Analytical Sampling, Lagrange-Type Interpolation Series and de Branges Spaces

Paulo E. Fernández Moncada

(joint work with A. G. García and M. A. Hernández-Medina.)

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Outline

1. Motivation
2. The Hilbert space $\mathcal{H}_K$.
3. Sampling in $\mathcal{H}_K$.
4. Lagrange-type interpolation series in $\mathcal{H}_K$.
5. The space $\mathcal{H}_K$ as de Branges space.

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Whittaker-Shannon-Kotelnikov sampling theorem

We consider the classical Whittaker-Shannon-Kotelnikov sampling theorem in the Paley-Wiener spaces

$$PW_\pi = \Big\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq [-\pi, \pi] \Big\}$$

where $\widehat{f}$ stands for the Fourier transform. Any function f in PW_π can be written as

$$\begin{aligned} f(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(x) e^{izx} dx \\ &= \left\langle \frac{e^{izt}}{\sqrt{2\pi}}, \overline{\widehat{f}} \right\rangle_{L^2[-\pi,\pi]} \end{aligned}$$

with $\widehat{f} \in L^2[-\pi, \pi]$ and the Fourier kernel (denoted K) is given by

$$\begin{array}{rccc} K & : & \mathbb{C} & \longrightarrow & L^2[-\pi,\pi] \\ & & z & \longrightarrow & K(z) \end{array}, \qquad [K(z)](x) = \frac{e^{izx}}{\sqrt{2\pi}}$$

Whittaker-Shannon-Kotelnikov sampling theorem

Whittaker-Shannon-Kotelnikov sampling theorem.

Any function f in the Paley-Wiener space PW_π can be recovered from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ as the cardinal series

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}$$

where the convergence in the series is absolute and uniform on horizontal strips of $\mathbb{C}$ since $\|K(z)\|_{L^2[-\pi,\pi]} \leq e^{\pi|y|}$ for all $z = x + iy \in \mathbb{C}$.

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The Hilbert space $\mathcal{H}_K$.

Given a complex, separable Hilbert space $\mathbb{H}$ and an kernel

$$\begin{array}{cccccc} K & : & \mathbb{C} & \longrightarrow & \mathbb{H} \\ & & z & \mapsto & K(z) \end{array}$$

we define a mapping between $\mathbb{H}$ and the set $\mathcal{F}(\mathbb{C},\mathbb{C}) := \{f : \mathbb{C} \longrightarrow \mathbb{C}\}$ as follows:

$$\begin{array}{cccccc} \mathcal{T}_K & : & \mathbb{H} & \longrightarrow & \mathcal{F}(\mathbb{C},\mathbb{C}) \\ & & x & \longrightarrow & f_x \end{array}$$

such that

$$f_x(z) = \langle K(z), x \rangle_{\mathbb{H}} \quad z \in \mathbb{C}.$$

and denote by $\mathcal{H}_K$ the linear space of all functions $f_x(z)$ in the range space of $\mathcal{T}_K$; i.e.,

$$\mathcal{T}_K(\mathbb{H}) = \mathcal{H}_K = \Big\{ f : \mathbb{C} \longrightarrow \mathbb{C} : f(z) = \langle K(z), x \rangle_{\mathbb{H}}, \ x \in \mathbb{H} \Big\}.$$

Some properties of $\mathcal{H}_K$.

- The space $\mathcal{H}_K$ endowed with the norm

$$\|f\|_{\mathcal{H}_K} := \inf\left\{\|x\|_{\mathbb{H}} :\ f = \mathcal{T}_K x\right\}.$$

  becomes a Hilbert Space.

- The mapping $\mathcal{T}_K$ is a bijective isometry from $\mathbb{H}$ to $\mathcal{H}_K$ if and only if $\{K(z) :\ z \in \mathbb{C}\}$ is complete in $\mathbb{H}$ or equivalently if and only if $\mathcal{T}_K$ is injective.

  In particular, if there exist $\{z_n\}_{n=1}^\infty$ in $\mathbb{C}$ such that $\{K(z_n)\}_{n=1}^\infty$ is a basis in $\mathbb{H}$, then $\mathcal{T}_K$ is an antilinear isometry from $\mathbb{H}$ onto $\mathcal{H}_K$.

