{Z}_s} X(n)\,.$$

If $G_1,G_2,\ldots,G_d$ are abelian discrete groups then the dual group of the product group is $\big(G_1\times G_2\times\ldots\times G_d\big)^\wedge\cong\widehat{G}_1\times\widehat{G}_2\times\ldots\times\widehat{G}_d$ with

$$\big\langle\,(g_1,g_2,\ldots,g_d)\,,\,(\xi_1,\xi_2\ldots,\xi_d)\,\big\rangle=\langle g_1,\xi_1\rangle\langle g_2,\xi_2\rangle\cdots\langle g_d,\xi_d\rangle\,.$$

For $x\in\ell^1(G)$ its *Fourier transform* is defined as

$$X(\xi)=\widehat{x}(\xi):=\sum_{g\in G} x(g)\overline{\langle g,\xi\rangle}=\sum_{g\in G} x(g)\langle -g,\xi\rangle\,,\quad \xi\in\widehat{G}\,.$$

The Plancherel theorem extends uniquely the Fourier transform on $\ell^1(G)\cap\ell^2(G)$ to a unitary isomorphism from $\ell^2(G)$ to $L^2(\widehat{G})$. For the details see, for instance, Ref. [9].

3 Convolution systems on discrete abelian groups

This section is devoted to collect some known results on discrete convolution systems, and to prove the new ones needed in the sequel. We will consider bounded operators $\mathcal{A}:\ell^2_N(G)\to\ell^2_M(G)$ expressed as $\mathcal{A}(\mathbf{x})=A*\mathbf{x}$ for each $\mathbf{x}\in\ell^2_N(G)$. For a fixed $g\in G$ we denote, as usually, the translation by g of any $\mathbf{x}\in\ell^2_N(G)$ as $T_g\mathbf{x}(h)=\mathbf{x}(h-g)$, $h\in G$. The first two results can be found in [18, Thms. 2-3]

Proposition 1. *Given $A\in\mathcal{M}_{M\times N}\big(\ell^2(G)\big)$, the operator $\mathcal{A}:\mathbf{x}\mapsto A*\mathbf{x}$ is a well defined bounded operator from $\ell^2_N(G)$ into $\ell^2_M(G)$ if and only if $\widehat{A}\in\mathcal{M}_{M\times N}\big(L^\infty(\widehat{G})\big)$, where*

$$\widehat{A}(\xi):=\big[\widehat{a}_{m,n}(\xi)\big]\,,\quad a.e.\ \xi\in\widehat{G}\,.$$

Proposition 2. *For a linear operator $\mathcal{A}:\ell^2_N(G)\to\ell^2_M(G)$ the following conditions are equivalent:*

(a) $\mathcal{A}$ is a bounded operator that conmutes with translations, i.e., $\mathcal{A}T_g=T_g\mathcal{A}$, for all $g\in G$.

(b) There exists a matrix $A\in\mathcal{M}_{M\times N}\big(\ell^2(G)\big)$ that satisfies $\widehat{A}\in\mathcal{M}_{M\times N}\big(L^\infty(\widehat{G})\big)$ and such that $\mathcal{A}(\mathbf{x})=A*\mathbf{x}$ for each $\mathbf{x}\in\ell^2_N(G)$.

(c) There exists a matrix $\Lambda\in\mathcal{M}_{M\times N}\big(L^\infty(\widehat{G})\big)$ such that $\widehat{\mathcal{A}(\mathbf{x})}=\Lambda\cdot\widehat{\mathbf{x}}$ for each $\mathbf{x}\in\ell^2_N(G)$.

The matrices A and Λ satisfying (b) and (c) are unique and satisfy $\Lambda=\widehat{A}$.

Under equivalent conditions in Prop. 2, we say that $\mathcal{A}$ is a *bounded convolution operator*, and the unique matrix $\widehat{A}$ which satisfies $\widehat{\mathcal{A}(\mathbf{x})} = \widehat{A}\cdot\widehat{\mathbf{x}}$ for each $\mathbf{x} \in \ell^2_{_N}(G)$, is called the *transfer matrix* of the operator $\mathcal{A}$.

Proposition 3. *Let $\mathcal{A} : \ell^2_{_N}(G) \to \ell^2_{_M}(G)$ be a bounded convolution operator with transfer matrix $\widehat{A}$. Then:*

(a) The adjoint operator $\mathcal{A}^*$ is a bounded convolution operator with transfer matrix $\widehat{A}^*$, the adjoint matrix (transpose conjugate) of $\widehat{A}$, i.e., $\widehat{A}^*(\xi) = [\widehat{A}(\xi)]^*$, a.e. $\xi \in \widehat{G}$ (in the sequel $\widehat{A}(\xi)^*$).

(b) If $\mathcal{B} : \ell^2_{_M}(G) \to \ell^2_{_K}(G)$ is other bounded convolution operator with transfer matrix $\widehat{B}$ then the composition $\mathcal{BA} : \ell^2_{_N}(G) \to \ell^2_{_K}(G)$ is a bounded convolution operator with transfer matrix $\widehat{B} \cdot \widehat{A}$.

(c) $\|\mathcal{A}\| = \operatorname{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{A}(\xi)\|_2$, where $\|\cdot\|_2$ denotes the spectral norm of the matrix.

(d) $\mathcal{A}$ is injective with a closed range if and only if $\operatorname{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)^*\widehat{A}(\xi)] > 0$.

(e) $\mathcal{A}$ is onto if and only if $\operatorname{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)\widehat{A}(\xi)^*] > 0$.

(f) $\mathcal{A}$ is an isomorphism if and only if $M = N$ and $\operatorname{ess\,inf}_{\xi\in\widehat{G}} |\det \widehat{A}(\xi)| > 0$. In this case, $\mathcal{A}^{-1}$ is a bounded convolution operator with transfer matrix $(\widehat{A}\,)^{-1}$ and

$$\|\mathcal{A}^{-1}\| = \big(\operatorname*{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]\big)^{-1/2}.$$

Proof. (a) Using Prop. 2, for each $\mathbf{x} \in \ell^2_N(G)$ and $\mathbf{y} \in \ell^2_M(G)$ we have

$$\langle \widehat{\mathbf{x}}, \widehat{\mathcal{A}^*\mathbf{y}} \rangle_{L^2_N(\widehat{G})} = \langle \mathbf{x}, \mathcal{A}^*\mathbf{y} \rangle_{\ell^2_N(G)} = \langle \mathcal{A}\mathbf{x}, \mathbf{y} \rangle_{\ell^2_M(G)} = \langle \widehat{A}\cdot\widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle_{L^2_M(\widehat{G})} = \langle \widehat{\mathbf{x}}, \widehat{A}^*\cdot\widehat{\mathbf{y}} \rangle_{L^2_N(\widehat{G})}\,.$$

Hence $\widehat{\mathcal{A}^*\mathbf{y}} = \widehat{A}^*\cdot\widehat{\mathbf{y}}$ for all $\mathbf{y} \in \ell^2_N(G)$, and the result follows from Prop. 2.

(b) For each $\mathbf{x} \in \ell^2_N(G)$ we have that $\widehat{\mathcal{BA}(\mathbf{x})} = \widehat{B}\cdot\widehat{\mathcal{A}(\mathbf{x})} = \widehat{B}\cdot\widehat{A}\cdot\widehat{\mathbf{x}}$. Since the entries of $\widehat{A}$ and $\widehat{B}$ belong to $L^\infty(\widehat{G})$, we get that $\widehat{B}\cdot\widehat{A} \in \mathcal{M}_{_{K\times N}}(L^\infty(\widehat{G}))$, and the result follows from Prop. 2.

