On the Analytic Sampling Theory (a link with the m-function theory)

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1. What does the Sampling theory mean?
2. The space $\mathcal{H}$
   - The Hilbert space structure of $\mathcal{H}$
   - Analyticity of the functions in $\mathcal{H}$
   - Sampling in $\mathcal{H}$
3. Generic examples
   - Sampling theory associated with the resolvent kernel
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What does the Sampling theory mean?

- Linear space $\mathcal{H} = \{f : \Omega \longrightarrow \mathbb{C}\}$
- $\{t_n\} \subset \Omega$
- $\{S_n\} \subset \mathcal{H}$

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In general

$$f(t) = \sum_n \left[f(t_n)\, S_n(t) + \widetilde{f}(t_n)\, \widetilde{S}_n(t)\right]$$

($\widetilde{f}$ a related function with f)

The Shannon's Example

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- $K : \Omega \subset \mathbb{C} \to \mathbb{H}$

For $x \in \mathbb{H}$ define

$$f(t) := \langle K(z), x \rangle_{\mathbb{H}} \,, \quad z \in \Omega$$

Definition

Consider the anti-linear mapping:

$$T : \mathbb{H} \ni x \mapsto f \in \mathcal{H}$$

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Properties of $\mathcal{H}$

- $\|f\|_{\mathcal{H}} := \inf\{\|x\|_{\mathbb{H}} : f = Tx\}$,
- T is one-to-one
  - $\Leftrightarrow$ $\{K(z)\}_{z\in\Omega}$ is a complete set in $\mathbb{H}$
  - $\Leftrightarrow$ T is an isometry onto $\mathcal{H}$.
- $\mathcal{H}$ is a RKHS: If $k(z,w) = \langle K(z), K(w)\rangle_{\mathbb{H}}$ then

$$f(w) = \langle f, k(\cdot, w)\rangle_{\mathcal{H}} \;\; \left(\Rightarrow |f(w)| \leq \|f\|_{\mathcal{H}}\|K(w)\|_{\mathbb{H}}\right)$$

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Analyticity of the functions in $\mathcal{H}$

The RKHS $\mathcal{H}$ is a space of analytic functions in Ω

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where

- $K(z) = \sum_{n=1}^{\infty} \langle K(z), x_n \rangle_{\mathbb{H}} \, x_n = \sum_{n=1}^{\infty} S_n(z) \, x_n$,
- $\{x_n\}_{n=1}^{\infty}$ is an orthonormal basis (Riesz basis or frame) in $\mathbb{H}$.

Sampling in $\mathcal{H}$

- Let $\{x_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathbb{H}$. Expanding $K(z)$ with respect to $\{x_n\}_{n=1}^{\infty}$

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Assume there exists $\{z_n\}_{n=1}^{\infty} \subset \Omega = \mathbb{C}$ such that $S_n(z_m) = a_n\, \delta_{n,m}$ $(a_n \neq 0)$

Sampling in $\mathcal{H}$

$$K(z) = \sum_{n=1}^{\infty} S_n(z)\, x_n\,; \quad S_n(z_m) = a_n\, \delta_{n,m}\ (a_n \neq 0)$$

Sampling theorem

Then, for all $f \in \mathcal{H}$

$$f(z) = \sum_{n=1}^{\infty} f(z_n)\, \frac{S_n(z)}{a_n}\,, \quad z \in \mathbb{C}\,.$$

Convergence of the series is absolute, and uniform in subsets of $\mathbb{C}$ where $\|K(z)\|_{\mathbb{H}}$ is bounded.

Sampling in $\mathcal{H}$

Suppose the orthonormal basis for $\mathbb{H}$ partitioned as

$$\{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}.$$

Now,

$$K(z) = \sum_{n=1}^{\infty} \Big[S_n(z)\ x_n + T_n(z)\ y_n \Big] ,$$

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Assume there exists $\{z_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that

- $S_n(z_m) = a_n\,\delta_{n,m}\,; \quad T_n(z_m) = b_n\,\delta_{n,m}$,
- $S'_n(z_m) = c_n\,\delta_{n,m}\,; \quad T'_n(z_m) = d_n\,\delta_{n,m}$, and
- $\Delta_n = a_n\,d_n - b_n\,c_n \neq 0$ for all $n \in \mathbb{N}$.

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- $K(z) = \sum_{n=1}^{\infty} \Big[S_n(z)\; x_n + T_n(z)\; y_n \Big]$,
- $S_n(z_m) = a_n\, \delta_{n,m}$; $\quad T_n(z_m) = b_n\, \delta_{n,m}$,
- $S_n'(z_m) = c_n\, \delta_{n,m}$; $\quad T_n'(z_m) = d_n\, \delta_{n,m}$, and
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Sampling theorem

Then, for all $f \in \mathcal{H}$

$$f(z) = \sum_{n=1}^{\infty} \left[f(z_n) \frac{d_n\, S_n(z) - c_n\, T_n(z)}{\Delta_n} + f'(z_n) \frac{a_n\, T_n(z) - b_n\, S_n(z)}{\Delta_n} \right].$$

Convergence of the series is absolute, and uniform in subsets of $\mathbb{C}$ where $\|K(z)\|_{\mathbb{H}}$ is bounded.

