On regular generalized sampling in T-invariant subspaces of a Hilbert space: an overview

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Abstract

A regular generalized sampling theory in some structured T-invariant subspaces of a Hilbert space $\mathcal{H}$, where T denotes a bounded invertible operator in $\mathcal{H}$, is established in this paper. This is done by walking through the most important cases which generalize the usual unitary sampling settings.

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1 Statement of the problem

The aim of this paper is to establish a regular generalized sampling theory in some structured T-invariant subspaces of an abstract separable Hilbert space $\mathcal{H}$, where T denotes a bounded invertible operator on $\mathcal{H}$. Concretely, for a fixed $a \in \mathcal{H}$ these subspaces $\mathcal{A}_a$ are constructed by using a representation $h \mapsto \Pi(h)$ of a discrete locally compact abelian (LCA) group H (with additive notation) into the space $\mathcal{B}(\mathcal{H})$ of the bounded invertible operators on $\mathcal{H}$ as

$$\mathcal{A}_a = \Big\{ \sum_{h\in H} \alpha_h \, \Pi(h)a \, : \, \{\alpha_h\}_{h\in H} \in \ell^2(H) \Big\}.$$

The vector a will be called the generator of $\mathcal{A}_a$. Two important and illustrative cases are those related with the representation of the groups $\mathbb{Z}$ or $\mathbb{Z}_N$ given by $n \mapsto T^n$ which yields the $\mathcal{A}_a$-subspaces

$$\Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, T^n a \, : \, \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\} \quad \text{or} \quad \Big\{ \sum_{n=0}^{N-1} \alpha(n) \, T^n a \, : \, \{\alpha(n)\}_{n=0}^{N-1} \in \mathbb{C}^N \Big\}.$$

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In the last case $T^N a = a$ and the cyclic group $\mathbb{Z}_N$ is represented on $\mathcal{A}_a$. Consequently, we have to describe the involved generalized samples and to exhibit the look of the obtained sampling formulas.

Motivated by the average sampling in classical shift-invariant subspaces of $L^2\mathbb{R}^d$), concerning the used samples we consider s fixed elements $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, that do not necessarily belong to $\mathcal{A}_a$, and we define for each $x \in \mathcal{A}_a$

$$\mathcal{L}_j x(h) := \big\langle x, \Pi^*(-h) b_j \big\rangle_{\mathcal{H}}, \quad h \in H\,, \tag{1}$$

where $\Pi^*(-h)$ denotes the adjoint operator of $\Pi(-h)$, and we restrict ourselves to the sequence of samples taken at a subgroup M of H, i.e., $\big\{\mathcal{L}_j x(m)\big\}_{m\in M;\, j=1,2,\dots,s}$. As it will be noticed in the remarks of Section 4, other types of samples yielding pointwise sampling can be considered.

Regarding the sampling formulas they look like

$$x = \sum_{j=1}^{s} \sum_{m\in M} \mathcal{L}_j x(m)\, \Pi(m) c_j \quad \text{in } \mathcal{H}\,,$$

where $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, and the sequence $\big\{\Pi(m)c_j\big\}_{m\in M;\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. Thus, we obtain a stable recovery of any $x \in \mathcal{A}_a$ from the data sequence $\big\{\mathcal{L}_j x(m)\big\}_{m\in M;\, j=1,2,\dots,s}$.

In the alluded examples the samples are, $\mathcal{L}_j x(rm) = \big\langle x, (T^*)^{-rm} b_j \big\rangle_{\mathcal{H}}$, $m \in \mathbb{Z}$ and $j = 1, 2, \dots, s$, or $\mathcal{L}_j x(rn) = \langle x, (T^*)^{-rn} b_j \rangle_{\mathcal{H}}$, $n = 0, 1, \dots, \ell - 1$ and $j = 1, 2, \dots, s$, respectively, where r is the sampling period, a positive integer in the first case, or a positive integer such that $r|N$ and $\ell = N/r$, in the second case. The above sampling formula reads

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, T^{rm} c_j \quad \text{or} \quad x = \sum_{j=1}^{s} \sum_{m=0}^{\ell-1} \mathcal{L}_j x(rm)\, T^{rm} c_j\,.$$

The used mathematical technique is very friendly: we express the given sequence of samples as frame coefficients in an auxiliary Hilbert space; the challenge problem is to obtain T-suitable dual frames yielding, via an isomorphism between the auxiliary Hilbert space and $\mathcal{A}_a$, the desired sampling formulas. There is not a unique way for implementing the above technique: it depends on the used expression for the samples (see, for instance, Refs. [27, 28]). The necessary background on Riesz bases or frame theory in a separable Hilbert space can be found, for instance, in Ref. [13].

The case where $T = U$ is a unitary operator in $\mathcal{H}$ has been studied in Refs. [18, 19, 25, 26, 35, 36], and it generalizes average sampling in shift-invariant subspaces in $L^2(\mathbb{R})$; whenever U is the shift operator, the samples given in (1) are nothing but samples of a convolution operator; in there words, samples of a filtered version of the function itself.

The present sampling study in the T-invariant subspace $\mathcal{A}_a$ has a double motivation: Firstly, in the recent paper [14] it is proved that any Riesz sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has a representation $\{T^n x_0\}_{n\in\mathbb{Z}}$ for a bounded and bijective operator $T : \overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{Z}} \to \overline{\operatorname{span}}\{x_n\}_{n\in\mathbb{Z}}$; besides, if $\{V^n g_0\}_{n\in\mathbb{Z}}$ with V bounded is a dual frame of $\{T^n f_0\}_{n\in\mathbb{Z}}$ where T is bounded and invertible, then $V = (T^*)^{-1}$. Secondly, it generalizes some previous work concerning the unitary case. Besides, this work can be seen as an introductory survey in abstract sampling theory.

As it was mentioned in Ref. [14], the idea of considering frames (or Riesz) sequences of the form $\{T^n a\}_{n\in\mathbb{Z}}$ (or $\{T^n a\}_{n=0}^{N-1}$) is closely related with dynamical sampling (see, for instance, Refs.[3, 4, 5, 6]), although the indexing of a frame in the dynamical sampling context is different from the one used here; the group structure is crucial in the sequel.

In writing this paper our choice in presenting the results is going from the particular to the general case due to mainly two reasons: firstly, the underlying theory can be easily followed by any interested reader without any knowledge on abstract harmonic analysis and, secondly, for the finite case as exhibited here any prior knowledge on finite Fourier analysis is needed: it is just a linear algebra approach.

The paper is organized as follows: in Section 2 we study the $\mathbb{Z}$ group case with a single generator, i.e., sampling formulas in $\mathcal{A}_a = \big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, T^n a \,:\, \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\big\}$; Section 3 is devoted to the finite case where the subspace $\mathcal{A}_a$ is finite dimensional; finally, in Section 4 the abstract case associated with an LCA group is exhibited. Once we have stated the problem and fixed the notation in each case, the mathematical development is very similar involving a very short proof which shares the pattern appearing in the case of a unitary operator U studied in previous works by the authors (see Refs. [18, 19, 25, 26]); for the sake of completeness we include here the close proofs. Putting all these cases together can help to exhibit the intrinsic nature of these sampling problems and their relationships. As the sections only include the essential sampling theory, they are accompanied with a pertinent list of notes and remarks enlightening the topic.

2 The $\mathbb{Z}$ group case

Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded invertible operator defined in a separable Hilbert space $\mathcal{H}$. For a fixed $a \in \mathcal{H}$ we consider the T-invariant subspace $\mathcal{A}_a$ in $\mathcal{H}$ defined as $\mathcal{A}_a := \overline{\operatorname{span}}\{T^n a \,:\, n \in \mathbb{Z}\}$. In case the sequence $\{T^n a\}_{n\in\mathbb{Z}}$ is a *Riesz sequence* in $\mathcal{H}$, i.e., a *Riesz basis* for $\mathcal{A}_a$, this subspace can be expressed as

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, T^n a \,:\, \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}.$$

The subspace $\mathcal{A}_a$ is the image of the usual Hilbert space $L^2(0,1)$ by means of the isomorphism

$$\begin{array}{llcl} \mathcal{T}_{T,a} : & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & f = \sum_{n\in\mathbb{Z}} \alpha_n\, \mathrm{e}^{2\pi i n w} & \longmapsto & x = \sum_{n\in\mathbb{Z}} \alpha_n\, T^n a\,. \end{array}$$

It is easy to check that this isomorphism $\mathcal{T}_{T,a}$ satisfies the *T-shifting property*

$$\mathcal{T}_{T,a}\big(f \mathrm{e}^{2\pi i m w}\big) = T^m\big(\mathcal{T}_{T,a} f\big) \quad \text{for any } f \in L^2(0,1) \text{ and } m \in \mathbb{Z}\,. \tag{2}$$

An expression for the samples

Firstly we introduce the used data to recover any $x \in \mathcal{A}_a$. Namely, given s fixed elements $b_j \in \mathcal{H}$, which do not necessarily belong to $\mathcal{A}_a$, and the *sampling period* r, an integer $r \geq 1$, we define the *generalized samples* $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ of any $x \in \mathcal{A}_a$ as

$$\mathcal{L}_j x(rm) := \big\langle x, (T^*)^{-rm} b_j \big\rangle_{\mathcal{H}}\,, \qquad m \in \mathbb{Z} \text{ and } \quad j = 1, 2, \dots, s\,, \tag{3}$$

where T^* denotes the (invertible) adjoint operator of T.