(c) The result is proved in [18, Cor. 6] for the case $N = M$. Hence, we obtain

$$\|\mathcal{A}\|^2 = \|\mathcal{A}^*\mathcal{A}\| = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{A}(\xi)^*\widehat{A}(\xi)\|_2 = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{A}(\xi)\|_2^2\,.$$

(d) A bounded operator $\mathcal{A}$ between Hilbert spaces is injective with a closed range if and only if the operator $\mathcal{A}^*\mathcal{A}$ is invertible. By using (a) and (b), we have that $\mathcal{A}^*\mathcal{A}$ is a bounded convolution operator with transfer matrix $\widehat{A}(\xi)^*\widehat{A}(\xi)$, and the result follows from [18, Thm. 7].

(e) A bounded operator $\mathcal{A}$ is onto if and only if its adjoint operator $\mathcal{A}^*$ is injective with a closed range; from (a), the transfer matrix of $\mathcal{A}^*$ is $\widehat{A}^*$. Thus, the result follows from (d).

(f) This characterization is a consequence of (d) and (e). From [18, Thm. 7], the inverse operator $\mathcal{A}^{-1}$ is a bounded convolution operator with transfer matrix $\widehat{A}(\xi)^{-1}$ and norm $\|\mathcal{A}^{-1}\| = \big(\operatorname{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]\big)^{-1/2}$

□

Note that from (a) the matrix associated with the adjoint operator $\mathcal{A}^*$ is not the adjoint matrix of A, but the one defined by means of the involution

$$A^* = \big[a^*_{m,n}\big]^\top \in \mathcal{M}_{N\times M}\big(\ell^2(G)\big) \quad \text{where} \quad a^*_{m,n}(g) := \overline{a_{m,n}(-g)}, \quad g\in G. \tag{4}$$

Indeed, since $\widehat{a^*_{m,n}}(\xi) = \overline{\widehat{a}_{m,n}(\xi)}$, we have $\widehat{A^*}(\xi) = \widehat{A}(\xi)^* = \widehat{A}^*(\xi)$, a.e. $\xi \in \widehat{G}$.

3.1 Dual frames in $\ell^2_N(G)$ having the form $\{T_g\,\mathbf{b}_m\}_{g\in G;\, m=1,2,\dots,M}$

Given a matrix $B \in \mathcal{M}_{N\times M}(\ell^2(G))$, the associated convolution operator $\mathcal{B} : \ell^2_M(G) \to \ell^2_N(G)$ can be written in terms of its M columns $\mathbf{b}_1, \mathbf{b}_2 \dots, \mathbf{b}_M$, as

$$\mathcal{B}(\mathbf{x}) = B * \mathbf{x} = \sum_{m=1}^{M}\sum_{g\in G} x_m(g) T_g \mathbf{b}_m\,, \quad \mathbf{x} \in \ell^2_M(G)\,, \tag{5}$$

where $T_g\mathbf{b}_m = \mathbf{b}_m(\cdot - g)$ denotes the translation operator for each $m = 1, 2, \dots, M$. In other words, operator $\mathcal{B}$ is the *synthesis operator* of the sequence $\big\{T_g\mathbf{b}_m\big\}_{g\in G;\, m=1,2,\dots,M}$ in $\ell^2_N(G)$.

Thus Props. 1, 2 and 3 can be translated (interchanging M by N) to the associated sequence $\big\{T_g\mathbf{b}_m\big\}_{g\in G;\, m=1,2,\dots,M}$. For instance, since a sequence in a Hilbert space is a Bessel sequence if and only if its synthesis operator is bounded and, in this case, its optimal Bessel bound is the square of the synthesis operator norm [7], from Props. 1 and 3 we get

Proposition 4. *The sequence $\big\{T_g\mathbf{b}_m\big\}_{g\in G;\, m=1,2,\dots,M}$ is a Bessel sequence for $\ell^2_N(G)$ if and only if the transfer matrix $\widehat{B}$ belongs to $\mathcal{M}_{N\times M}\big(L^\infty(\widehat{G})\big)$. In this case the optimal Bessel bound is $\beta_B = \operatorname{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{B}(\xi)\|_2^2$.*

Let $\mathbf{a}^*_m$ denote the m-th column of the matrix A^*, the associated matrix of $\mathcal{A}^*$, given in (4). The convolution operator $\mathcal{A} : \ell^2_N(G) \to \ell^2_M(G)$ can also be written as

$$[\mathcal{A}(\mathbf{x})]_m(g) = [A * \mathbf{x}]_m(g) = \big\langle \mathbf{x}, T_g\mathbf{a}^*_m\big\rangle_{\ell^2_N(G)}\,. \tag{6}$$

In other words, operator $\mathcal{A}$ is the *analysis operator* for the sequence $\big\{T_g\mathbf{a}^*_m\big\}_{g\in G;\, m=1,2,\dots,M}$ in $\ell^2_N(G)$.

Proposition 5. *Assume that $\widehat{A} \in \mathcal{M}_{M\times N}(L^\infty(\widehat{G}))$. Then:*

(a) *The sequence* $\{T_g \mathbf{a}^*_m\}_{g\in G;\, m=1,2,\ldots,M}$ *is a frame for* $\ell^2_{_N}(G)$ *if and only if*

$$\operatorname*{ess\,inf}_{\xi\in\widehat{G}} \det\big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big] > 0.$$

In this case, the optimal frame bounds are

$$\alpha_{_A} = \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)] \quad \text{and} \quad \beta_{_A} = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \lambda_{\max}[\widehat{A}(\xi)^*\widehat{A}(\xi)]\,.$$

(b) *The sequence* $\{T_g \mathbf{a}^*_m\}_{g\in G;\, m=1,2,\ldots,M}$ *is a Riesz basis for* $\ell^2_{_N}(G)$ *if and only if* $N = M$ *and* $\operatorname*{ess\,inf}_{\xi\in\widehat{G}} \big|\det[\widehat{A}(\xi)]\big| > 0$.

