Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

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Statement of the problem

Let V_Φ^2 be a shift-invariant subspace of $L^2(\mathbb{R}^d)$ with a stable set of generators $\Phi := \{\varphi_1 \dots \varphi_r\} \in L^2(\mathbb{R}^d)$, i.e.,

$$V_\Phi^2 = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\varphi_k(t-\alpha) \ : \ d_k \in \ell^2(\mathbb{Z}^d),\ k = 1,2,\dots,r \Big\}$$

Consider s linear-time invariant systems $\mathcal{L}_j$, $j = 1,2,\dots,s$ defined on V_Φ^2.

The problem: Recover any function $f \in V_\Phi^2$ from the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ taken at the sub-lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$ by means of a sampling formula as

$$f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha), \qquad t \in \mathbb{R}^d$$

where the sequence $\{S_j(t - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for V_Φ^2 (The sampling rate $1/r(\det M)$ necessarily satisfies $r(\det M) \le s$)

Assume that $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ satisfy the following stability condition: There exist two positive constants $0 < A \le B$ such that

$$A\|f\|^2 \le \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \le B\|f\|^2 \quad \text{for all } f \in V_\Phi^2 .$$

A fresh approach to the problem

- We define a Fourier–type duality $\mathcal{T}_\Phi$ between $L_r^2[0,1)^d$ and V_Φ^2
- We express the generalized samples of f in terms of a suitable frame for $L_r^2[0,1)^d$ and $\mathbf{F} := \mathcal{T}_\Phi^{-1}(f) \in L_r^2[0,1)^d$
- We obtain its dual frames in $L_r^2[0,1)^d$
- Finally, a generalized sampling expansion for $f \in V_\Phi$ comes out by using $\mathcal{T}_\Phi$ in the frame expansion for $\mathbf{F}$

The working hypotheses

Hypotheses on V_Φ^2

Assume the following hypotheses on the generador Φ :

- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$ is a Riesz basis for V_Φ^2
  Recall that the sequence $\{\varphi_k(\cdot-\alpha)\}_{\alpha\in\mathbb{Z}^d,\, k=1,2,\dots,r}$ is a Riesz basis for V_Φ^2 if and only if $\exists M, m > 0$ such that $m\mathbb{I}_r \le G_\Phi(w) \le M\mathbb{I}_r$ a.e. $w \in [0,1)^d$ where
  $$G_\Phi(w) := \sum_{\alpha\in\mathbb{Z}^d} \widehat{\Phi}(w+\alpha)\overline{\widehat{\Phi}(w+\alpha)}^\top$$
  is the Gramian matrix-function
- The generators φ_k are continuous on $\mathbb{R}^d$
- The series $\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$
  Thus, the pointwise sum $f(t) = \sum_{k=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} d_k(\alpha)\varphi_k(t-\alpha)$ defines a continuous function on $\mathbb{R}^d$

As a consequence, V_Φ^2 becomes a RKHS (reproducing kernel Hilbert space) and, in particular, the $L^2(\mathbb{R}^d)$-norm convergence in V_Φ^2 implies uniform convergence on $\mathbb{R}^d$

Hypotheses on the systems $\mathcal{L}_j$

We consider two types of linear-time invariant systems:

1. The impulse response $h \in L^2(\mathbb{R}^d)$
2. The impulse response h_j is a linear combination of partial derivatives of shifted delta functionals, i.e.,
$$(\mathcal{L}_j f)(t) := \sum_{|\beta|\le N_j} c_{j,\beta} D^\beta f(t + d_{j,\beta}), \quad t \in \mathbb{R}^d .$$

(If there is a system of this type, we also assume that $\sum_{\alpha\in\mathbb{Z}^d} |D^\beta \varphi(t-\alpha)|^2$ is uniformly bounded on $\mathbb{R}^d$ for $|\beta| \le N_j$.)

Lattices in $\mathbb{Z}^d$

Given a matrix M with integer entries and $\det M > 0$, we consider the lattice in $\mathbb{Z}^d$ generated by M, i.e.,

$$\Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d .$$

The set $\mathcal{N}(M^\top) := \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\}$ has $\det M$ elements . One of these elements is zero, say $i_1 = 0$; we denote the rest of elements by $i_2, \dots, i_{\det M}$ ordered in any form

An expression for the samples

The isomorphism $\mathcal{T}_\Phi$

Let $\mathcal{T}_\Phi : L_r^2[0,1)^d \longrightarrow V_\Phi^2$ be the isomorphism defined by

$$\mathcal{T}_\Phi(e^{-2\pi i\alpha^\top w}\mathbf{e}_k) := \varphi_k(t-\alpha), \quad \alpha \in \mathbb{Z}^d, k = 1,2,\dots,r$$

Then:

- For any $f \in V_\Phi^2$ we have
  $$f(t) = \langle \mathbf{F}, \mathbf{K}_t \rangle_{L_r^2[0,1)^d}, \quad t \in \mathbb{R}^d,$$
  where $F = \mathcal{T}_\Phi^{-1}(f)$ and
  $$\mathbf{K}_t(x) := \overline{\mathbf{Z}\Phi}(t,x)$$
  ($\mathbf{Z}\Phi$ is the Zak transform of Φ)
- $\mathcal{T}_\Phi\big[\mathbf{F}(\cdot)e^{-2\pi i\alpha^\top\cdot}\big](t) = \mathcal{T}_\Phi\mathbf{F}(t-\alpha), \quad t \in \mathbb{R}^d .$

An expression for the samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$

Let $\mathcal{L}$ be a system as those considered here. For any $f \in V_\Phi^2$ we have

$$(\mathcal{L}f)(t) = \langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}\Phi)}(t,\cdot) \rangle_{L_r^2[0,1)^d},.$$

where $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$. In particular, for each $j = 1,2,\dots,s$:

$$\begin{aligned}(\mathcal{L}_j f)(M\alpha) &= \langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(M\alpha,\cdot) \rangle \\ &= \langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(0,\cdot)e^{-2\pi i\alpha^T M^T\cdot} \rangle_{L_r^2[0,1)^d}\end{aligned}$$

Denoting by $g_j(x) := \mathbf{Z}\mathcal{L}_j\Phi(0,x) \quad j = 1,2,\dots,s$, we consider the $s \times r(\det M)$ matrix

$$\mathbf{G}(x) := \begin{bmatrix} g_1^\top(x) & g_1^\top(x+M^{-\top}i_2) & \cdots & g_1^\top(x+M^{-\top}i_{\det M}) \\ g_2^\top(x) & g_2^\top(x+M^{-\top}i_2) & \cdots & g_2^\top(x+M^{-\top}i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ g_s^\top(x) & g_s^\top(x+M^{-\top}i_2) & \cdots & g_s^\top(x+M^{-\top}i_{\det M}) \end{bmatrix}$$

(we are considering $\mathbb{Z}^d$-periodic extensions of the functions g_j) and its related constants

$$A_\mathbf{G} := \operatorname*{ess\,inf}_{x\in[0,1)^d} \lambda_{\min}[\mathbf{G}^*(x)\mathbf{G}(x)], \quad B_\mathbf{G} := \operatorname*{ess\,sup}_{x\in[0,1)^d} \lambda_{\max}[\mathbf{G}^*(x)\mathbf{G}(x)],$$

$\lambda_{\min}$ ($\lambda_{\max}$) the smallest (the largest) eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(x)\mathbf{G}(x)$

Two problems to solve:

- To characterize when the sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for $L_r^2[0,1)^d$

  The sequence $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for $L_r^2[0,1)^d$ if and only if $0 < A_\mathbf{G} \le B_\mathbf{G} < \infty$. In this case, the optimal frame bounds are $A_\mathbf{G}/(\det M)$ and $B_\mathbf{G}/(\det M)$.

- To find its dual frames

  If there exists an $r\times s$ matrix $\mathbf{a}(x) := [\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that
  $$[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]\, \mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{a.e. in } [0,1)^d .$$
  then the sequence $\{(\det M)\mathbf{a}_j(x)e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a dual frame of $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$

Thus, for $\mathbf{F} = \mathcal{T}_\Phi^{-1}(f)$ the following expansion holds:

$$\mathbf{F}(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\, \mathbf{a}_j(x)e^{-2\pi i\alpha^\top M^\top x} \quad \text{in } L_r^2[0,1)^d$$

The generalized regular sampling result

The main result: Assume that $g_j \in L_r^\infty[0,1)^d$ for each $j = 1,2,\dots,s$ ($\Leftrightarrow B_\mathbf{G} < \infty$). The following statements are equivalent:

1. $A_\mathbf{G} > 0$
2. There exists a vector $[a_1(x),\dots,a_s(x)]$ with entries $a_j \in L_r^\infty[0,1)^d$ satisfying
$$[\mathbf{a}_1(x),\dots,\mathbf{a}_s(x)]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{a.e. in } [0,1)^d$$
3. There exists a frame for V_Φ^2 having the form $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ such that for any $f \in V_\Phi^2$,
$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)$$

In case the equivalent conditions are satisfied:

- $S_{j\mathbf{a}} = \mathcal{T}_\Phi(\mathbf{a}_j), \ j = 1,2,\dots,s$
- Convergence of the sampling series is absolute and uniform on $\mathbb{R}^d$
- If $r(\det M) = s$ then $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a Riesz basis for V_Φ^2.
  The functions $\mathbf{a}_j$ form the first r rows of the matrix $\mathbf{G}^{-1}$
  In this case, the following interpolation property holds:
  $$(\mathcal{L}_l S_{j,\beta})(M\alpha) = \delta_{j,l}\delta_{\alpha\beta}, \qquad \alpha,\beta \in \mathbb{Z}^d \quad j,l = 1,2,\dots,s$$