For any $x = \sum_{n\in\mathbb{Z}} \alpha_n\, T^n a$ we obtain the following expression for its samples

$$\begin{aligned} \mathcal{L}_j x(rm) &= \big\langle \sum_{n\in\mathbb{Z}} \alpha_n\, T^n a, (T^*)^{-rm} b_j \big\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \alpha_n\, \overline{\big\langle (T^*)^{-rm} b_j, T^n a \big\rangle}_{\mathcal{H}} \\ &= \Big\langle f, \sum_{n\in\mathbb{Z}} \big\langle (T^*)^{n-rm} b_j, a \big\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w} \Big\rangle_{L^2(0,1)} = \big\langle f, g_j(w)\, \mathrm{e}^{2\pi i r m w} \big\rangle_{L^2(0,1)}\,, \end{aligned} \tag{4}$$

where the functions $f(w) = \sum_{n\in\mathbb{Z}} \alpha_n\, \mathrm{e}^{2\pi \mathrm{i} n w}$ and $g_j(w) = \sum_{k\in\mathbb{Z}} \big\langle (T^*)^k b_j, a\big\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}$, $j = 1, 2, \ldots, s$, belong to $L^2(0,1)$.

Thus the stable recovery of $F \in L^2(0,1)$ (and consequently of $x = \mathcal{T}_{T,a} f \in \mathcal{A}_a$) from the sequence of generalized samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ depends on whether the sequence $\{g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r m w}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ forms a *frame* for $L^2(0,1)$. Moreover, in order to derive an associated sampling formula we also need to know *dual frames* having the same structure (in order to apply the T-shifting property (2)).

For the first question, consider the $s \times r$ matrix-valued function in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{pmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{pmatrix} = \Big(g_j\Big(w + \frac{k-1}{r}\Big)\Big)_{\substack{j=1,2,\ldots,s \\ k=1,2,\ldots,r}} \qquad (5)$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)]\,; \qquad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

As usual, the symbol $*$ denotes the transpose conjugate matrix and $\lambda_{\min}$ (respectively $\lambda_{\max}$) the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbb{G}^*(w)\mathbb{G}(w)$.

A characterization of the sequence $\big\{g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ as a complete system, Bessel sequence, frame or Riesz basis for $L^2(0,1)$ is well known (see, for instance, Refs. [22, 23]). In particular,

The sequence $\{g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r m w}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ *is a frame for* $L^2(0,1)$ *if and only if* $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$*. In this case, the optimal frame bounds are* $\alpha_{\mathbb{G}}/r$ *and* $\beta_{\mathbb{G}}/r$*.*

For the second question, the existence of dual frames of $\{g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r m w}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ with its same structure, choose functions h_j in $L^\infty(0,1)$, $j = 1, 2, \ldots, s$, such that

$$\big(h_1(w), h_2(w), \ldots, h_s(w)\big)\mathbb{G}(w) = \big(1, 0, \ldots, 0\big) \quad \text{a.e. in } (0,1)\,. \qquad (6)$$

In [22] it was proven that

The sequence $\{r\overline{h_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} r m w}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$*, where the functions* h_j *in* $L^\infty(0,1)$ *satisfy* (6)*, is a dual frame of the sequence* $\{g_j(w)\, \mathrm{e}^{2\pi \mathrm{i} r m w}\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ *in* $L^2(0,1)$*.*

Concerning to the existence of the functions h_j, $j = 1, 2, \ldots, s$, satisfying (6), consider the first row of the $r \times s$ *Moore-Penrose pseudo-inverse* $\mathbb{G}^\dagger(w)$ of $\mathbb{G}(w)$ given by $\mathbb{G}^\dagger(w) = \big[\mathbb{G}^*(w)\, \mathbb{G}(w)\big]^{-1}\, \mathbb{G}^*(w)$. Its entries are essentially bounded in $(0,1)$ since the functions g_j, $j = 1, 2, \ldots, s$, and $\det^{-1}\big[\mathbb{G}^*(w)\, \mathbb{G}(w)\big]$ are essentially bounded in $(0,1)$, and (6) trivially holds. In fact, all the possible solutions of (6) are given by the first row of the $r \times s$ matrices given by

$$\mathbb{H}(w) := \mathbb{G}^\dagger(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^\dagger(w)\big]\,,$$

where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^\infty(0,1)$, and $\mathbb{I}_s$ is the identity matrix of order s.

A regular sampling formula in $\mathcal{A}_a$

Taking into account the expression for the samples (4), for any $f \in L^2(0,1)$ we have the frame expansion

$$f = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, r\overline{h_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw} \quad \text{in } L^2(0,1)\,. \tag{7}$$

In case $x = \mathcal{T}_{T,a} f \in \mathcal{A}_a$, applying the isomorphism $\mathcal{T}_{T,a}$ to the expansion (7) for f one obtains the sampling expansion

$$\begin{aligned} x &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, \mathcal{T}_{T,a}\big[r\overline{h_j(\cdot)}\, \mathrm{e}^{2\pi \mathrm{i} rm\,\cdot}\big] = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, T^{rm}\big[\mathcal{T}_{T,a}(r\overline{h}_j)\big] \\ &= \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, T^{rm} c_{j,\mathbf{h}} \quad \text{in } \mathcal{H}\,, \end{aligned}$$

where $c_{j,\mathbf{h}} := \mathcal{T}_{T,a}(r\overline{h}_j) \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, and we have used the T-shifting property (2). Besides, the sequence $\big\{T^{rm} c_{j,\mathbf{h}}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds:

Theorem 1. *For any $x \in \mathcal{A}_a$ consider the sequence of samples $\{\mathcal{L}_j x(rm)\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ defined in* (3)*. Assume that the functions g_j, $j = 1, 2, \ldots, s$, given in* (4) *belong to $L^\infty(0,1)$, and consider the associated $\mathbb{G}(w)$ matrix given in* (5)*. The following statements are equivalent:*

(a) The constant $\alpha_{\mathbb{G}} > 0$.

(b) There exists a vector-function $\mathbf{h}(w) = \big(h_1(w), h_2(w), \ldots, h_s(w)\big)$ with entries in $L^\infty(0,1)$ and satisfying

$$\big(h_1(w), h_2(w), \ldots, h_s(w)\big)\mathbb{G}(w) = \big(1, 0, \ldots, 0\big) \quad \text{a.e. in } (0,1)\,.$$

(c) There exist $c_{j,\mathbf{h}} \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, such that the sequence $\big\{T^{rm} c_{j,\mathbf{h}}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, T^{rm} c_{j,\mathbf{h}} \quad \text{in } \mathcal{H} \tag{8}$$

holds.

(d) There exists a frame $\big\{C_{j,m}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ for $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, C_{j,m} \quad \text{in } \mathcal{H}$$

holds.

Proof. First notice that the equivalence between the spectral and Frobenius norms [29] proves that the functions g_j, $j = 1, 2, \ldots, s$, belong to $L^\infty(0,1)$ if and only if $\beta_{\mathbb{G}} < \infty$. We have already proved that (a) implies (b), and that (b) implies (c). Obviously, (c)

implies (d). We only need to prove that (d) implies (a). Applying the isomorphism $\mathcal{T}_{T,a}^{-1}$ to the expansion in (d), and taking into account (4) we obtain

$$f = \mathcal{T}_{T,a}^{-1}x = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, \mathcal{T}_{T,a}^{-1}(C_{j,m})$$
$$= \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \big\langle f, g_j(w)\, \mathrm{e}^{2\pi i rmw}\big\rangle_{L^2(0,1)}\, \mathcal{T}_{T,a}^{-1}(C_{j,m}) \quad \text{in } L^2(0,1)\,,$$

where the sequence $\big\{\mathcal{T}_{T,a}^{-1}(C_{j,m})\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for $L^2(0,1)$. The sequence $\big\{g_j(w)\,\mathrm{e}^{2\pi i rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Bessel sequence in $L^2(0,1)$ since $\beta_{\mathbb{G}} < \infty$, and satisfying the above expansion in $L^2(0,1)$. According to [13, Lemma 6.3.2] the sequences $\big\{\mathcal{T}_{T,a}^{-1}(C_{j,m})\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ and $\big\{g_j(w)\,\mathrm{e}^{2\pi i rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ form a pair of dual frames in $L^2(0,1)$; in particular, we deduce that $\alpha_{\mathbb{G}} > 0$ which concludes the proof. □

Notes and remarks

Some comments on the result appearing in this section are pertinent:

1. The fact of considering subspaces as $\mathcal{A}_a$ is reinforced by a result proved in Ref.[14, Corollary 2.4]: Any Riesz sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ has a representation $\{T^n x_0\}_{n\in\mathbb{Z}}$ for a bounded and bijective operator $T : \overline{\text{span}}\{x_n\}_{n\in\mathbb{Z}} \to \overline{\text{span}}\{x_n\}_{n\in\mathbb{Z}}$.