Proof. (a) Since $\widehat{A} \in \mathcal{M}_{_{M\times N}}(L^\infty(\widehat{G}))$, from Prop. 4 the sequence $\{T_g \mathbf{a}^*_m\}_{g\in G;\, m=1,2,\ldots,M}$, whose corresponding transfer matrix is $\widehat{A}^*$, is a Bessel sequence of $\ell^2_{_N}(G)$. Since a Bessel sequence is a frame if and only if its analysis operator $\mathcal{A}$ is injective with a closed range (see, for instance, Ref. [7]), the result is a consequence of Prop. 3(d). Since the optimal upper frame bound $\beta_{_A}$ is the squared norm of the analysis operator $\mathcal{A}$, and the optimal lower frame bound $\alpha_{_A}$ is the reciprocal of the norm of the inverse of the frame operator $\mathcal{A}^*\mathcal{A}$ (see, for instance, Ref. [7]), from Prop. 3 we get

$$\beta_{_A} = \|\mathcal{A}\|^2 = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{A}(\xi)\|_2^2 = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \lambda_{\max}[\widehat{A}(\xi)^*\widehat{A}(\xi)],$$

$$\alpha_{_A} = \|(\mathcal{A}^*\mathcal{A})^{-1}\|^{-1} = \big(\operatorname*{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)\widehat{A}(\xi)^*\widehat{A}(\xi)]\big)^{1/2} = \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]\,.$$

(b) The Bessel sequence $\{T_g \mathbf{a}^*_m\}_{g\in G;\, m=1,2,\ldots,M}$ is a Riesz basis for $\ell^2_{_N}(G)$ if and only if its synthesis operator $\mathcal{A}^*$ is an isomorphism (see, for instance, Ref. [7]). Hence, the result is a consequence of Prop. 3(f). □

Proposition 6. *Assume that* $\widehat{A} \in \mathcal{M}_{_{M\times N}}(L^\infty(\widehat{G}))$ *and* $\widehat{B} \in \mathcal{M}_{_{N\times M}}(L^\infty(\widehat{G}))$. *Then the sequences* $\{T_g \mathbf{a}^*_n\}_{g\in G;\, n=1,2,\ldots,M}$ *and* $\{T_g \mathbf{b}_n\}_{g\in G;\, n=1,2,\ldots,M}$ *form a pair of dual frames for* $\ell^2_{_N}(G)$ *if and only if*

$$\widehat{B}(\xi)\,\widehat{A}(\xi) = I_{_N}\,, \quad \text{a.e. } \xi \in \widehat{G}\,.$$

Proof. Having in mind that the analysis operator of $\{T_g \mathbf{a}^*_n\}_{g\in G;\, n=1,2,\ldots,M}$ is $\mathcal{A}$ and that the synthesis operator of $\{T_g \mathbf{b}_n\}_{g\in G;\, n=1,2,\ldots,M}$ is $\mathcal{B}$, we obtain that these two Bessel sequences form a pair of dual frames if and only if $\mathcal{B}\mathcal{A} = \mathcal{I}_{\ell^2_N(G)}$ [7, Lemma 6.3.2] or, equivalently, $\widehat{B}(\xi)\,\widehat{A}(\xi) = I_{_N}$, a.e. $\xi \in \widehat{G}$. (see Prop. 3(b)). □

4 The resulting sampling theory

In this section we propose a regular sampling theory for a multiply generated U-invariant subspace $\mathcal{V}_\Phi$ in a separable Hilbert space $\mathcal{H}$. This theory includes most of classical well known regular sampling results for shift-invariant subspaces of $L^2(\mathbb{R}^d)$. Besides, we obtain new sampling results; for instance, those associated with crystallographic groups.

4.1 Sampling in a U-invariant subspace with multiple generators

Suppose that $g \mapsto U(g)$ is a unitary representation of the countable discrete abelian group G on a separable Hilbert space $\mathcal{H}$, and assume that for a fixed set of generators $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ in $\mathcal{H}$ the sequence $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ is a Riesz sequence for $\mathcal{H}$. For necessary and sufficient conditions see Ref. [18]; see also Refs. [1, 3, 5, 15, 16]. Thus, we consider the U-invariant subspace in $\mathcal{H}$

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n \,:\, x_n \in \ell^2(G),\ n = 1, 2, \dots, N \Big\}. \tag{7}$$

For a given matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$, we consider the vector samples $\boldsymbol{\mathcal{L}} f$ of any $f = \sum_{n=1}^{N} \sum_{g\in G} x_n(g)U(g)\varphi_n \in \mathcal{V}_\Phi$ defined by

$$\boldsymbol{\mathcal{L}} f(g) := \big(\mathcal{L}_1 f(g), \mathcal{L}_2 f(g), \dots, \mathcal{L}_M f(g)\big)^\top = (A * \mathbf{x})(g) = \big[\mathcal{A}(\mathbf{x})\big](g)\,, \quad g \in G\,. \tag{8}$$

Assume $\widehat{A} \in \mathcal{M}_{M\times N}(L^\infty(\widehat{G}))$ and $\operatorname{ess\,inf}_{\xi\in\widehat{G}} \det\big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big] > 0$. Since $[\mathcal{A}(\mathbf{x})]_m(g) = \big\langle \mathbf{x}, T_g \mathbf{a}_m^* \big\rangle_{\ell^2_N(G)}$, the optimal frame bounds given in Prop. 5 provide relevant information about the stability of the recovering. Namely,

$$\alpha_A \|\mathbf{x}\|^2 \le \sum_{m=1}^{M} \sum_{g\in G} |\mathcal{L}_m f(g)|^2 \le \beta_A \|\mathbf{x}\|^2\,, \quad \mathbf{x} \in \ell^2_N(G)\,,$$

where $\alpha_A = \operatorname{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]$ and $\beta_A = \operatorname{ess\,sup}_{\xi\in\widehat{G}} \lambda_{\max}[\widehat{A}(\xi)^*\widehat{A}(\xi)]$. Moreover, denoting by α_Φ and β_Φ the Riesz bounds for $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ (see [18, Thm. 9]) we have

$$\alpha_\Phi \alpha_A \|f\|^2 \le \sum_{m=1}^{M} \sum_{g\in G} |\mathcal{L}_m f(g)|^2 \le \beta_\Phi \beta_A \|f\|^2\,, \quad f \in \mathcal{V}_\Phi\,.$$

Now, for the recovery of any $f \in \mathcal{V}_\Phi$ from its generalized samples (8), the idea is to find a $N \times M$ matrix $\widehat{B} \in \mathcal{M}_{N\times M}(L^\infty(\widehat{G}))$ such that $\widehat{B}(\xi)\,\widehat{A}(\xi) = I_N$, a.e. $\xi \in \widehat{G}$. In other words, the corresponding convolution operator $\mathcal{B}(\mathbf{x}) = B * \mathbf{x}$ should satisfy $\mathbf{x} = \mathcal{B}\mathcal{A}(\mathbf{x}) = \mathcal{B}(\boldsymbol{\mathcal{L}} f)$, that is

$$\mathbf{x} = B * \boldsymbol{\mathcal{L}} f \quad \text{and} \quad f = \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n\,, \quad f \in \mathcal{V}_\Phi\,.$$

Moreover, an explicit structured sampling formula can be obtained. Namely, the recovering of $\mathbf{x} = B * \boldsymbol{\mathcal{L}} f$ from the samples $\boldsymbol{\mathcal{L}} f$ can be written as an expansion in terms of a pair of dual frames (see Eqs. (5)–(6) and Prop. 6)

$$\mathbf{x} = \sum_{m=1}^{M} \sum_{g\in G} \big\langle \mathbf{x}, T_g \mathbf{a}_m^* \big\rangle_{\ell^2_N(G)} T_g \mathbf{b}_m = \sum_{m=1}^{M} \sum_{g\in G} \mathcal{L}_m f(g)\, T_g \mathbf{b}_m \quad \text{in } \ell^2_N(G)\,. \tag{9}$$

Besides, we consider the natural isomorphism $\mathcal{T}_{U,\Phi} : \ell^2_N(G) \to \mathcal{V}_\Phi$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ for $\ell^2_N(G)$ onto the Riesz basis $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ for $\mathcal{V}_\Phi$. This isomorphism satisfies the *shifting property*:

$$\mathcal{T}_{U,\Phi}\big(T_g \mathbf{b}\big) = U(g)\big(\mathcal{T}_{U,\Phi} \mathbf{b}\big) \quad \text{for each } g \in G \text{ and } \mathbf{b} \in \ell^2_N(G)\,. \tag{10}$$