Some properties of $\mathcal{H}_K$.

- $\mathcal{H}_K$ is an Hilbert space of reproducing kernel (RKHS in short); i.e., the evaluation functionals

$$\begin{array}{rcccc} E_z & : & \mathcal{H}_K & \rightarrow & \mathbb{C} \\ & & f & \rightarrow & f(z) \end{array}$$

  are bounded. For fixed $z \in \mathbb{C}$, for any $f \in \mathcal{H}_K$, since $f(z) = \langle K(z), x \rangle_{\mathbb{H}}$ $x \in \mathbb{H}$, using the Cauchy-Schwarz inequality we obtain

$$|f(z)| \leq \|K(z)\|_{\mathbb{H}} \|x\|_{\mathbb{H}} = C_z \|f\|_{\mathcal{H}_K}$$

- As a consequence, convergence in the norm $\|\cdot\|_{\mathcal{H}_K}$ implies pointwise convergence which will be uniform on subsets of $\mathbb{C}$ where $\|K(\cdot)\|_{\mathbb{H}}$ is bounded.
- The reproducing kernel of $\mathcal{H}_K$ is

$$\kappa(z, \omega) = \langle K(z), K(\omega) \rangle_{\mathbb{H}}$$

  which verifies the reproducing property

$$f(\omega) = \langle f(\cdot), \kappa(\cdot, \omega) \rangle_{\mathcal{H}} \text{ for each } \omega \in \mathbb{C} \text{ and } f \in \mathcal{H}$$

Analyticity of the functions in $\mathcal{H}_K$.

Theorem

$\mathcal{H}_K$ is a RKHS of entire functions if and only if the kernel K is analytic in $\mathbb{C}$.

Characterization of the analyticity of the functions in $\mathcal{H}_K$ in terms of Riesz bases.

- A Riesz basis for $\mathbb{H}$ a separable Hilbert space is a sequence of the form $\{Ue_n\}_{n=1}^{\infty}$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\mathbb{H}$ and $U: \mathbb{H} \to \mathbb{H}$ is a bounded biyective operator.
- If $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis for $\mathbb{H}$, there exists a unique sequence $\{x_n^*\}_{n=1}^{\infty}$ in $\mathbb{H}$ such that

$$x = \sum_{n=1}^{\infty} \langle x, x_n^* \rangle_{\mathbb{H}} \, x_n = \sum_{n=1}^{\infty} \langle x, x_n \rangle_{\mathbb{H}} \, x_n^*, \quad x \in \mathbb{H}.$$

  $\{x_n^*\}_{n=1}^{\infty}$ is also Riesz basis (called the dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$) and these series converges unconditionally for each x in $\mathbb{H}$.
- $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ are biorthogonal bases, i.e, $\langle x_n, x_m^* \rangle_{\mathbb{H}} = \delta_{n,m}$.

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Analyticity of the functions in $\mathcal{H}_K$.

Suppose that a Riesz basis $\{x_n\}_{n=1}^{\infty}$ is given and let $\{x_n^*\}_{n=1}^{\infty}$ be its dual Riesz basis. Expanding $K(z)$ for $z \in \mathbb{C}$ fixed with respect to this basis we obtain

$$K(z) = \sum_{n=1}^{\infty} \langle K(z), x_n^* \rangle_{\mathbb{H}} \, x_n$$

where the sequence of coefficients

$$S_n(z) := \langle K(z), x_n^* \rangle_{\mathbb{H}}$$

as functions in z are in $\mathcal{H}_K$. The following result holds

Theorem

$\mathcal{H}_K$ is a RKHS of entire functions if and only if the functions $\{S_n\}_{n=1}^{\infty}$ are entire and the function $z \mapsto \|K(z)\|_{\mathbb{H}}$ is bounded on compact sets of $\mathbb{C}$.

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Sampling in $\mathcal{H}_K$.

Definition

An analytic kernel $K : \mathbb{C} \longrightarrow \mathbb{H}$ is said to be an **analytic Kramer kernel** if there are sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$, $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$ and a Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathbb{H}$, such that

$$K(z_n) = a_n x_n \quad \forall n \in \mathbb{N},$$

Analytic Kramer sampling theorem.

Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic Kramer kernel as in above definition and $\mathcal{H}_K$ its corresponding RKHS of entire functions.