2. In case the operator $T = U$ is unitary, the *auto-covariance* $\langle U^n a, a\rangle_{\mathcal{H}}$, $n \in \mathbb{Z}$, of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ admits the integral representation (see Ref.[32])

$$\langle U^n a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{in\theta} d\mu_a(\theta)\,, \quad n \in \mathbb{Z}\,,$$

   where μ_a is a positive Borel measure on $(-\pi,\pi)$ called the *spectral measure* of the sequence $\{U^n a\}_{n\in\mathbb{Z}}$. The spectral measure μ_a can be decomposed into an absolute continuous and a singular part as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$ with respect to Lebesgue measure. A necessary and sufficient condition for the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence for $\mathcal{H}$ is that the singular part $\mu_a^s \equiv 0$ and the *spectral density* ϕ_a satisfies

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,;$$

   see, for instance, Ref. [18]. In particular, whenever U is the shift operator $f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$, the above condition yields the classical condition (see, for instance, Ref. [13])

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 \leq \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(\theta+2\pi n)|^2 < \infty\,.$$

   As far as we know, a characterization, allowing to really check whether $\{T^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $\mathcal{H}$ or not, remains an open question, even when the operator T is selfadjoint or normal. However, a theoretical characterization has been recently proposed in Ref. [15, Proposition 4.3].

3. In case the sequence $\{T^n a\}_{n\in\mathbb{Z}}$ is a frame sequence for $\mathcal{H}$, the operator $\mathcal{T}_{T,a}$ is bounded and surjective. The sampling formula (8) still remains valid as a frame expansion in $\mathcal{A}_a$.

4. The choice of the generalized samples as in (3) is motivated by a result in Ref.[14, Lemma 3.3]: If $\{V^n g_0\}_{n\in\mathbb{Z}}$ with V bounded is a dual frame of $\{T^n f_0\}_{n\in\mathbb{Z}}$ where T is bounded and invertible, then $V = (T^*)^{-1}$.

5. In Theorem 1 it can be added the equivalent condition

$$\operatorname*{ess\,inf}_{w\in(0,1/r)} \det[\mathbb{G}^*(w)\mathbb{G}(w)] > 0\,.$$

   In case the 1-periodic extension of the functions g_j, $j = 1, 2, \dots, s$, are continuous on $\mathbb{R}$, this condition reduces to say that rank $\mathbb{G}(w) = r$ for all $w \in \mathbb{R}$.

6. In the *overcomplete* setting we have that $s > r$; this oversampling technique allows to obtain reconstruction formulas $T^{rm}c_{j,\mathbf{h}}$ with prescribed properties (see, for instance, Refs.[18, 24]). In case $r = s$, the frame condition (c) in Theorem 1 becomes a Riesz basis condition: There exist r unique elements $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, r$, such that the sequence $\{T^{rm}c_j\}_{m\in\mathbb{Z};\, j=1,2,\dots,r}$ is a *Riesz basis* for $\mathcal{A}_a$, and the sampling expansion (8) holds. Moreover, due to the uniqueness of the coefficients in a Riesz basis expansion, the *interpolation property* $\mathcal{L}_{j'}c_j(rm) = \delta_{j,j'}\,\delta_{m,0}$, where $m \in \mathbb{Z}$ and $j, j' = 1, 2, \dots, r$, holds (for a similar result, see [18, Corollary 3.3]).

7. Let us to take a closer look at the analyzing sequence $\{(T^*)^{-rm}b_j\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ which appears in the definition of the generalized samples (3). Having in mind (4) and the isomorphism $\mathcal{T}_{T,a}$, we have the inequalities

$$\frac{\alpha_{\mathbb{G}}}{r}\|\mathcal{T}_{T,a}\|^{-2}\|x\|^2 \le \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \big|\langle x, (T^*)^{-rm}b_j\rangle\big|^2 \le \frac{\beta_{\mathbb{G}}}{r}\|\mathcal{T}_{T,a}^{-1}\|^2\|x\|^2 \quad \text{for all } x \in \mathcal{A}_a\,.$$

   The sequence $\{(T^*)^{-rm}b_j\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is not contained in $\mathcal{A}_a$ except for some particular cases such as whenever all $b_j \in \mathcal{A}_a$ and operator T is selfadjoint or unitary. Therefore, as a consequence of the above inequalities, the sequence $\{(T^*)^{-rm}b_j\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a *pseudo-dual frame* of $\{T^{rm}c_{j,\mathbf{h}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ in $\mathcal{A}_a$ (see Refs. [33, 34]). In other words, denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, we derive that the sequence $\{P_{\mathcal{A}_a}\big((T^*)^{-rm}b_j\big)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a dual frame of $\{T^{rm}c_{j,\mathbf{h}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ in $\mathcal{A}_a$.

   Whenever $r = s$, the sequence $\{(T^*)^{-rm}b_j\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is, except for some particular cases, a *pseudo-Riesz basis* for $\mathcal{A}_a$.

8. In the d-dimensional case we consider the group $\mathbb{Z}^d$ and the samples are taken at a lattice $M\mathbb{Z}^d$, where M denotes a $d \times d$ matrix with integer entries and positive determinant; in this case we necessarily have $s \ge \det M$ (see, for instance, Ref. [24]).

9. The case of L generators $\{a_1, a_2, \dots, a_L\}$ can be analogously handled, obtaining similar results. Multiple generators in the shift-invariant case leads to the multiwavelet setting. Multiwavelets lead to multiresolution analyses and fast algorithms just as scalar wavelets, but they have some advantages: they can have short support coupled with high smoothness and high approximation order, and they can be both symmetric and orthogonal (see, for instance, Ref. [30]). Classical sampling in multiwavelet subspaces has been studied in Refs. [23, 38, 40]. An example of the formula (8) in the shift-invariant subspace of $L^2(\mathbb{R})$ generated by the Hermite cubic splines can be found in [23]; see also Ref. [16].

Theorem 1 comprises all the known results concerning average regular sampling in shift-invariant spaces which appear in the mathematical literature. For a few selected references see, for instance, Refs. [2, 11, 12, 22, 23, 31, 41, 42, 43, 44].

3 The $\mathbb{Z}_N$ cyclic group case

For a fixed $a \in \mathcal{H}$, assume that there exists a nonnegative integer N such that $T^N a = a$; let N be the smallest index with this property. Next, we consider the finite dimensional subspace $\mathcal{A}_a := \operatorname{span}\big\{a, Ta, T^2a, \dots, T^{N-1}a\big\}$ in $\mathcal{H}$. Assuming that this set of vectors is linearly independent in $\mathcal{H}$ we have the N-dimensional subspace of $\mathcal{H}$

$$\mathcal{A}_a = \Big\{ \sum_{k=0}^{N-1} \alpha(k)\, T^k a \,:\, (\alpha(0), \alpha(1), \dots, \alpha(N-1))^\top \in \mathbb{C}^N \Big\},$$

and the isomorphism $\mathcal{T}_{N,a}$ between $\mathbb{C}^N$ and $\mathcal{A}_a$

$$\begin{array}{rccc} \mathcal{T}_{N,a} : & \mathbb{C}^N & \longrightarrow & \mathcal{A}_a \\ & \boldsymbol{\alpha} = \displaystyle\sum_{k=0}^{N-1} \alpha(k)\, \mathbf{e_k} & \longmapsto & x = \displaystyle\sum_{k=0}^{N-1} \alpha(k)\, T^k a\,, \end{array}$$

where $\big\{\mathbf{e_0}, \mathbf{e_1}, \dots, \mathbf{e_{N-1}}\big\}$ denotes the canonical basis for $\mathbb{C}^N$.

Let $\{\alpha(k)\}_{k\in\mathbb{Z}}$ be an N-periodic sequence in $\mathbb{C}$. For $1 \le m \le N-1$ consider the vectors in $\mathbb{C}^N$

$$\begin{aligned} \boldsymbol{\alpha}_0 &:= \big(\alpha(0), \alpha(1), \dots, \alpha(N-1)\big)^\top \quad \text{and} \\ \boldsymbol{\alpha}_{N-m} &:= \big(\alpha(N-m), \alpha(N-m+1), \dots, \alpha(N-m+N-1)\big)^\top. \end{aligned}$$

Then, the following T-*shifting property* holds (its proof is analogous to that in [25, Proposition 2])

$$\mathcal{T}_{N,a}(\boldsymbol{\alpha}_{N-m}) = T^m\big(\mathcal{T}_{N,a}(\boldsymbol{\alpha}_0)\big) \quad \text{for any } \ 1 \le m \le N-1\,. \tag{9}$$

An expression for the samples

Let r be a positive integer such that $r|N$, and define $\ell := N/r$. Fixed s elements $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, which do not necessarily belong to $\mathcal{A}_a$, for each $x \in \mathcal{A}_a$ we consider its generalized samples defined by

$$\mathcal{L}_j x(rn) := \big\langle x, (T^*)^{-rn} b_j \big\rangle_{\mathcal{H}}, \quad n = 0, 1, \dots, \ell-1 \text{ and } j = 1, 2, \dots, s\,. \tag{10}$$