Finally, for each $f = \mathcal{T}_{U,\Phi}\mathbf{x} \in \mathcal{V}_\Phi$, applying the isomorphism $\mathcal{T}_{U,\Phi}$ on (9) and the shifting property (10) we obtain the sampling expansion in $\mathcal{V}_\Phi$

$$\begin{aligned} f &= \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\,\mathcal{T}_{U,\Phi}\big(T_g\mathbf{b}_m\big) = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\,U(g)\big(\mathcal{T}_{U,\Phi}\mathbf{b}_m\big) \\ &= \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\,U(g)S_m \quad \text{in } \mathcal{H}\,, \end{aligned} \tag{11}$$

where the reconstruction elements are given by $S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m \in \mathcal{V}_\Phi$, $m = 1, 2, \dots, M$, and the sequence $\{U(g)S_m\}_{g\in G;\, m=1,2,\dots,M}$ is a frame for $\mathcal{V}_\Phi$. In fact, the following sampling theorem in the subspace $\mathcal{V}_\Phi$ holds:

Theorem 1. *Let $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$ be the matrix defining the samples $\boldsymbol{\mathcal{L}}f(g)$, $g \in G$, for each $f \in \mathcal{V}_\Phi$ as in (8), and assume that its transfer matrix $\widehat{A}$ has all its entries in $L^\infty(\widehat{G})$. Then, the following statements are equivalent:*

(a) The constant $\delta_A := \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)^*\widehat{A}(\xi)] > 0$.

(b) There exist constants $0 < \alpha \le \beta$ such that

$$\alpha\|f\|^2 \le \sum_{m=1}^{M}\sum_{g\in G}|\mathcal{L}_m f(g)|^2 \le \beta\|f\|^2\,, \quad f \in \mathcal{V}_\Phi\,.$$

(c) There exists a matrix $\widehat{B} \in \mathcal{M}_{N\times M}\big(L^\infty(\widehat{G})\big)$ such that $\widehat{B}(\xi)\,\widehat{A}(\xi) = I_N$, a.e. $\xi \in \widehat{G}$.

(d) There exists a matrix $B \in \mathcal{M}_{N\times M}(\ell^2(G))$ with $\widehat{B} \in \mathcal{M}_{N\times M}(L^\infty(\widehat{G}))$, such that

$$\mathbf{x} = B * \boldsymbol{\mathcal{L}}f \quad \text{and} \quad f = \sum_{n=1}^{N}\sum_{g\in G} x_n(g)U(g)\varphi_n\,, \quad f \in \mathcal{V}_\Phi\,.$$

In other words, there exists a bounded convolution system $\mathcal{B} : \ell^2_M(G) \to \ell^2_N(G)$ such that $\mathcal{B}\mathcal{A} = \mathcal{I}_{\ell^2_N(G)}$.

(e) There exist M elements $S_m \in \mathcal{V}_\Phi$ such that the sequence $\big\{U(g)S_m\big\}_{g\in G;\, m=1,2,\dots,M}$ is a frame for $\mathcal{V}_\Phi$ and for each $f \in \mathcal{V}_\Phi$ the reconstruction formula

$$f = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\,U(g)S_m \quad \text{in } \mathcal{H} \tag{12}$$

holds.

(f) There exists a frame $\big\{S_{g,m}\big\}_{g\in G;\, m=1,2,\dots,M}$ for $\mathcal{V}_\Phi$ such that for each $f \in \mathcal{V}_\Phi$ the expansion

$$f = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\,S_{g,m} \quad \text{in } \mathcal{H}$$

holds.

In this case, the reconstruction elements $\{S_m\}_{m=1,2,\ldots,M}$ *in* $\mathcal{V}_\Phi$ *in formula* (12) *are necessarily obtained from the columns* $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_M$ *of a matrix* B *satisfying* (*c*), *i.e.,* $S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m = \sum_{n=1}^{N}\sum_{g\in G} b_{n,m}\, U(g)\varphi_n$, $m = 1, 2, \ldots, M$.

Proof. First we note that, since $\beta_{_A} < \infty$, condition $\delta_{_A} > 0$ is equivalent to condition $\alpha_{_A} > 0$. Now we prove that (a) and (b) are equivalent. Indeed, since $\mathcal{T}_{U,\Phi}$ is an isomorphism condition (b) is equivalent to the existence of $0 < \alpha_1 \le \beta_1$ such that

$$\alpha_1 \|\mathbf{x}\|^2 \le \sum_{m=1}^{M}\sum_{g\in G} |\mathcal{L}_m f(g)|^2 \le \beta_1 \|\mathbf{x}\|^2\,, \quad \mathbf{x} \in \ell^2_{_N}(G)\,.$$

Since Eq. (6), this is equivalent to be the sequence $\big\{T_g \mathbf{a}^*_m\big\}_{g\in G;\, m=1,2,\ldots,M}$ a frame for $\ell^2_N(G)$. Therefore, the result follows from Prop. 5.

Assume now that (a) holds. Then, from Prop. 3, operator $\mathcal{A}^*\mathcal{A}$ is invertible, and $\mathcal{B} := (\mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^*$ is a bounded convolution operator satisfying $\mathcal{B}\mathcal{A} = \mathcal{I}_{\ell^2_{_N}(G)}$. From Prop. 3, its transfer matrix satisfies the requirement in (c).

If $\widehat{B}$ satisfies (c), the bounded convolution operator $\mathcal{B}$ whose transfer matrix is $\widehat{B}$ satisfies $\mathcal{B}\mathcal{A} = \mathcal{I}_{\ell^2_{_N}(G)}$ from Prop. 3, that is, condition (d).

We have proved that condition (d) implies a sampling expansion as (12), where $S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m$, $m = 1, 2, \ldots, M$, and $\mathbf{b}_1, \ldots, \mathbf{b}_M$ are the columns of a matrix B satisfying (d). Besides, the sequence $\big\{U(g)S_m\big\}_{g\in G;\, m=1,2,\ldots,M} = \mathcal{T}_{U,\Phi}\big\{T_g\mathbf{b}_m\big\}_{g\in G;\, m=1,2,\ldots,M}$ is a frame since (9) is a frame expansion in $\ell^2_{_N}(G)$ and $\mathcal{T}^{-1}_{U,\Phi}$ an isomorphism. This proves condition (e) which trivially implies condition (f).