Then, any $f \in \mathcal{H}_K$ can be recovered from its samples $\{f(z_n)\}_{n=1}^{\infty}$ by means of the sampling series

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C}.$$

This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$

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Lagrange-type interpolation series

In the Whittaker-Shannon-Kotelnikov sampling formula for each f in the Paley-Wiener space PW_π, and $z \in \mathbb{C}$,

$$f(z) = \sum_{n\in\mathbb{Z}} f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} = \sum_{n\in\mathbb{Z}} f(n) \frac{G(z)}{(z-n)G'(n)}, \quad \text{where} \ \ G(z) = \frac{\sin \pi z}{\pi}$$

Problem

In the Analytic Kramer sampling theorem, a more difficult question concerns whether the sampling expansion

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C},$$

in $\mathcal{H}_K$ (K an analytic Kramer kernel), can be written as a Lagrange-type interpolation series.

A necessary and sufficient condition involves the following algebraic property:

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Definition (Zero removing property)

A space $\mathcal{H}$ of entire functions has the zero-removing property (ZR in short) if for any $g \in \mathcal{H}$ and any zero ω of g the function $\frac{g(z)}{z-\omega}$ belongs to $\mathcal{H}$.

Theorem (Lagrange-type interpolation series)

Let $\mathcal{H}_K$ be a RKHS of entire functions obtained from an analytic Kramer kernel K with respect to the sequences $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{C}$ and $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{C}\setminus\{0\}$, i.e., for some Riesz basis $\{x_n\}_{n=1}^{\infty}$ for $\mathbb{H}$, $K(z_n) = a_n x_n, \quad n \in \mathbb{N}$.

Then, the sampling formula $f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z)$, $z \in \mathbb{C}$, for $\mathcal{H}_K$ can be written as a Lagrange-type interpolation series

$$f(z) = \sum_{n=1}^{\infty} f(z_n) \frac{Q(z)}{Q'(z_n)(z-z_n)}.$$

where Q denotes an entire function having only simple zeros at $\{z_n\}_{n=1}^{\infty}$, if and only if the space $\mathcal{H}_K$ satisfies the ZR property.

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Lagrange-type interpolation series. (Examples)

Note: The entire function Q is such that $(z - z_n)S_n(z) = \sigma_n Q(z)$ for some nonzero constants σ_n $n \in \mathbb{N}$

Example 1. (The entire functions in the Pólya class.)

The entire function $F(z)$ is said to be of Pólya class if:

- It has no zeros in the upper half-plane.
- $|F(x - iy)| \leq |F(x + iy)|$, for $y > 0$.
- $|F(x + iy)|$ is a nondecreasing function of $y > 0$, for each fixed x.

Example 2. (The Paley-Wiener class.)

The Paley-Wiener class PW_π :

$$PW_\pi = \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \text{supp } \widehat{f} \subseteq [-\pi, \pi] \right\}$$

satisfy the ZR property. Using the classical Paley-Wiener theorem, the space PW_π also is expressable as

$$PW_\pi = \left\{ f \text{ entire function} : \quad |f(z)| \leq Ae^{\pi|z|}, \quad f\,|_{\mathbb{R}} \in L^2(\mathbb{R}) \right\}$$

From this characterization the ZR property inmediately comes out

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Lagrange-type interpolation series. (Examples)

Example 3.

Let $K : \mathbb{C} \longrightarrow \mathbb{H}$ be an analytic kernel such that $K(z_0) = 0$ for some $z_0 \in \mathbb{C}$. Then all the functions in the associated space $\mathcal{H}_K$ have a zero at z_0 and the ZR property does not hold in $\mathcal{H}_K$. Let f be a nonzero entire function in $\mathcal{H}_K$ and let r denote the order of its zero z_0. The function

$$\frac{f(z)}{(z - z_0)^r}$$

is not in $\mathcal{H}_K$.