For $x = \sum_{k=0}^{N-1} \alpha(k)\, T^k a$ we obtain a more convenient expression for its samples

$$\begin{aligned} \mathcal{L}_j x(rn) &= \Big\langle \sum_{k=0}^{N-1} \alpha(k)\, T^k a, (T^*)^{-rn} b_j \Big\rangle_{\mathcal{H}} = \sum_{k=0}^{N-1} \alpha(k)\, \langle T^k a, (T^*)^{-rn} b_j \rangle_{\mathcal{H}} \\ &= \Big\langle \sum_{k=0}^{N-1} \alpha(k)\, \mathbf{e_k}, \sum_{k=0}^{N-1} \overline{\langle T^{k-rn} a, b_j \rangle_{\mathcal{H}}}\, \mathbf{e_k} \Big\rangle_{\mathbb{C}^N} = \big\langle \boldsymbol{\alpha}, \mathbf{g}_{j,n} \big\rangle_{\mathbb{C}^N}, \end{aligned} \tag{11}$$

where $\mathbf{g}_{j,n} = \sum_{k=0}^{N-1} \overline{\langle T^{k-rn} a, b_j \rangle_{\mathcal{H}}}\, \mathbf{e_k}$. In terms of the N-periodic sequence in $\mathbb{C}$ defined as

$$r_{a,b_j}(k) := \big\langle T^k a, b_j \big\rangle_{\mathcal{H}}, \ k \in \mathbb{Z}\,, \tag{12}$$

we can write

$$\mathbf{g}_{j,n} = \sum_{k=0}^{N-1} \overline{\langle T^{N+k-rn}a, b_j \rangle_{\mathcal{H}}}\, \mathbf{e_k} = \sum_{k=0}^{N-1} \overline{r_{a,b_j}(N+k-rn)}\, \mathbf{e_k}\,. \tag{13}$$

Having in mind the expression (11) for the samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$, and the isomorphism $\mathcal{T}_{N,a}$, any $x \in \mathcal{A}_a$ can be recovered from its samples if and only if the set of vectors $\big\{\mathbf{g}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^N$ forms a spanning set for $\mathbb{C}^N$, i.e., a frame for $\mathbb{C}^N$ (see, for instance, Refs. [10, 13]). This is equivalent to the condition $\operatorname{rank} \mathbb{G}_{a,\mathbf{b}} = N$, where $\mathbb{G}_{a,\mathbf{b}}$ denotes the $N \times s\ell$ matrix whose columns are precisely the vectors $\big\{\mathbf{g}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ written as

$$\mathbb{G}_{a,\mathbf{b}} := \big(\mathbf{g}_{1,0} \;\; \dots \;\; \mathbf{g}_{1,\ell-1} \;\; \mathbf{g}_{2,0} \;\; \dots \;\; \mathbf{g}_{2,\ell-1} \;\; \dots \;\; \mathbf{g}_{s,0} \;\; \dots \;\; \mathbf{g}_{s,\ell-1}\big)\,.$$

In particular, we have that $N \le s\ell$, that is, $s \ge r$. From expression (13), $N = r\ell$ and the N-periodic character of $r_{a,b_j}(k)$ we obtain that

$$\mathbb{G}_{a,\mathbf{b}} = \big(\mathbb{R}^*_{a,b_1} \;\; \mathbb{R}^*_{a,b_2} \;\; \dots \;\; \mathbb{R}^*_{a,b_s}\big) := \mathbb{R}^*_{a,\mathbf{b}}\,, \tag{14}$$

where each $\ell \times N$ block $\mathbb{R}_{a,b_j}$, $j = 1, 2, \dots, s$, is given by

$$\mathbb{R}_{a,b_j} = \begin{pmatrix} r_{a,b_j}(0) & r_{a,b_j}(1) & \dots & r_{a,b_j}(N-1) \\ r_{a,b_j}(N-r) & r_{a,b_j}(N-r+1) & \dots & r_{a,b_j}(2N-r-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_{a,b_j}(r) & r_{a,b_j}(r+1) & \dots & r_{a,b_j}(r+N-1) \end{pmatrix}.$$

For $j = 1, 2, \dots, s$, we have the following expression for the samples $\big\{\mathcal{L}_j x(rn)\big\}_{n=0}^{\ell-1}$ of $x = \sum_{k=0}^{N-1} \alpha(k)\, T^k a \in \mathcal{A}_a$

$$\big(\mathcal{L}_j x(0), \mathcal{L}_j x(r), \cdots, \mathcal{L}_j x(r(\ell-1))\big)^\top = \mathbb{R}_{a,b_j}\, \big(\alpha(0), \alpha(1), \dots, \alpha(N-1)\big)^\top.$$

In other words, denoting the vectors $\boldsymbol{\alpha} := \big(\alpha(0), \alpha(1), \dots, \alpha(N-1)\big)^\top$ in $\mathbb{C}^N$ and

$$\boldsymbol{\mathcal{L}}_{\text{sam}} x := \big(\mathcal{L}_1 x(0), \mathcal{L}_1 x(r), \dots, \mathcal{L}_1 x(r(\ell-1)), \dots, \mathcal{L}_s x(0), \dots, \mathcal{L}_s x(r(\ell-1))\big)^\top \tag{15}$$

in $\mathbb{C}^{s\ell}$, the matrix relationship $\boldsymbol{\mathcal{L}}_{\text{sam}} x = \mathbb{R}_{a,\mathbf{b}}\, \boldsymbol{\alpha}$ holds where $\mathbb{R}_{a,\mathbf{b}}$ is the $s\ell \times N$ matrix deduced from (14).

As $\operatorname{rank} \mathbb{R}_{a,\mathbf{b}} = \operatorname{rank} \mathbb{G}_{a,\mathbf{b}} = N$, the *Moore-Penrose pseudo-inverse* of $\mathbb{R}_{a,\mathbf{b}}$ is the $N \times s\ell$ matrix $\mathbb{R}^\dagger_{a,\mathbf{b}} = \big[\mathbb{R}^*_{a,\mathbf{b}}\, \mathbb{R}_{a,\mathbf{b}}\big]^{-1} \mathbb{R}^*_{a,\mathbf{b}}$. Any dual frame of $\big\{\mathbf{g}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ in $\mathbb{C}^N$ is given by the columns of any left-inverse $\mathbb{H}$ of the matrix $\mathbb{R}_{a,\mathbf{b}}$; i.e., $\mathbb{H}\, \mathbb{R}_{a,\mathbf{b}} = \mathbb{I}_N$. All these matrices are expressed as

$$\mathbb{H} = \mathbb{R}^\dagger_{a,\mathbf{b}} + \mathbb{U}\big[\mathbb{I}_{s\ell} - \mathbb{R}_{a,\mathbf{b}}\, \mathbb{R}^\dagger_{a,\mathbf{b}}\big]\,, \tag{16}$$

where $\mathbb{U}$ denotes any arbitrary $N \times s\ell$ matrix. Let $\mathbb{H}$ be any *left-inverse* of $\mathbb{R}_{a,\mathbf{b}}$, and denote $\mathbf{h}_{j,n}$ its $(j-1)\ell+n+1$ column where $j = 1, 2, \dots, s$ and $n = 0, 1, \dots, \ell-1$; thus, $\big\{\mathbf{h}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is a dual frame of $\big\{\mathbf{g}_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$. Given any $x = \sum_{k=0}^{N-1} \alpha(k)\, T^k a$

in $\mathcal{A}_a$, from the matrix relationship $\boldsymbol{\mathcal{L}}_{\text{sam}}x = \mathbb{R}_{a,\mathbf{b}}\,\boldsymbol{\alpha}$ for the corresponding $\boldsymbol{\alpha} = \sum_{k=0}^{N-1}\alpha(k)\,\mathbf{e_k} \in \mathbb{C}^N$ we obtain

$$\boldsymbol{\alpha} = \big(\alpha(0), \alpha(1), \dots, \alpha(N-1)\big)^\top = \mathbb{H}\,\boldsymbol{\mathcal{L}}_{\text{sam}}x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1}\mathcal{L}_j x(rn)\,\mathbf{h}_{j,n}\,.$$

Applying the isomorphism $\mathcal{T}_{N,a}$, for any $x = \sum_{k=0}^{N-1}\alpha(k)\,T^k a \in \mathcal{A}_a$ we get

$$x = \mathcal{T}_{N,a}\big(\boldsymbol{\alpha}\big) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1}\mathcal{L}_j x(rn)\,\mathcal{T}_{N,a}\big(\mathbf{h}_{j,n}\big)\,. \tag{17}$$

The column $\mathbf{h}_{j,n}$ in the above formula do not have, in principle, any suitable structure for applying the T-shifting property (9). Although we will see that the columns of the Moore-Penrose pseudo-inverse $\mathbb{R}^\dagger_{a,\mathbf{b}}$ fulfil the required attribute, we will construct all the left-inverses of $\mathbb{R}_{a,\mathbf{b}}$ allowing it.