Finally, condition (f) implies (a). Applying $\mathcal{T}^{-1}_{U,\Phi}$ to the formula in (f) we obtain that $\big\{T_g\mathbf{a}^*_m\big\}_{g\in G;\, m=1,2,\ldots,M}$ and $\{\mathcal{T}^{-1}_{U,\Phi}S_{g,m}\}_{g\in G;\, m=1,2,\ldots,M}$ form a pair of dual frames for $\ell^2_{_N}(G)$; in particular, by using Prop. 5(a) we obtain that $\delta_{_A} > 0$. □

All the possible solutions of $\widehat{B}(\xi)\widehat{A}(\xi) = I_{_N}$ a.e. $\xi \in \widehat{G}$ with entries in $L^\infty(\widehat{G})$ are given in terms of the Moore-Penrose pseudo-inverse $\widehat{A}(\xi)^\dagger = \big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$ by means of the $N\times M$ matrices $\widehat{B}(\xi) := \widehat{A}(\xi)^\dagger + C(\xi)\big[I_M - \widehat{A}(\xi)\widehat{A}(\xi)^\dagger\big]$, where $C(\xi)$ denotes any $N\times M$ matrix with entries in $L^\infty(\widehat{G})$. Since $\mathbf{x} = B * \boldsymbol{\mathcal{L}} f$, from Prop. 1 we have that

$$\|\mathbf{x}\|^2_{\ell^2_N(G)} \le C\sum_{m=1}^{M}\sum_{g\in G}|\mathcal{L}_m f(g)|^2 \quad \text{where } \ C = \operatorname*{ess\,sup}_{\xi\in\widehat{G}} \|\widehat{B}(\xi)\|_2^2\,.$$

The best possible bound we can get is $C = \alpha_{_A}^{-1} = \operatorname{ess\,inf}_{\xi\in\widehat{G}} \lambda_{\min}[\widehat{A}(\xi)^*\widehat{A}(\xi)]^{-1}$, which correspond to choosing $\widehat{B} = \widehat{A}^\dagger$ or, equivalently, $B = (A^*A)^{-1}A^*$, the pseudo-inverse of A (see Ref. [7]).

Notice that in Thm. 1 necessarily $M \ge N$ where N is the number of generators in $\mathcal{V}_\Phi$. In case $M = N$, we have:

Corollary 2. *In case* $M = N$, *assume that the transfer matrix* $\widehat{A}(\xi)$ *has all entries in* $L^\infty(\widehat{G})$. *The following statements are equivalent:*

1. *The constant* $\operatorname*{ess\,inf}_{\xi\in\widehat{G}} \big|\det[\widehat{A}(\xi)]\big| > 0$.

2. *There exist N unique elements S_n, $n = 1, 2, \dots, N$, in $\mathcal{V}_\Phi$ such that the associated sequence $\big\{U(g)S_n\big\}_{g\in G;\, n=1,2,\dots,N}$ is a Riesz basis for $\mathcal{V}_\Phi$ and the sampling formula*

$$f = \sum_{n=1}^{N} \sum_{g\in G} \mathcal{L}_n f(g)\, U(g) S_n \qquad \text{in } \mathcal{H}$$

holds for each $f \in \mathcal{V}_\Phi$.

Moreover, the interpolation property $\mathcal{L}_n S_{n'}(g) = \delta_{n,n'}\delta_{g,0_G}$, where $g \in G$ and $n, n' = 1, 2, \dots, N$, holds.

Proof. In this case, the square matrix $\widehat{A}(\xi)$ is invertible and the result comes out from Prop. 5(b). The uniqueness of the coefficients in a Riesz basis expansion gives the interpolation property. □

4.1.1 A more general framework

A slightly more general setting is motivated by condition (f) in Thm. 1. Namely, let $F := \{f_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ be a Riesz sequence in a separable Hilbert space $\mathcal{H}$, and let $\mathcal{V}_F := \overline{\operatorname{span}}_{\mathcal{H}}\{f_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ be its associated subspace, that is,

$$\mathcal{V}_F = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g)\, f_{g,n} \;:\; x_n \in \ell^2(G),\ n = 1, 2, \dots, N \Big\}.$$

Given a matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$, for each $f = \sum_{n=1}^{N}\sum_{g\in G} x_n(g)\, f_{g,n}$ in $\mathcal{V}_F$ we define its data samples $\boldsymbol{\mathcal{L}} f$ by means of A and $\mathbf{x} = (x_1, x_2, \dots, x_N)^\top \in \ell^2_N(G)$ as

$$\boldsymbol{\mathcal{L}} f(g) := \big(\mathcal{L}_1 f(g), \mathcal{L}_2 f(g), \dots, \mathcal{L}_M f(g)\big)^\top = (A * \mathbf{x})(g)\,, \quad g \in G\,.$$

As before, the aim is the stable recovery of any $f \in \mathcal{V}_F$ from data $\boldsymbol{\mathcal{L}} f \in \ell^2_M(G)$. Under the hypotheses on the matrix A in Thm.1 there exists a frame $\{S_{g,m}\}_{g\in G;\, m=1,2,\dots,M}$ for $\mathcal{V}_F$ such that for each $f \in \mathcal{V}_F$ the reconstruction formula

$$f = \sum_{m=1}^{M} \sum_{g\in G} \mathcal{L}_m f(g)\, S_{g,m} \quad \text{in } \mathcal{H}$$

holds. Moreover, there exist $\mathbf{b}_m$ in $\ell^2_N(G)$, $m = 1, 2, \dots, M$, such that $S_{g,m} = \mathcal{T}_F\big(T_g \mathbf{b}_m\big)$, $g \in G$ and $m = 1, 2, \dots, M$, where $\mathcal{T}_F : \ell^2_N(G) \to \mathcal{V}_F$ stands for the natural isomorphism which maps the standard orthonormal basis $\{\boldsymbol{\delta}_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ for $\ell^2_N(G)$ on the Riesz basis $\{f_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ for $\mathcal{V}_F$. Since the subspace $\mathcal{V}_F$ has not any a priori structure, the same occurs for the reconstruction functions $S_{g,m}$.

4.2 Some regular sampling settings as particular examples

In this section, we illustrate the result in Thm. 1 with some average sampling examples.

• Choose $\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big(U(p)f\big)(t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$. Under the hypotheses in Thm. 1 for the average samples given by (1), i.e., for the associated matrix

$A = [a_{m,n}]$ where $a_{m,n}(p) = \langle \varphi_n, \psi_m(\cdot - p)\rangle_{L^2(\mathbb{R}^d)}$, we obtain oversampled *average sampling* in the classical shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$ described as

$$V_\Phi^2 = \Big\{ \sum_{n=1}^{N} \sum_{p\in\mathbb{Z}^d} x_n(p)\, \varphi_n(t-p) \,:\, x_n \in \ell^2(\mathbb{Z}^d),\, n = 1,2,\dots,N \Big\}.$$

Under mild hypotheses, the space V_Φ^2 is a reproducing kernel Hilbert space (RKHS). For each $f \in V_\Phi^2$ a sampling expansion having the form

$$f(t) = \sum_{m=1}^{M} \sum_{p\in\mathbb{Z}^d} \big\langle f, \psi_m(\cdot - p)\big\rangle_{L^2(\mathbb{R}^d)} S_m(t-p) \quad \text{in } L^2(\mathbb{R}^d)\,,$$

holds, for some sampling functions $S_m \in V_\Phi^2$, $m = 1,2,\dots,M$. Moreover, the sequence $\{S_m(t-p)\}_{p\in\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for V_Φ^2. As a consequence of the RKHS setting the convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$. As we will see later (see Section 4.4), this oversampling can be reduced by sampling on a sublattice $P\mathbb{Z}^d$ of $\mathbb{Z}^d$, where P denotes a $d\times d$ matrix with integer entries and positive determinant.