Example 4. Consider $\mathbb{H} = L^2[-\pi, \pi]$ and $K : \mathbb{C} \longrightarrow L^2[-\pi, \pi]$ be the analytic Kramer kernel defined by:

$$[K(z)](t) := \frac{e^{iz^2 t}}{\sqrt{2\pi}}$$

Its Taylor series around $z = 0$ is given by:

$$[K(z)](t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} z^{2n}$$

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$$[K(z)](t) := \frac{e^{iz^2 t}}{\sqrt{2\pi}}$$

Its Taylor series around $z = 0$ is given by:

$$[K(z)](t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} z^{2n}$$

Lagrange-type interpolation series. (Examples)

The Taylor series for any function $f(z) = \langle K(z), F \rangle_{L^2[-\pi,\pi]}$ in $\mathcal{H}_K$ where $F \in L^2[-\pi, \pi]$ is of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{\langle (it)^n, F \rangle}{n!} z^{2n}, \quad z \in \mathbb{C}.$$

f is an even function.

However, there is a function $g \in L^2[-\pi, \pi]$ such that $g(0) = 0$. Therefore,

$$\frac{g(z)}{z} = \sum_{n=0}^{\infty} \frac{\langle (it)^n, G \rangle}{n!} z^{2n-1}$$

and clearly, $\frac{g(z)}{z} \notin \mathcal{H}_K$ does not belong to $\mathcal{H}_K$.

Outline

1 Motivation

2 The Hilbert space $\mathcal{H}_K$.

3 Sampling in $\mathcal{H}_K$.

4 Lagrange-type interpolation series in $\mathcal{H}_K$.

5 The space $\mathcal{H}_K$ as de Branges space.

The space $\mathcal{H}_K$ as de Branges space

The Paley-Wiener spaces can be seen as special cases of a more general class of Hilbert spaces of entire functions: **The de Branges spaces**:

Definition

Let E be an entire function verifiying $|E(\bar{z})| < |E(z)|, \quad \text{Im}(z) > 0$. The de Branges space $\mathcal{H}(E)$ is the set of all entire functions f such that

$$\|f\|^2_{\mathcal{H}(E)} = \int_{-\infty}^{+\infty} \left| \frac{f(x)}{E(x)} \right|^2 dx < \infty$$

and such that both ratios $\frac{f(z)}{E(z)}$ and $\frac{\overline{f(\bar{z})}}{E(z)}$ are of **bounded type** and **nonpositive mean type** in $\mathbb{C}^+ := \{z : \text{Im}(z) > 0\}$.

- $h(z)$ is of **bounded type** if it can be written as a quotient of two bounded analytic functions in $\mathbb{C}^+$.
- $h(z)$ is of **nonpositive mean type** if it grows no fasther than $e^{\epsilon y}$ for each $\epsilon > 0$ on the possitive imaginary axis $\{iy : y > 0\}$

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Some properties of the de Branges spaces

- Any de Branges function E can be written as $E(z) = A(z) - iB(z)$ where A and B are entire functions which are real when z is real, given by

$$A(z) = \tfrac{1}{2}(E(z) + \overline{E(\overline{z})}), \quad B(z) = \tfrac{i}{2}(E(z) - \overline{E(\overline{z})})$$

  and the functions $A(z)$ and $B(z)$ have only real zeros and these zeros interlace.

- $\mathcal{H}(E)$ is a RKHS. The reproducing kernel is

$$\kappa(\omega, z) := \frac{\overline{A(\omega)}B(z) - A(z)\overline{B(\omega)}}{\pi(z - \overline{\omega})}, \quad z, \omega \in \mathbb{C}$$

  This kernel has the property that for each $f(z) \in \mathcal{H}(E)$, there holds

$$f(\omega) = \langle f(\cdot), \kappa(\omega, \cdot)\rangle_{\mathcal{H}(E)} \qquad \text{for all } \omega \in \mathbb{C}$$

- If E is a strict de Branges function, then de Branges space $\mathcal{H}(E)$ **satisfies the ZR property**.

Sampling in $\mathcal{H}(E)$

The existence of orthogonal sequences in $\mathcal{H}(E)$ is conditioned by so-called phase functions, which implies a sampling formula in this space.

Definition

The continuos function $\varphi(x)$ of real x is said be a phase function associated with $E(z)$ if $E(x)e^{i\varphi(x)}$ is real-valued for all $x \in \mathbb{R}$.