Note that each $\ell \times N$ block $\mathbb{R}_{a,b_j}$ has an r-circulant character in the sense that each row of $\mathbb{R}_{a,b_j}$ is the previous row moved to the right r places and wrapped around. In general, and in terms of a matrix C of order $s\ell \times N$ partitioned into s submatrices of order $\ell \times N$, each block has an r-circulant character if and only if $CP_N^r = \mathbb{P}C$, or equivalently,

$$C = \mathbb{P}^* C P_N^r$$

where P_i denotes the 1-circulant square matrix of order $i \in \mathbb{N}$ with first row $(0, 1, 0, \cdots, 0)$ and $\mathbb{P}$ is the square matrix of order $s\ell$ given by $\mathbb{P} = \text{diag}(P_\ell, \cdots, P_\ell)$, the direct sum of s times the matrix P_ℓ. The above characterization allows to conclude easily that $(C^\dagger)^*$ inherits, and consequently the transpose $(C^\dagger)^\top$, the r-circulant character from C. Indeed,

Lemma 1. *If C is a composite matrix of s r-circulant matrices of order $\ell \times N$, then so is $(C^\dagger)^*$. As a consequence, the matrix $(\mathbb{R}^\dagger_{a,\mathbf{b}})^\top$ is a composite matrix of s r-circulant matrices of order $\ell \times N$.*

Proof. Having in mind that $\mathbb{P}^*$ and P_N^r are orthogonal matrices we get

$$(C^\dagger)^* = ((\mathbb{P}^* C P_N^r)^\dagger)^* = \big((P_N^r)^* C^\dagger \mathbb{P}\big)^* = \mathbb{P}^*(C^\dagger)^* P_N^r\,.$$

□

For more details on pseudoinverses of circulant matrices see Refs. [37, 39]. In these sources is to be found the above lemma although for a square matrix C.

For any left-inverse $\mathbb{H}$ given by (16) we proceed to construct a specific left-inverse $\mathbb{H}_\mathbb{S}$ of $\mathbb{R}_{a,\mathbf{b}}$ allowing to apply the T-shifting property (9). Namely, we denote as $\mathbb{S}$ the first r rows of the matrix $\mathbb{H}$, i.e., $\mathbb{S}\,\mathbb{R}_{a,\mathbf{b}} = \big[\mathbb{I}_r, \mathbb{O}_{r\times(N-r)}\big]$, where $\mathbb{I}_r$ and $\mathbb{O}_{r\times(N-r)}$ denote, respectively, the identity matrix of order r and the zero matrix of order $r \times (N-r)$. According to the structure of the matrix $\mathbb{R}_{a,\mathbf{b}}$ (see (14)), we partition the $r \times s\ell$ matrix $\mathbb{S}$ into $\big(\mathbb{S}_1\ \mathbb{S}_2\ \dots\ \mathbb{S}_s\big)$, where each $\mathbb{S}_j$, $j = 1, 2, \dots, s$, is an $r \times \ell$ block. Now, we form the $N \times s\ell$ matrix $\mathbb{H}_\mathbb{S} := \big(\widetilde{\mathbb{S}}_1\ \widetilde{\mathbb{S}}_2\ \dots\ \widetilde{\mathbb{S}}_s\big)$ by using the columns of $\mathbb{S}_j$, $j = 1, 2, \dots, s$ in the following manner:

- The first column of $\widetilde{\mathbb{S}}_j$ is a concatenation of the columns 1, ℓ, $\ell - 1$, …, and 2 of $\mathbb{S}_j$;
- The second column of $\widetilde{\mathbb{S}}_j$ is a concatenation of the columns 2, 1, ℓ, …, and 3 of $\mathbb{S}_j$;

• The third column of $\widetilde{\mathbb{S}}_j$ is a concatenation of the columns 3, 2, 1, …, and 4 of $\mathbb{S}_j$; Repeating the process, finally,
• The column ℓ of $\widetilde{\mathbb{S}}_j$ is a concatenation of the columns ℓ, $\ell-1$, $\ell-2$, …, and 1 of $\mathbb{S}_j$.

The elements of each column of $\widetilde{\mathbb{S}}_j$ are identical to the previous column of $\mathbb{S}_j$ but are moved r positions down with wraparound.

With the above procedure, we obtain a left-inverse matrix $\mathbb{H}_\mathbb{S}$ for $\mathbb{R}_{a,\mathbf{b}}$, i.e., $\mathbb{H}_\mathbb{S}\,\mathbb{R}_{a,\mathbf{b}} = \mathbb{I}_N$ (see the proof in [25, Lemma 2]). By using these left-inverses $\mathbb{H}_\mathbb{S}$ for $\mathbb{R}_{a,\mathbf{b}}$ we can obtain structured sampling formulas involving the samples $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$.

A regular sampling formula in $\mathcal{A}_a$

Now denote the columns of a left-inverse matrix $\mathbb{H}_\mathbb{S}$ for $\mathbb{R}_{a,\mathbf{b}}$ as

$$\mathbb{H}_\mathbb{S} = \big(\mathbf{h}_{1,0} \quad \dots \quad \mathbf{h}_{1,\ell-1} \quad \mathbf{h}_{2,0} \quad \dots \quad \mathbf{h}_{2,\ell-1} \quad \dots \quad \mathbf{h}_{s,0} \quad \dots \quad \mathbf{h}_{s,\ell-1}\big)$$

Due to the structure of the columns of $\mathbb{H}_\mathbb{S}$, by using the T-shifting property (9), we have that $\mathcal{T}_{N,a}\big(\mathbf{h}_{j,n}\big) = T^{rn}\mathcal{T}_{N,a}\big(\mathbf{h}_{j,0}\big)$, $j = 1,2,\dots,s$ and $n = 0,1,\dots,\ell-1$. As a consequence, formula (17) reads

$$x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, T^{rn} c_j\,,$$

where $c_j := \mathcal{T}_{N,a}\big(\mathbf{h}_{j,0}\big) \in \mathcal{A}_a$, $j = 1,2,\dots,s$. In fact, the following result holds:

Theorem 2. *Given the $s\ell \times N$ matrix $\mathbb{R}_{a,\mathbf{b}}$ defined in* (14)*, the following statements are equivalent:*

(a) rank $\mathbb{R}_{a,\mathbf{b}} = N$

(b) There exists an $r \times s\ell$ matrix $\mathbb{S}$ such that

$$\mathbb{S}\,\mathbb{R}_{a,\mathbf{b}} = \big(\mathbb{I}_r, \mathbb{O}_{r\times(N-r)}\big)\,,$$

where $\mathbb{I}_r$ and $\mathbb{O}_{r\times(N-r)}$ denote, respectively, the identity matrix of order r and the zero matrix of order $r \times (N-r)$.

(c) There exist $c_j \in \mathcal{A}_a$, $j = 1,2,\dots,s$ such that the sequence $\big\{T^{rn}c_j\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, T^{rn} c_j \tag{18}$$

holds.

(d) There exists a frame $\big\{C_{j,n}\big\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ for $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, C_{j,n}$$

holds.

Proof. We have already proved that (a) implies (b), and that (b) implies (c), where we have completed the matrix $\mathbb{S}$ to a left-inverse $\mathbb{H}_{\mathbb{S}}$ of $\mathbb{R}_{a,\mathbf{b}}$ as the described procedure. Obviously, (c) implies (d). Finally, we prove that condition (d) implies condition (a). Indeed, let $x = \sum_{k=0}^{N-1} \alpha(k)\, T^k a$ be an arbitrary element in $\mathcal{A}_a$, and let define $\mathbf{k}_{j,n} := \mathcal{T}_{T,a}^{-1}\big(C_{j,n}\big) \in \mathbb{C}^N$ for $j = 1, 2, \ldots, s$ and $n = 0, 1, \ldots, \ell-1$. Applying $\mathcal{T}_{N,a}^{-1}$ in $x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, C_{j,n}$ we get

$$\boldsymbol{\alpha} = \big(\alpha(0), \alpha(1), \ldots, \alpha(N-1)\big)^\top = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \mathbf{k}_{j,n} = \mathbb{K}\, \boldsymbol{\mathcal{L}}_{\text{sam}} x\,,$$

where $\mathbb{K}$ is the $N \times s\ell$ matrix having $\mathbf{k}_{j,n}$ as columns, and $\boldsymbol{\mathcal{L}}_{\text{sam}} x$ is defined in (15). Since we have $\boldsymbol{\mathcal{L}}_{\text{sam}} x = \mathbb{R}_{a,\mathbf{b}}\, \boldsymbol{\alpha}$ we deduce that $\boldsymbol{\alpha} = \mathbb{K}\, \mathbb{R}_{a,\mathbf{b}}\, \boldsymbol{\alpha}$ for all $\boldsymbol{\alpha} \in \mathbb{C}^N$, i.e., $\mathbb{K}\, \mathbb{R}_{a,\mathbf{b}} = \mathbb{I}_N$. This implies that rank $\mathbb{R}_{a,\mathbf{b}} = N$ which completes the proof. $\square$

A filter-bank implementation

The sampling formulas (8) and (18) can be implemented as *filter-banks*, i.e., by using convolution systems. Let us show it, for instance, in the case included in this section. Assume that rank of $\mathbb{R}_{a,\mathbf{b}}$ equals N, and let $\mathbb{H}$ be a structured left-inverse of $\mathbb{R}_{a,\mathbf{b}}$ with columns $\mathbf{h}_{j,n}$, $j = 1, 2, \ldots, s$ and $n = 0, 1, \ldots, \ell-1$. In the corresponding sampling formula (17) we have $c_j = \mathcal{T}_{N,a}\big(\mathbf{h}_{j,0}\big)$, $j = 1, 2, \ldots, s$; denote the components of $\mathbf{h}_{j,0}$ as the N dimensional vector

$$\mathbf{h}_{j,0} = \big(\beta_j(0), \beta_j(1), \ldots, \beta_j(N-1)\big)^\top, \quad j = 1, 2, \ldots, s\,.$$