• The case where the group G is the *semi-direct product of two groups* can be easily reduced to the described situation in Section 4.1. Suppose that $(k,h) \mapsto U(k,h)$ is a unitary representation of the semi-direct product group $G = K \rtimes_\sigma H$ (or, in particular, the direct product $G = K \times H$) on a separable Hilbert space $\mathcal{H}$, where K is a countable discrete group and H a finite not necessarily abelian group; the subscript σ denotes the action of the group H on the group K, i.e., a homomorphism $\sigma : H \to Aut(K)$ mapping $h \mapsto \sigma_h$. The composition law in G is $(k_1,h_1)\,(k_2,h_2) := (k_1\sigma_{h_1}(k_2), h_1h_2)$ for $(k_1,h_1), (k_2,h_2) \in G$. In general, the group $G = K \rtimes_\sigma H$ is not abelian. In case $\sigma_h \equiv Id_K$ for each $h \in H$ we recover the direct product group $G = K \times H$.

Assume that for a fixed $\varphi \in \mathcal{H}$ the sequence $\big\{U(k,h)\varphi\big\}_{(k,h)\in G}$ is a Riesz sequence for $\mathcal{H}$. Thus, the U-invariant subspace in $\mathcal{H}$ spanned by $\big\{U(k,h)\varphi\big\}_{(k,h)\in G}$ can be described as

$$\mathcal{V}_\varphi = \Big\{ \sum_{(k,h)\in G} x(k,h)\, U(k,h)\varphi \;:\; \{x(k,h)\}_{(k,h)\in G} \in \ell^2(G) \Big\}.$$

Since $U(k,h)\varphi = U[(k,1_H)(0_K,h)]\varphi = U(k,1_H)\varphi_h$, where $\varphi_h := U(0_K,h)\varphi$ for $h \in H$. Assuming that the order of the group H is N, the subspace $\mathcal{V}_\varphi$ coincides with the subspace $\mathcal{V}_\Phi$ generated by the set $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$, i.e.,

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{k\in K} x_n(k) U(k,1_H)\varphi_n \,:\, x_n \in \ell^2(N),\; n = 1,2,\dots,N \Big\},$$

where $x_n(k) := x(k,h_n)$ and $\varphi_n := \varphi_{h_n}$, $\;n = 1,2,\dots,N$. For M fixed elements $\psi_m \in \mathcal{H}$, $m = 1,2,\dots,M$, not necessarily in $\mathcal{V}_\Phi$, we consider for each $f \in \mathcal{V}_\Phi$ its generalized samples defined as

$$\mathcal{L}_m f(k) := \big\langle f, U(k,1_H)\,\psi_m\big\rangle_{\mathcal{H}}\,, \quad k \in K\,, \quad m = 1,2,\dots,M\,. \tag{13}$$

Notice that these samples are a particular case of samples (1). Then, under the hypotheses in Thm.1 on the matrix $A=[a_{m,n}]$ where $a_{m,n}(k)=\big\langle \varphi, U[(-k,h_n)^{-1}]\psi_m\big\rangle_{\mathcal{H}}$, there exist M elements $S_m\in\mathcal{V}_\Phi$ such that the sequence $\big\{U(k,1_H)S_m\big\}_{k\in K;\, m=1,2,\dots,M}$ is a frame for $\mathcal{V}_\Phi$, and for each $f\in\mathcal{V}_\Phi$ we have the reconstruction formula

$$f=\sum_{m=1}^{M}\sum_{k\in K}\mathcal{L}_m f(k)\, U(k,1_H)S_m \quad \text{in } \mathcal{H}\,. \tag{14}$$

• An important case of the example above is given by *crystallographic groups*. Namely, the Euclidean motion group $E(d)$ is the semi-direct product $\mathbb{R}^d\rtimes_\sigma O(d)$ corresponding to the homomorphism $\sigma: O(d)\to Aut(\mathbb{R}^d)$ given by $\sigma_\gamma(x)=\gamma x$, where $\gamma\in O(d)$ and $x\in\mathbb{R}^d$; $O(d)$ denotes the orthogonal group of order d. The composition law on $E(d)=\mathbb{R}^d\rtimes_\sigma O(d)$ reads $(x,\gamma)\cdot(x',\gamma')=(x+\gamma x',\gamma\gamma')$.

Let P be a non-singular $d\times d$ matrix and Γ a finite subgroup of $O(d)$ of order N such that $\gamma(P\mathbb{Z}^d)=P\mathbb{Z}^d$ for each $\gamma\in\Gamma$. We consider the *crystallographic group* $\mathcal{C}_{P,\Gamma}:=P\mathbb{Z}^d\rtimes_\sigma\Gamma$ and its *quasi regular representation* (see Ref. [3]) on $L^2(\mathbb{R}^d)$

$$U(p,\gamma)f(t)=f[\gamma^\top(t-p)]\,,\quad p\in P\mathbb{Z}^d,\ \gamma\in\Gamma \text{ and } f\in L^2(\mathbb{R}^d)\,.$$

For a fixed $\varphi\in L^2(\mathbb{R}^d)$ such that the sequence $\big\{U(p,\gamma)\varphi\big\}_{(p,\gamma)\in\mathcal{C}_{P,\Gamma}}$ is a Riesz sequence for $L^2(\mathbb{R}^d)$ we consider the U-invariant subspace in $L^2(\mathbb{R}^d)$

$$\mathcal{V}_\varphi=\Big\{\sum_{(p,\gamma)\in\mathcal{C}_{P,\Gamma}} x(p,\gamma)\,\varphi[\gamma^\top(t-p)]\ :\ \{x(p,\gamma)\}\in\ell^2(\mathcal{C}_{P,\Gamma})\Big\} \tag{15}$$

Choosing M functions $\psi_m\in L^2(\mathbb{R}^d)$, $m=1,2,\dots,M$, we consider the average samples of $f\in\mathcal{V}_\varphi$

$$\mathcal{L}_m f(p)=\langle f, U(p,I)\psi_m\rangle=\langle f,\psi_m(\cdot-p)\rangle\,,\quad p\in P\mathbb{Z}^d\,.$$

Denoting $\{\gamma_1=I,\gamma_2,\dots,\gamma_N\}$ the elements of the group Γ, under the hypotheses of Thm. 1 on the matrix $A=[a_{m,n}]$ where $a_{m,n}(p)=\big\langle\varphi(t),\psi_m(\gamma_n t-p)\big\rangle_{L^2(\mathbb{R}^d)}$, there exist $M\geq N$ sampling functions $S_m\in\mathcal{V}_\varphi$ for $m=1,2,\dots,M$, such that the sequence $\{S_m(\cdot-p)\}_{p\in P\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for $\mathcal{V}_\varphi$, and the sampling expansion

$$f(t)=\sum_{m=1}^{M}\sum_{p\in P\mathbb{Z}^d}\big\langle f,\psi_m(\cdot-p)\big\rangle_{L^2(\mathbb{R}^d)}\, S_m(t-p) \quad \text{in } L^2(\mathbb{R}^d) \tag{16}$$

holds. If the generator φ is continuous in $\mathbb{R}^d$ and the function $t\mapsto\sum_{p\in\mathbb{Z}^d}|\varphi(t-p)|^2$ is bounded on $\mathbb{R}^d$, a standard argument shows that $\mathcal{V}_\varphi$ is a RKHS of bounded continuous functions in $L^2(\mathbb{R}^d)$ (see, for instance, Ref. [14]). As a consequence, convergence in $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