Sampling theory associated with an invertible symmetric operator with resolvent kernel

$$\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \longrightarrow \mathbb{H}$$

- $\mathcal{A}$ is a symmetric operator, densely defined on $\mathbb{H}$:
- Exists $\mathcal{T} = \mathcal{A}^{-1}$,
- The resolvent operator $R_z = (zI - \mathcal{A})^{-1}$ is a compact operator.

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For any $x \in \mathbb{H}$ we have

$$R_z(x) = \sum_{n=1}^{\infty} \Big(\frac{1}{z - z_n} \sum_{i=1}^{k_n} \langle x, e_{n,i} \rangle_{\mathbb{H}}\; e_{n,i} \Big)$$

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- $\{z_n\}_{n=1}^{\infty}$ are the eigenvalues of $\mathcal{A}$.
- $\{\{e_{n,i}\}_{i=1}^{k_n}\}_{n=1}^{\infty}$, are the associated orthonormal basis of eigenvectors of $\mathcal{A}$.

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Definition

For a fixed $a \in \mathbb{H}$ define

$$\begin{array}{rccl} K_a : & \mathbb{C} & \longrightarrow & \mathbb{H} \\ & z & \longrightarrow & K_a(z) := P(z)\, R_z(a) \end{array}$$

P any entire function having simple zeros at $\{z_n\}_{n=1}^{\infty}$

Sampling theory associated with an invertible symmetric operator with resolvent kernel

Sampling result

For $x \in \mathbb{H}$, let f be the function given by $f(z) := \langle K_a(z), x \rangle_{\mathbb{H}}$, $z \in \mathbb{C}$. Then,

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{P(z)}{(z - z_n)P'(z_n)} .$$

Convergence of the series is absolute, and uniform in compact subsets of $\mathbb{C}$.

Sampling theory associated with an invertible symmetric operator with resolvent kernel

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Remarks

- Classical sampling results associated with differential problems are derived from this result. ▸ References

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Remarks

- Classical sampling results associated with differential problems are derived from this result. ▸ References
- The corresponding $\mathcal{H}_a$ space is a *de Branges space* of entire functions. ▸ de Branges space

Sampling associate with Indeterminate moment problem

- $s = \{s_n\}_{n=0}^{\infty}$ indeterminate Hamburger moment sequence

$$V_s = \left\{ \mu \geq 0 \text{ Borel } \;\middle|\; \int_{-\infty}^{\infty} x^n \, d\mu(x) = s_n \, , n \geq 0 \right\}$$

- $\{P_n(x)\}_{n=0}^{\infty}$ orthonormal polynomials (with positive leading coefficient) with respect to any $\mu \in V_s$
- $\{Q_n(x)\}_{n=0}^{\infty}$ second kind orthogonal polynomials associated with $\{P_n(x)\}_{n=0}^{\infty}$.

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Differential problems and Sampling

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Differential problems and Sampling

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de Branges Spaces

Definition

An operator J defined on a Hilbert space $\mathbb{H}$ is a conjugation operator if, for all $x, y \in \mathbb{H}$,

$$\langle Jx, Jy\rangle_{\mathbb{H}} = \langle y, x\rangle_{\mathbb{H}}\,, \text{ and } J^2x = x\,.$$

Assume that the operator $\mathcal{A}$ is real with respect to J, i.e., the relationship $J\mathcal{A}J = \mathcal{A}$ is satisfied.

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Assume that the operator $\mathcal{A}$ is real with respect to J, i.e., the relationship $J\mathcal{A}J = \mathcal{A}$ is satisfied.

The following properties, which will be used later, hold:

1. The sequence $\{(Je_{n,i})_{i=1}^{k_n}\}_{n=1}^{\infty}$ is also an orthonormal basis of eigenfunctions in $\mathbb{H}$.
2. Since $\overline{P(\overline{z})} = P(z)$ for $z \in \mathbb{C}$, we have that $JK_a(\overline{z}) = K_{Ja}(z)$ for each $z \in \rho(\mathcal{A})$.
3. $JK_a(z_m) = K_{Ja}(z_m)$ for each $z_m \in \sigma(\mathcal{A})$.

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de Branges Spaces

Definition

A Hilbert space $\mathcal{H}$ of entire functions is a de Branges space if the following conditions hold:

B1. Whenever $f \in \mathcal{H}$ and ω is a nonreal zero of f, the function

$$g(z) := f(z)\frac{z-\overline{\omega}}{z-\omega}$$

belongs to $\mathcal{H}$ and $\|g\| = \|f\|$.

B2. For each $\omega \notin \mathbb{R}$ the linear mapping $\mathcal{H} \ni f \to f(\omega)$ is continuous.

B3. The function $f^*(z) := \overline{f(\overline{z})}$ belongs to the space, and $\|f^*\| = \|f\|$.