Substituting in (18), for $x \in \mathcal{A}_a$ we get

$$x = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, T^{rn}\Big(\sum_{k=0}^{N-1} \beta_j(k)\, T^k a\Big) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{k=0}^{N-1} \beta_j(k)\, T^{rn+k} a\Big).$$

The change of index $m := rn + k$ and the N-periodicity of each $\mathbf{h}_{j,0}$, $j = 1, 2, \ldots, s$, give

$$\begin{aligned} x &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{m=rk}^{rk+N-1} \beta_j(m-rn)\, T^m a\Big) \\ &= \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \Big(\sum_{m=0}^{N-1} \beta_j(m-rn)\, T^m a\Big) \\ &= \sum_{m=0}^{N-1} \Big\{\sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta_j(m-rn)\Big\} T^m a\,. \end{aligned}$$

In other words, for each $x = \sum_{m=0}^{N-1} \alpha(m)\, T^m a$ in $\mathcal{A}_a$, its coefficients $\alpha(m)$, $m = 0, 1, \ldots, N-1$, can be obtained as the output of a filter-bank

$$\alpha(m) = \sum_{j=1}^{s}\sum_{n=0}^{\ell-1} \mathcal{L}_j x(rn)\, \beta_j(m-rn)\,, \quad m = 0, 1, \ldots, N-1\,,$$

involving the input data $\big\{\mathcal{L}_j x(rn)\big\}_{\substack{j=1,2,\ldots,s \\ n=0,1,\ldots,\ell-1}}$ and the columns $\mathbf{h}_{j,0}$, $j = 1, 2, \ldots, s$, of the matrix $\mathbb{H}$, in the usual engineering jargon, as the impulse responses.

Notes and remarks

Next we list some specific comments for this section:

1. The vectors $\{T^k a\}_{k=0}^{N-1}$ in $\mathcal{H}$ are linearly independent if and only if the $N \times N$ *Gram matrix* $\Big(\langle T^l a, T^k a\rangle\Big)_{0\le k,l\le N-1}$ has non zero determinant.

2. The pseudo-inverse $\mathbb{R}_{a,\mathbf{b}}^{\dagger}$ is computed by using the singular value decomposition (SVD) of the matrix $\mathbb{R}_{a,\mathbf{b}}$; the singular values are the square root of the eigenvalues of the $N \times N$ invertible and positive semidefinite matrix $\mathbb{R}_{a,\mathbf{b}}^{*}\,\mathbb{R}_{a,\mathbf{b}}$ (see, for instance, [13, 29]). Note that the singular value decomposition of $\mathbb{R}_{a,\mathbf{b}}$ is the most reliable method to reveal its rank in practice.

   Besides, the optimal frame bound of the frame $\{\mathbf{g}_{j,n}\}_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,\ell-1}}$ for $\mathbb{C}^N$ defined in (13) are σ_1^2 and σ_N^2 where σ_1 (respectively σ_N) denotes the smallest (respectively the largest) singular value of the matrix $\mathbb{R}_{a,\mathbf{b}}$.

3. In the *overcomplete* setting we have that $s > r$. Whenever $r = s$, the $N\times N$ matrix $\mathbb{R}_{a,\mathbf{b}}$ is invertible and there exist r unique elements $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, r$, such that the sequence $\{T^{rn} c_j\}_{\substack{j=1,2,\dots,r\\ n=0,1,\dots,\ell-1}}$ is a basis for $\mathcal{A}_a$, and the sampling expansion (18) holds. Notice that in this case the inverse matrix $\mathbb{R}_{a,\mathbf{b}}^{-1}$ has necessarily the structure of the matrix $\mathbb{H}_{\mathbb{S}}$. Moreover, due to the uniqueness of the coefficients in a basis expansion, the *interpolation property* $\mathcal{L}_{j'} c_j(rn) = \delta_{j,j'}\,\delta_{n,0}$, where $n = 0, 1, \dots, \ell-1$ and $j, j' = 1, 2, \dots, r$, holds (for a similar result, see [25, Corollary 4]).

4. The matrix $\mathbb{H}_{\mathbb{S}}$ constructed from the first r rows of a left-inverse $\mathbb{H}$ of $\mathbb{R}_{a,\mathbf{b}}$ belongs to the family described by (16); indeed, $\mathbb{H}_{\mathbb{S}} = \mathbb{R}_{a,\mathbf{b}}^{\dagger} + \mathbb{U}_{\mathbb{S}}[\mathbb{I}_{s\ell} - \mathbb{R}_{a,\mathbf{b}}\,\mathbb{R}_{a,\mathbf{b}}^{\dagger}]$ for $\mathbb{U}_{\mathbb{S}} = \mathbb{U} + \mathbb{H}_{\mathbb{S}} - \mathbb{H}$ where $\mathbb{U}$ is a matrix such that $\mathbb{H} = \mathbb{R}_{a,\mathbf{b}}^{\dagger} + \mathbb{U}[\mathbb{I}_{s\ell} - \mathbb{R}_{a,\mathbf{b}}\,\mathbb{R}_{a,\mathbf{b}}^{\dagger}]$.

5. An easy example involving the cyclic shift in the Hilbert space $\ell_N^2(\mathbb{Z})$ of N-periodic sequences of complex numbers can be found in [25].

6. Similarly we can deal with the case of multiple generators. Namely, for a set of generators as $\mathbf{a} := \{a_1, a_2, \dots, a_L\} \subset \mathcal{H}$ with orders $N_1, N_2, \dots, N_L$, i.e., $T^{N_l} a_l = a_l$, $l = 1, 2, \dots, L$ respectively, we consider the subspace $\mathcal{A}_{\mathbf{a}}$ in $\mathcal{H}$ spanned by $\{a_l, T a_l, T^2 a_l, \dots, T^{N_l-1} a_l\}_{l=1}^{L}$. In case the Gram matrix of these vectors has non zero determinant, the subspace $\mathcal{A}_{\mathbf{a}}$ can be described as

   $$\mathcal{A}_{\mathbf{a}} = \Big\{ \sum_{l=1}^{L} \sum_{k=0}^{N_l-1} \alpha^l(k)\, T^k a_l \ :\ \alpha^l(k) \in \mathbb{C} \Big\}.$$

   Under suitable hypotheses sampling formulas like (17) can be obtained for the data (10); here the sampling period r divides $N = \text{l.c.m.}\,(N_1, N_2, \dots N_L)$ and $\ell := N/r$. Multiply generated subspaces $\mathcal{A}_{\mathbf{a}}$ involving the shift operator are very natural in signal theory. There are examples where a single generator fails to describe the appropriate signal subspace: for instance to describe subspaces of periodic extensions of finite signals, several generators are required (see [19, Section IV]).

4 The general case associated with an LCA group G

Let $(G,+)$ be a second countable *locally compact abelian* (LCA) Hausdorff group. Let $M < H < G$ be countable (finite or countably infinite) *uniform lattices* in G. Recall that a uniform lattice K in G is a discrete subgroup of G such that the quotient group G/K is compact (see, for instance, Ref. [8]). It is known that if $M < H$ are uniform lattices in G then H/M is a finite group (see [9, Remark 2.2]).

The dual group of the subgroup $H < G$, that is, the set of continuous characters on H is denoted by $\widehat{H}$. Since H is discrete, its dual $\widehat{H}$ is compact. We assume that its Haar measure $m_{\widehat{H}}$ is normalized to $m_{\widehat{H}}(\widehat{H}) = 1$. The value of the character $\gamma \in \widehat{H}$ at the point $h \in H$ is denoted by $(h,\gamma) \in \mathbb{T}$. With this Haar measure normalization the sequence $\{\chi_h\}_{h\in H}$ defined by

$$\widehat{H} \ni \gamma \mapsto \chi_h(\gamma) = (h,\gamma) \in \mathbb{T}$$

turns out to be an orthonormal basis for $L^2(\widehat{H})$ (see, for instance, [21, Prop. 4.3]).

Let $g \in G \mapsto \Pi(g)$ a *group representation* of G on a complex separable Hilbert space $\mathcal{H}$; i.e., Π is a mapping from G into the space of bounded invertible operators on $\mathcal{H}$, satisfying that $\Pi(g+g') = \Pi(g)\Pi(g')$ for all $g, g' \in G$.