4.3 The case of pointwise samples whenever $\mathcal{H}=L^2(\mathbb{R}^d)$

Assume here that G is a countable discrete subgroup of a locally compact abelian group $\widetilde{G}$ and let $t\in\widetilde{G}\mapsto U(t)\in\mathcal{U}(L^2(\widetilde{G}))$ be a unitary representation of $\widetilde{G}$ on $L^2(\widetilde{G})$. Let $\mathcal{V}_\Phi$ be the corresponding U-invariant subspace of $\mathcal{H} = L^2(\widetilde{G})$ given in (7); for any $f \in \mathcal{V}_\Phi$ we consider the samples defined in (3) from M fixed points $t_m \in \widetilde{G}$, $m = 1, 2, \dots, M$, i.e.,

$$\mathcal{L}_m f(g) := \big[U(-g)f\big](t_m), \quad g \in G, \; m = 1, 2, \dots, M. \tag{17}$$

Let $A = [a_{m,n}]$ be the $M \times N$ matrix where $a_{m,n}(g) = \big[U(-g)\varphi_n\big](t_m)$, $g \in G$; assuming that, for each $t \in \widetilde{G}$, the sequence $\{[U(g)\varphi_n](t)\}_{g \in G}$ belongs to $\ell^2(G)$ for each $n = 1, 2, \dots, N$, the matrix A has its entries in $\ell^2(G)$. Moreover, if the functions $[U(g)\varphi_n](t)$, $g \in G$ and $n = 1, 2, \dots, N$, are continuous on $\widetilde{G}$, and the condition

$$\sup_{t \in \widetilde{G}} \sum_{g \in G} \big|[U(g)\varphi_n](t)\big|^2 < +\infty\,, \quad n = 1, 2, \dots, N\,, \tag{18}$$

holds, then the subspace $\mathcal{V}_\Phi$ is a reproducing kernel Hilbert space of continuous bounded functions in $L^2(\widetilde{G})$. In fact, it is a necessary and sufficient condition as the following result shows; its proof is analogous to that in [14, Lemma 4.2].

Proposition 7. *For any $\{x_n(g)\}_{g \in G;\, n=1,2,\dots,N} \in \ell^2_N(G)$ the series*

$$\sum_{n=1}^{N} \sum_{g \in G} x_n(g)\, [U(g)\varphi_n](t)$$

converges pointwise to a continuous bounded function on $\widetilde{G}$ if and only if for each $g \in G$ and $n = 1, 2, \dots, N$, the function $U(g)\varphi_n$ is continuous on $\widetilde{G}$, and condition (18) holds.

Notice that, whenever $\mathcal{H} = L^2(\mathbb{R}^d)$ and $[U(p)f](t) := f(t-p)$, $t \in \mathbb{R}^d$, $p \in \mathbb{Z}^d$, the samples in (17) read

$$\mathcal{L}_m f(p) = \big[U(-p)f\big](t_m) = f(p + t_m)\,, \quad p \in \mathbb{Z}^d \;\text{ and }\; m = 1, 2, \dots, M\,.$$

• Choosing $\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big(U(p)f\big)(t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$. Thus, under hypotheses in Thm. 1 on the matrix $A = [a_{m,n}]$ where $a_{m,n}(p) = \varphi_n(t_m + p)$ we obtain oversampled pointwise sampling in the shift-invariant subspace V^2_Φ of $L^2(\mathbb{R}^d)$, i.e., for each $f \in V^2_\Phi$ a sampling expansion having the form

$$f(t) = \sum_{m=1}^{M} \sum_{p \in \mathbb{Z}^d} f(p + t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d$$

holds, for some functions $S_m \in V^2_\Phi$, $m = 1, 2, \dots, M$. The convergence of the series in $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

• In the case of the quasi regular representation of the crystallographic group $\mathcal{C}_{P,\Gamma} = P\mathbb{Z}^d \rtimes_\sigma \Gamma$, for each $f \in \mathcal{V}_\varphi$ defined in (15) the samples (3) read

$$\mathcal{L}_m f(p) = \big[U(-p, I)f\big](t_m) = f(p + t_m)\,, \quad p \in P\mathbb{Z}^d \text{ and } m = 1, 2, \dots, M\,.$$

Under hypotheses in Thm. 1 on the matrix $A = [a_{m,n}]$ where $a_{m,n}(p) = \varphi[\gamma_n^\top(t_m - p)]$, there exist M functions $S_m \in \mathcal{V}_\varphi$, $m = 1, 2, \dots, M$, such that for each $f \in \mathcal{V}_\varphi$ the sampling formula

$$f(t) = \sum_{m=1}^{M} \sum_{p \in P\mathbb{Z}^d} f(p + t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

4.4 Sampling in a subgroup H of G

Let $(G,+)$ be a countable discrete LCA group, and let H be a subgroup of G with finite index L. We fix a set $\{g_0, g_1, \dots, g_L\}$ of representatives of the cosets of H, i.e., the group G can be decomposed as

$$G = (g_1 + H) \cup (g_2 + H) \cup \cdots \cup (g_L + H) \text{ with } (g_l + H) \cap (g_{l'} + H) = \varnothing \text{ for } l \neq l' .$$

Given a unitary representation $g \mapsto U(g)$ of the group G on a separable Hilbert space $\mathcal{H}$ and a set of generators $\Phi = \{\varphi_1, \varphi_{2,} \dots, \varphi_N\}$ in $\mathcal{H}$, we consider the subspace $\mathcal{V}_\Phi = \overline{\operatorname{span}}_{\mathcal{H}}\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$. In case $\{U(g)\varphi_n\}_{g\in G;\, n=1,2,\dots,N}$ is a Riesz sequence in $\mathcal{H}$, it can be expressed as

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g)\, U(g)\varphi_n \,:\, x_n \in \ell^2(G) \Big\} = \Big\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{h\in H} x_n(g_l + h)\, U(g_l + h)\varphi_n \Big\} ,$$

where the sequence

$$\mathbf{x}(h) := \big(x_{11}(h), \dots, x_{1L}(h), x_{21}(h), \dots, x_{2L}(h), \dots, x_{N1}(h), \dots, x_{NL}(h)\big)^\top \in \ell^2_{_{NL}}(H) ,$$

with $x_{nl}(h) := x_n(g_l + h)$. From now on we consider a new index nl, from 11 to NL, whose order is the indicated above. Next, for M fixed elements $\psi_m \in \mathcal{H}$, $m = 1, 2, \dots, M$, not necessarily in $\mathcal{V}_\Phi$, for each $f \in \mathcal{V}_\Phi$ we define its generalized samples

$$\mathcal{L}_m f(h) = \big\langle f, U(h)\psi_m \big\rangle_{\mathcal{H}}, \quad h \in H \text{ and } m = 1, 2, \dots, M . \tag{19}$$

For $f = \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{k\in H} x_n(g_l + k)\, U(g_l + k)\varphi_n$ in $\mathcal{V}_\Phi$, the samples (19) can be expressed as

$$\mathcal{L}_m f(h) = \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{k\in H} x_n(g_l + k) \big\langle \varphi_n, U(h - g_l - k)\psi_m \big\rangle = \sum_{n=1}^{N} \sum_{l=1}^{L} \big(a_{m,nl} *_H x_{nl}\big)(h) ,$$

where $a_{m,nl}(h) := \big\langle \varphi_n, U(h - g_l)\psi_m \big\rangle_{\mathcal{H}}$, $h \in H$, for $m = 1, 2, \dots, M$, $n = 1, 2, \dots, N$ and $l = 1, 2, \dots, L$. Notice that each $a_{m,nl} \in \ell^2(H)$. The subscript $*_H$ means convolution over the subgroup H.