Some remarks:

- The first r rows of the pseudo-inverse matrix
  $$\mathbf{G}^\dagger(x) = [\mathbf{G}^*(x)\mathbf{G}(x)]^{-1}\mathbf{G}^*(x)$$
  gives the $\mathbf{a}_j$ functions defining the canonical frame dual of $\{\overline{g_j(x)}e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$
- Other suitable a_j functions are given by the first row of the matrix $\mathbf{G}^\dagger(x) + \mathbf{U}(x)[\mathbf{I}_s - \mathbf{G}(x)\mathbf{G}^\dagger(x)]$, where $\mathbf{U}(x)$ is any $r(\det M) \times s$ matrix function with entries in $L^\infty[0,1)^d$
- We can take advantage of the oversampling setting $s > r(\det M)$. There exist many solutions for the $\mathbf{a}_j$ functions: One may use this flexibility to obtain appropriate sampling functions $S_{j,\alpha}$. For instance, if the generators φ_k and the impulse responses of the filters $\mathcal{L}_j$ have compact support we could choose the $\mathbf{a}_j$ functions in order to obtain sampling functions $S_{j,\alpha}$ with compact support

Generalized irregular sampling

Assume that we have at our disposal the sequence of irregular samples $\{(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$. Our starting point is:

- The expression of the irregular samples
  $$(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha})) = \big\langle \mathbf{F}, \overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},\cdot)e^{-2\pi i\alpha^\top M^\top\cdot} \big\rangle, \quad \alpha \in \mathbb{Z}^d,$$
  for each $j = 1,2,\cdots,s$.
- The sequence $\{\overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(0,x)e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ is a frame for $L_r^2[0,1)^d$ if and only if $0 < A_\mathbf{G} \le B_\mathbf{G} < \infty$

A suitable approach is to consider the sequence

$$\{\overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(\varepsilon_{j,\alpha},x)e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$$

as a perturbation of the frame $\{\overline{(\mathbf{Z}\mathcal{L}_j\Phi)}(0,x)e^{-2\pi i\alpha^\top M^\top x}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ for the error sequence $\varepsilon := \{\varepsilon_{j,\alpha}\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$

We define on $\ell_r^2(\mathbb{Z}^d)$ the linear operator $D_\varepsilon := [D_{\varepsilon 1},\dots,D_{\varepsilon s}]$ where

$$D_{\varepsilon,j}\, c := \Big\{ \sum_{k=1}^{r} \sum_{\beta\in\mathbb{Z}^d} \big[\mathcal{L}_j\varphi_k(M\alpha-\beta+\varepsilon_{j,\alpha}) - \mathcal{L}_j\varphi_k(M\alpha-\beta)\big] c_{k,\beta} \Big\}_{\alpha\in\mathbb{Z}^d}$$

for each $c = (\{c_{1\beta}\}_{\beta\in\mathbb{Z}^d},\dots,\{c_{r\beta}\}_{\beta\in\mathbb{Z}^d}) \in \ell_r^2(\mathbb{Z}^d)$, and consider the norm

$$\|D_\varepsilon\| := \sup_{\|c\|_{\ell_r^2(\mathbb{Z}^d)}=1} \sqrt{\sum_{j=1}^{s} \|D_{\varepsilon,j}c\|^2_{\ell^2(\mathbb{Z}^d)}}$$

By using a standard result on perturbation of frames we get:

Let $\varepsilon := \{\varepsilon_{j,\alpha}\}$ be a sequence error such that $\|D_\varepsilon\|^2 < A_\mathbf{G}/\det M$. Then, there exists a frame $\{S_{j,\alpha}^\varepsilon\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$ for V_φ such that, for any $f \in V_\Phi^2$

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S_{j,\alpha}^\varepsilon(t), \quad t \in \mathbb{R}^d$$

Two practical problems:

- To estimate $\sum_{j=1}^{s} \|D_{\varepsilon,j}\|^2$ in terms of $\delta := \sup_{j,\alpha} |\varepsilon_{j,\alpha}|$. This can be done, for instance, in Spline spaces.
- The sampling functions $S_{j,\alpha}^\varepsilon$ are impossible to determine: They depend on the error sequence. However, a frame algorithm can be implemented in $\ell_r^2(\mathbb{Z}^d)$ to approximate $f \in V_\Phi^2$ from the samples $\{(\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha})\}_{\alpha\in\mathbb{Z}^d,\, j=1,2,\dots,s}$

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References

The proofs of the results above exhibited can be found in the references below. See also references therein for previous and related works.

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