Therefore, the mapping $h \in H \mapsto \Pi(h)$ is a group representation of H on $\mathcal{H}$. For a fixed $a \in \mathcal{H}$ let define the subspace in $\mathcal{H}$

$$\mathcal{A}_a := \overline{\operatorname{span}}\big\{\Pi(h)a \,:\, h \in H\big\} \subset \mathcal{H}\,.$$

We assume that $\big\{\Pi(h)a\big\}_{h\in H}$ is a Riesz sequence in $\mathcal{H}$. Thus, the subspace $\mathcal{A}_a$ can be expressed as

$$\mathcal{A}_a = \Big\{ \sum_{h\in H} \alpha_h \Pi(h)a \,:\, \{\alpha_h\}_{h\in H} \in \ell^2(H) \Big\} \subset \mathcal{H}\,.$$

As usual, $\{\alpha_h\}_{h\in H} \in \ell^2(H)$ means that $\sum_{h\in H} |\alpha_h|^2 < \infty$. The subspace $\mathcal{A}_a$ is the image of the Hilbert space $L^2(\widehat{H})$ by means of the isomorphism:

$$\begin{array}{rccc} \mathcal{T}_{H,a} : & L^2(\widehat{H}) & \longrightarrow & \mathcal{A}_a \\ & f = \displaystyle\sum_{h\in H} \alpha_h \chi_h & \longmapsto & x = \displaystyle\sum_{h\in H} \alpha_h \Pi(h)a \end{array}$$

This isomorphism $\mathcal{T}_{H,a}$ has the following Π-*shifting property* (its proof is analogous to that in [26, Proposition 1]):

$$\mathcal{T}_{H,a}(f\chi_k) = \Pi(k)(\mathcal{T}_{H,a}f) \quad \text{for any } f \in L^2(\widehat{H}) \text{ and } k \in H\,. \tag{19}$$

An expression for the samples

Suppose that s vectors $b_j \in \mathcal{H}$, $j = 1, 2, \dots, s$, which do not necessarily belong to $\mathcal{A}_a$, are given. For each $x \in \mathcal{A}_a$ we define the sequence of its samples taken at the subgroup M, as

$$\mathcal{L}_j x(m) := \big\langle x, \Pi^*(-m)b_j \big\rangle_{\mathcal{H}}, \;\; m \in M \text{ and } j = 1, 2, \dots, s\,, \tag{20}$$

where $\Pi^*(-m)$ denotes the adjoint operator of $\Pi(-m)$. For each $x \in \mathcal{A}_a$, let f be the element in $L^2(\widehat{H})$ such that $\mathcal{T}_{H,a} f = x$. An alternative expression for the sample $\mathcal{L}_j x(m)$, $j = 1, 2, \dots, s$ and $m \in M$ is

$$\mathcal{L}_j x(m) = \Big\langle \sum_{h\in H} \alpha_h \Pi(h)a, \Pi^*(-m)b_j \Big\rangle_{\mathcal{H}} = \sum_{h\in H} \alpha_h \overline{\big\langle \Pi^*(h-m)b_j, a \big\rangle}_{\mathcal{H}}\,.$$

Therefore, for any fixed $m \in M$ we have

$$\begin{aligned}\mathcal{L}_j x(m) =& \Big\langle f, \sum_{h\in H} \langle \Pi^*(h-m) b_j, a\rangle_{\mathcal{H}}\, \chi_h \Big\rangle_{L^2(\widehat{H})} \\ =& \Big\langle f, \Big(\sum_{k\in H} \langle \Pi^*(-k) b_j, a\rangle_{\mathcal{H}}\, \chi_{-k} \Big) \chi_m \Big\rangle_{L^2(\widehat{H})} ,\end{aligned}$$

where $k = m - h$ runs over H. Hence, we obtain the expression

$$\mathcal{L}_j x(m) = \big\langle f, \overline{f}_j\, \chi_m \big\rangle_{L^2(\widehat{H})}, \quad m \in M \text{ and } j = 1, 2, \dots, s\,, \tag{21}$$

where the function $f_j \in L^2(\widehat{H})$ is given by $f_j = \sum_{k\in H} \mathcal{L}_j a(k)\, \chi_k$, $j = 1, 2, \dots, s$. As a consequence of expression (21), the recovery of any $x \in \mathcal{A}_a$ depends on the frame property of the sequence $\big\{\overline{f}_j\, \chi_m\big\}_{m\in M;\, j=1,2,\dots,s}$ in $L^2(\widehat{H})$. This study was done in [26, Proposition 2]. In order to state the needed result we need to introduce some necessary preliminaries. The *annihilator* of M in $\widehat{H}$ is the closed subgroup

$$M^\perp = \{\gamma \in \widehat{H} \,:\, (m, \gamma) = 1 \text{ for all } m \text{ in } M\}\,.$$

Since $M^\perp$ is isomorphic to $\widehat{H/M}$, and H/M is finite, the annihilator $M^\perp$ is a finite subgroup of $\widehat{H}$. Let r be the order of $M^\perp$ and set $M^\perp = \big\{\mu_0^\perp = 0, \mu_1^\perp, \dots, \mu_{r-1}^\perp\big\}$. It is known that there exists a measurable (Borel) *section* Ω of $\widehat{H}/M^\perp$ (see the seminal Ref. [20]), i.e., a measurable set Ω such that

$$\widehat{H} = \bigcup_{n=0}^{r-1} (\mu_n^\perp + \Omega) \quad \text{and} \quad (\mu_n^\perp + \Omega) \cap (\mu_{n'}^\perp + \Omega) = \emptyset, \quad \text{for } n \neq n'\,.$$

Notice that $m_{\widehat{H}}(\Omega) = 1/r$. Besides, the sequence $\{\chi_m\}_{m\in M}$ is an orthogonal basis for $L^2(\Omega)$.

For $f_j \in L^2(\widehat{H})$, $j = 1, 2, \dots, s$, we consider the associated $s \times r$ matrix-valued function defined by

$$\mathbb{F}(\xi) := \Big(f_j\big(\xi + \mu_n^\perp\big) \Big)_{\substack{j=1,2,\dots,s\\ n=0,1,\dots,r-1}}, \qquad \xi \in \Omega\,, \tag{22}$$

and the related constants

$$\alpha_{\mathbb{F}} := \operatorname*{ess\,inf}_{\xi\in\Omega} \lambda_{\min}[\mathbb{F}^*(\xi)\mathbb{F}(\xi)]\,; \quad \beta_{\mathbb{F}} := \operatorname*{ess\,sup}_{\xi\in\Omega} \lambda_{\max}[\mathbb{F}^*(\xi)\mathbb{F}(\xi)]\,.$$

Thus we have ([26, Proposition 2]):

The sequence $\{\overline{f}_j\, \chi_m\}_{m\in M;\, j=1,2,\dots,s}$ is a frame for $L^2(\widehat{H})$ if and only if $0 < \alpha_{\mathbb{F}} \leq \beta_{\mathbb{F}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{F}}/r$ and $\beta_{\mathbb{F}}/r$.

We also need to characterize its dual frames having the same structure. This is done in [26, Proposition 4]:

Assume that the functions $h_j \in L^\infty(\widehat{H})$, $j = 1, 2, \dots, s$, satisfy

$$\big(h_1(\xi), h_2(\xi), \dots, h_s(\xi)\big)\, \mathbb{F}(\xi) = (1, 0, \dots, 0)\,, \quad a.e.\ \xi \in \widehat{H}\,.$$

Then, the sequences $\{\overline{f}_j\, \chi_m\}_{m\in M;\, j=1,2,\dots,s}$ and $\{r h_j\, \chi_m\}_{m\in M;\, j=1,2,\dots,s}$ form a pair of dual frames for $L^2(\widehat{H})$.

All the possible vector-functions $\mathbf{h}(\xi) := \big(h_1(\xi), h_2(\xi), \dots, h_s(\xi)\big)$ satisfying the above condition, and with entries in $L^\infty(\widehat{H})$ are given by the first row of the $r \times s$ matrices

$$\mathbb{H}(\xi) := \mathbb{F}^\dagger(\xi) + \mathbb{U}(\xi)\big[\mathbb{I}_s - \mathbb{F}(\xi)\mathbb{F}^\dagger(\xi)\big]$$

where $\mathbb{F}^\dagger(\xi) = \big[\mathbb{F}^*(\xi)\,\mathbb{F}(\xi)\big]^{-1}\,\mathbb{F}^*(\xi)$ denotes the *Moore-Penrose pseudo-inverse* of $\mathbb{F}(\xi)$, and $\mathbb{U}(\xi)$ denotes any $r \times s$ matrix with entries in $L^\infty(\widehat{H})$.

A regular sampling formula in $\mathcal{A}_a$

For $x \in \mathcal{A}_a$ let $f \in L^2(\widehat{H})$ be such that $\mathcal{T}_{H,a} f = x$. Expanding f with respect to the pair of dual frames $\{\overline{f}_j\,\chi_m\}_{m\in M;\, j=1,2,\dots,s}$ and $\{rh_j\,\chi_m\}_{m\in M;\, j=1,2,\dots,s}$, where f_j is given in (21), we obtain

$$f = \sum_{j=1}^{s}\sum_{m\in M} \big\langle f, \overline{f}_j\,\chi_m\big\rangle_{L^2(\widehat{H})}\, rh_j\,\chi_m = \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\, rh_j\,\chi_m \quad \text{in } L^2(\widehat{H}).$$

The isomorphism $\mathcal{T}_{H,a}$ and the Π-shifting property (19) give, for any $x \in \mathcal{A}_a$, the sampling formula

$$\begin{aligned} x &= \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\mathcal{T}_{H,a}\big(rh_j\chi_m\big) = \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\,\Pi(m)\mathcal{T}_{H,a}\big(rh_j\big) \\ &= \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\,\Pi(m) c_{j,\mathbf{h}}\,, \end{aligned}$$

where $c_{j,\mathbf{h}} := \mathcal{T}_{H,a}(rh_j) \in \mathcal{A}_a$, $j = 1, 2, \dots, s$. Besides, $\big\{\Pi(m)c_{j,\mathbf{h}}\big\}_{m\in M;\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. In fact, the following result holds:

Theorem 3. *For $x \in \mathcal{A}_a$ consider the sequence of samples $\{\mathcal{L}_j x(m)\}_{m\in M;\, j=1,2,\dots,s}$ defined in* (20)*. Assume that the functions f_j, $j = 1, 2, \dots, s$, in* (21) *belong to $L^\infty(\widehat{H})$, and consider the associated $\mathbb{F}(\xi)$ matrix-valued function given in* (22)*. The following statements are equivalent:*

(a) The constant $\alpha_{\mathbb{F}} > 0$.