If we consider the $M \times NL$ matrix $A = [a_{m,nl}]$, the hypotheses in Thm.1 on matrix A proves, with slight differences, that $M \geq NL$ and there exists a frame sequence $\big\{T_h \mathbf{b}_m\big\}_{h\in H;\, m=1,2,\dots,M}$ for $\ell^2_{_{NL}}(H)$ which is a dual frame of $\big\{T_h \mathbf{a}^*_m\big\}_{h\in H;\, m=1,2,\dots,M}$. Thus, for any $\mathbf{x} \in \ell^2_{_{NL}}(G)$ we have

$$\mathbf{x} = \sum_{m=1}^{M} \sum_{h\in H} \big\langle \mathbf{x}, T_h \mathbf{a}^*_m \big\rangle_{\ell^2_{_{NL}}(H)} T_h \mathbf{b}_m = \sum_{m=1}^{M} \sum_{h\in H} \mathcal{L}_m f(h)\, T_h \mathbf{b}_m \quad \text{in } \ell^2_{_{NL}}(H) . \tag{20}$$

Next, the natural isomorphism $\mathcal{T}_{U,\Phi} : \ell^2_{_{NL}}(H) \to \mathcal{V}_\Phi$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_{h,nl}\}$ for $\ell^2_{_{NL}}(H)$ on the Riesz basis $\big\{U(g_l + h)\varphi_n\big\}$ for $\mathcal{V}_\Phi$, and satisfies the *shifting property* $\mathcal{T}_{U,\Phi}\big(T_h \mathbf{b}\big) = U(h)\big(\mathcal{T}_{U,\Phi}\mathbf{b}\big)$ for each $h \in H$ and $\mathbf{b} \in \ell^2_{_{NL}}(H)$.

Applying the isomorphism $\mathcal{T}_{U,\Phi}$ in (20) we obtain that any $f = \mathcal{T}_{U,\Phi}\mathbf{x} \in \mathcal{V}_\Phi$ can be recovered from data $\{\mathcal{L}_m f(h)\}_{h\in H;\, m=1,2,\dots,M}$ by means of the sampling formula

$$f = \sum_{m=1}^{M} \sum_{h\in H} \mathcal{L}_m f(h)\, U(h) S_m \quad \text{in } \mathcal{H}\,, \tag{21}$$

for some sampling functions $S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m \in \mathcal{V}_\Phi,\ m = 1,2,\dots,M$. Moreover, the sequence $\big\{U(h)S_m\big\}_{h\in H;\, m=1,2,\dots,M}$ is a frame for $\mathcal{V}_\Phi$.

• In particular, consider $\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big[U(p)f\big](t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$. Let $P\mathbb{Z}^d$ be a sublattice in $\mathbb{Z}^d$ where P denotes a $d\times d$ matrix of integer entries with positive determinant $L := \det P$. Under the hypotheses in Thm. 1 on the matrix $A = [a_{m,nl}]$ where $a_{m,nl}(p) = \big\langle \varphi_n(t), \psi_m(t-p+g_l)\big\rangle_{L^2(\mathbb{R}^d)}$, the sampling formula (21) gives an average sampling formula in the classical shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$, i.e., for each $f \in V_\Phi^2$ formula (21) reads

$$f(t) = \sum_{m=1}^{M} \sum_{p\in P\mathbb{Z}^d} \langle f, \psi_m(\cdot - p)\rangle_{L^2(\mathbb{R}^d)}\, S_m(t-p)\,, \quad t \in \mathbb{R}^d\,,$$

for some sampling functions $S_m \in V_\Phi^2,\ m = 1,2,\dots,M$. Moreover, the sampling sequence $\{S_m(t-p)\}_{p\in P\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for V_Φ^2. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

• Whenever $\mathcal{H} = L^2(\mathbb{R}^d)$, for M fixed points $t_m \in \mathbb{R}^d$, we consider in the shift-invariant subspace V_Φ^2 the samples $\mathcal{L}_m f(h) := \big[U(-h)f\big](t_m),\ h \in H,\ m = 1,2,\dots,M$, for any $f = \sum_{n=1}^N \sum_{l=1}^L \sum_{k\in H} x_n(g_l+k)\, U(g_l+k)\varphi_n$ in $\mathcal{V}_\Phi$ the samples can be expressed as

$$\mathcal{L}_m f(h) = \sum_{n=1}^{N}\sum_{l=1}^{L}\sum_{k\in H} x_n(g_l+k)\big[U(k-h)U(g_l)\varphi_n\big](t_m) = \sum_{n=1}^{N}\sum_{l=1}^{L}\big(a_{m,nl} *_H x_{nl}\big)(h)\,,$$

where $a_{m,nl}(h) := \big[U(-h+g_l)\varphi_n\big](t_m),\ h \in H$, for $m = 1,2,\dots,M,\ n = 1,2,\dots,N$ and $l = 1,2,\dots,L$.

In particular, if we sample any function in V_Φ^2 on a sublattice $P\mathbb{Z}^d$ of $\mathbb{Z}^d$, under the hypotheses in Thm.1 on the $M \times NL$ matrix $A = [a_{m,nl}]$ where $a_{m,nl}(p) = \varphi_n(t_m + p - g_l)$, there exist $M \geq NL$ sampling functions $S_m \in V_\Phi^2,\ m = 1,2,\dots,M$, such that we recover any $f \in V_\Phi^2$ from the samples $\{\mathcal{L}_m f(p) = f(p+t_m)\}_{p\in P\mathbb{Z}^d;\, m=1,2,\dots,M}$ by means of the sampling formula

$$f(t) = \sum_{m=1}^{M}\sum_{p\in P\mathbb{Z}^d} f(p+t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d\,.$$

Moreover, the sampling sequence $\{S_m(t-p)\}_{p\in P\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for V_Φ^2. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

4.5 Some final comments

Our main sampling result, Theorem 1, involves some sampling conditions appearing in the mathematical literature; thus, condition (b) says that $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_M)$ is a *stable averaging sampler* for $\mathcal{V}_\Phi$ as it was introduced in Ref. [2]. Besides, formula (12) is the expected reconstruction formula in the U-invariant subspace $\mathcal{V}_\Phi$. The differences can be found in conditions (a)-(c)-(d) used here since these conditions are directly related to the filtering process defining the samples (8). Consequently, these conditions are given in terms of the convolution system $\mathcal{A}$ and its transfer matrix $\widehat{A}$. In references concerning shift-invariant subspaces these conditions are given in terms of some Gram matrices (see, for instance, Refs. [1, 2]), or in terms of the so called *modulation matrix* whose entries are given in terms of the Zak transform as $\big(Z\mathcal{L}_m\varphi_n\big)(0, w)$ (see, for instance, Refs. [8, 11, 20]).

As it was mentioned before, some previous sampling results can be seen as particular examples of this approach. As a non-exhaustive sample of such examples we can cite sampling in shift-invariant subspaces Refs. [2, 4, 10, 11, 17, 20, 21], and sampling in U-invariant subspaces Refs.[8, 12, 13, 19]. Besides, as it was showed in Section 4.2, the present approach opens new sampling settings: for instance, those related with crystallographic groups involving examples of practical interest.

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