(b) There exist functions $h_j(\xi)$ in $L^\infty(\widehat{H})$, $j = 1, 2, \dots, s$, satisfying

$$\big(h_1(\xi), h_2(\xi), \dots, h_s(\xi)\big)\,\mathbb{F}(\xi) = (1, 0, \dots, 0) \quad \text{a.e. } \xi \text{ in } \widehat{H}\,.$$

(c) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, such that the sequence $\big\{\Pi(m)c_j\big\}_{m\in M;\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$, and for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\,\Pi(m)c_j \quad \text{in } \mathcal{H}\,, \tag{23}$$

holds.

(d) There exists a frame $\big\{C_{j,m}\big\}_{m\in M;\, j=1,2,\dots s}$ for $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{m\in M} \mathcal{L}_j x(m)\, C_{j,m} \quad \text{in } \mathcal{H}\,,$$

holds.

In case the equivalent conditions are satisfied, for the elements c_j in (c) we have $c_j = \mathcal{T}_{H,a}(rh_j)$, for some functions h_j in $L^\infty(\widehat{H})$, $j = 1, 2, \dots, s$, and satisfying the condition in (b).

Proof. We have already proved that (a) implies (b), and that (b) implies (c). Obviously, (c) implies (d). Finally, we prove that condition (d) implies condition (a). Indeed, applying the isomorphism $\mathcal{T}_{H,a}^{-1}$ to the expansion in (d), and taking into account (21) we obtain

$$
\begin{aligned}
f = \mathcal{T}_{H,a}^{-1} x &= \sum_{j=1}^{s} \sum_{m \in M} \mathcal{L}_j x(m)\, \mathcal{T}_{H,a}^{-1}\big(C_{j,m}\big) \\
&= \sum_{j=1}^{s} \sum_{m \in M} \big\langle f, \overline{f}_j\, \chi_m \big\rangle_{L^2(\widehat{H})}\, \mathcal{T}_{H,a}^{-1}\big(C_{j,m}\big) \quad \text{in } L^2(\widehat{H})\,,
\end{aligned}
$$

where the sequence $\big\{\mathcal{T}_{H,a}^{-1}\big(C_{j,m}\big)\big\}_{m \in M;\, j=1,2,\dots s}$ is a frame for $L^2(\widehat{H})$. The sequence $\big\{\overline{f}_j\, \chi_m\big\}_{m \in M;\, j=1,2,\dots s}$ is a Bessel sequence in $L^2(\widehat{H})$ since $\beta_{\mathbb{G}} < \infty$, and it satisfies the above expansion in $L^2(\widehat{H})$. As a consequence, according to [13, Lemma 6.3.2] the sequences $\big\{\overline{f}_j\, \chi_m\big\}_{m \in M;\, j=1,2,\dots s}$ and $\big\{\mathcal{T}_{H,a}^{-1}\big(C_{j,m}\big)\big\}_{m \in M;\, j=1,2,\dots s}$ form a pair of dual frames in $L^2(\widehat{H})$. In particular, by using ([26, Proposition 2]) we obtain that $\alpha_{\mathbb{F}} > 0$ which concludes the proof. □

Notes and remarks

Some specific comments for this general case treated here are the following:

1. The LCA group approach is not just a unified way of dealing with the four classical groups $\mathbb{R}, \mathbb{Z}, \mathbb{T}, \mathbb{Z}_N$: signal processing often involves products of these groups which are also LCA groups. For example, multichannel video signal involves the group $\mathbb{Z}^d \times \mathbb{Z}_N$, where d is the number of channels and N the number of pixels of each image. The availability of an abstract sampling theory for unitary invariant spaces becomes a useful tool to handle these problems in a unified way. Moreover, any notational complication is avoided especially in the multidimensional setting.

2. Notice that the case exhibited in this section is more general than those in the former sections. Indeed, let G be an LCA group and let $H := \{ng\}_{n \in \mathbb{Z}}$ the (infinite) cyclic group generated by some fix element $g \in G$. Then, the subspace

   $$\mathcal{A}_a := \overline{\operatorname{span}}\big\{\Pi(ng)a \,:\, n \in \mathbb{Z}\big\} = \overline{\operatorname{span}}\big\{[\Pi(g)]^n a \,:\, n \in \mathbb{Z}\big\}\,,$$

   is obtained from $T := \Pi(g)$.

3. The cyclid case in Section 3 involving the group $\mathbb{Z}_N$ may be a particular case of this section. Although there we have used the canonical basis of $\mathbb{C}^N$ to define the isomorphism $\mathcal{T}_{N,a}$ instead of the Fourier basis of the characters used here. So in the finite case we have not used, explicitly, Fourier analysis as in [5, Proposition 3.1]; just linear algebra.

4. In case the operator $T = U$ is unitary, a necessary and sufficient condition for $\{T(h)a\}_{h \in H}$ to be a Riesz sequence in $\mathcal{H}$ can be found in [7].

5. In the *overcomplete* setting we have that $s > r$. In case $r = s$, the frame condition in Theorem 3 becomes a Riesz basis condition: There exist r unique elements $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, r$, such that the sequence $\big\{\Pi(m)c_j\big\}_{m\in M;\, j=1,2,\dots r}$ is a *Riesz basis* for $\mathcal{A}_a$, and the sampling expansion (23) holds. Moreover, due to the uniqueness of the coefficients in a Riesz basis expansion, the *interpolation property* $\mathcal{L}_{j'}c_j(m) = \delta_{j,j'}\,\delta_{m,0}$, where $m \in M$ (0 denotes the null element in G) and $j, j' = 1, 2, \dots, r$, holds (for a similar result, see [26, Corollary 2]).

6. Concerning pointwise sampling, whenever $\mathcal{H} = L^2(G)$, instead, we formally consider the samples of any $x \in \mathcal{A}_a := \big\{ \sum_{h\in H} \alpha_h\, \Pi(h)a \,:\, \{\alpha_h\} \in \ell^2(H)\big\}$ as

$$\mathcal{L}_j x(m) := \big[\Pi(-m)x\big](g_j), \;\; m \in M \text{ and } j = 1, 2, \dots, s\,,$$

where g_j, $j = 1, 2, \dots, s$, denote s fixed points in the group G. Under mild hypotheses: the functions $\Pi(h)a$, $h \in H$, are continuous in G, and

$$\sup_{g\in G} \sum_{h\in H} \big|[\Pi(h)a](g)\big|^2 < \infty\,,$$

one can prove that the subspace $\mathcal{A}_a$ is a RKHS (reproducing kernel Hilbert space) of bounded continuous functions in $L^2(G)$ (see [27, Lemma 4.2]). Moreover, for any $x = \mathcal{T}_{H,a} f = \sum_{h\in H} \alpha_h \Pi(h)a$ in $\mathcal{A}_a$, we easily obtain the expression for the samples

$$\big[\Pi(-m)x\big](g_j) = \Big\langle f, \Big(\sum_{k\in H} \overline{[\Pi(k)a](g_j)}\chi_k\Big)\chi_m\Big\rangle_{L^2(\widehat{H})} = \big\langle f, \overline{f}_j \chi_m\big\rangle_{L^2(\widehat{H})}\,,$$

where $f_j := \sum_{k\in H} [\Pi(k)a](g_j)\chi_{-k} \in L^2(\widehat{H})$, $j = 1, 2, \dots, s$; thus, the above samples are well defined. Under the hypotheses on the corresponding matrix-valued function $\mathbb{F}(\xi)$ in Theorem 3, a sampling formula like (23) holds in $\mathcal{A}_a$ for these new samples.

In particular, for $\big[\Pi(m)x\big](g) := L_m x(g) = x(g - m)$, i.e., the left regular unitary representation $g \in G \mapsto L_g$ of the group G in $L^2(G)$, the above sample $\mathcal{L}_j x(m)$ is nothing but the pointwise sample $x(m + g_j)$. In this case, the sampling formula (23) in $\mathcal{A}_a = \big\{ \sum_{h\in H} \alpha_h\, L_h a(g) \,:\, \{\alpha_h\} \in \ell^2(H)\big\}$ reads

$$x(g) = \sum_{j=1}^{s} \sum_{m\in M} x(m + g_j)\, c_j(g - g_j)\,, \quad g \in G\,,$$

for some functions $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$. The convergence of the series in the $L^2(G)$-norm sense implies pointwise convergence which is absolute (due to the unconditional character of a frame expansion) and uniform on G. Thus, a generalization of Kluvánek's sampling theorem in $\mathcal{A}_a$ can be deduced from Theorem 3; see [26, Theorem 3] for the details. See also Ref. [17].

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