If α is a given real number such that the function $e^{i\alpha}E(z) - e^{-i\alpha}\overline{E(\overline{z})}$ does not belong to $\mathcal{H}(E)$, then the sequence of real numbers $\{t_n\}$ satisfying $\varphi(t_n) = \alpha \mod \pi$ gives an orthogonal basis $\{\kappa(t_n, \cdot)\}$ for $\mathcal{H}(E)$.

Consequently, the following result holds:

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Consequently, the following result holds:

Sampling in $\mathcal{H}(E)$ (Theorem and Example)

Theorem.

Let $\mathcal{H}(E)$ be de Branges space, $\{t_n\}$ a sequence of real numbers and $\{\kappa(t_n,\cdot)\}$ an orthogonal basis in $\mathcal{H}(E)$. Then, any function $f \in \mathcal{H}(E)$ can be recovered from its samples $\{f(t_n)\}$ through the sampling formula

$$f(z) = \sum_{n\in\mathbb{N}} f(t_n)\frac{\kappa(t_n,z)}{\kappa(t_n,t_n)} = \sum_{n\in\mathbb{N}} f(t_n)\frac{Q(z)}{(z-t_n)Q'(t_n)}, \quad z \in \mathbb{C}$$

Where Q is an entire function having only simple zeros at $\{t_n\}$. This series converges absolutely and uniformly on compact subsets of $\mathbb{C}$.

Note: The entire function Q is such that $(z-t_n)\kappa(t_n,z) = \sigma_n Q(z)$ for some nonzero constants σ_n $n \in \mathbb{N}$.

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Sampling in $\mathcal{H}(E)$. (Examples)

Example 1. The Paley-Wiener spaces PW_π corresponds to the de Branges space $\mathcal{H}(E)$ where the structure function is $E(z) = e^{-i\pi z}$; $A(z) = \cos(\pi z)$, $B(z) = \sin(\pi z)$ and the phase function is $\varphi(x) = \pi x$.

Example 2. (Makarov and Poltoratski.)

For $\nu \geq 1/2$ consider the second order differential Bessel equation:

$$-u'' + \left(\frac{\nu^2 - 1/4}{t^2}\right) u = zu, \quad 0 < t < 1, \tag{1}$$

and the boundary condition which is satisfied by the solution

$$u_z(t) = \sqrt{t} J_\nu(t\sqrt{z}) \quad \text{of } (1).$$

Then:

- The associated Weyl inner function is

$$\Theta_\nu(z) = \frac{\sqrt{z} J'_\nu(\sqrt{z}) + (\frac{1}{2} + i) J_\nu(\sqrt{z})}{\sqrt{z} J'_\nu(\sqrt{z}) + (\frac{1}{2} - i) J_\nu(\sqrt{z})}$$

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Sampling in $\mathcal{H}(E)$ (Example)

Given an inner function Θ, we say that a strict de Branges function E is a de Branges function of Θ if

$$\Theta(z) = \frac{\overline{E(\overline{z})}}{E(z)}$$

- There is an even real entire function $G_\nu(z)$ such that $J_\nu(z) = z^\nu G_\nu(z)$ and $G_\nu(0) \neq 0$.
- The function $F_\nu(z) = zG'_\nu(z)$ is an even real entire function. Therefore,

$$\Theta_\nu(z) = \frac{F_\nu(\sqrt{z}) + (\frac{1}{2} + \nu + i)G_\nu(\sqrt{z})}{F_\nu(\sqrt{z}) + (\frac{1}{2} + \nu - i)G_\nu(\sqrt{z})}$$

- The function $E_\nu(z) := F_\nu(\sqrt{z}) + (\frac{1}{2} + \nu - i)G_\nu(\sqrt{z})$ is a de Branges function of Θ_ν, which defines a de Branges space $\mathcal{H}(E_\nu)$.

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Sampling in $\mathcal{H}(E)$ (Example)

We assume that $\nu = 1/2$. In this case, in the de Branges space $\mathcal{H}(E_{1/2})$ are obtained:

- $J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z$
- $G_{1/2}(z) = z^{-1/2} J_{1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{\sin z}{z}$.
- $F_{1/2}(z) = z{G'}_{1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{z \cos z - \sin z}{z}$.

Finally, assuming that $E_{1/2}(z) = A_{1/2}(z) - iB_{1/2}(z)$, in our case,

$$A_{1/2}(z) = F_{1/2}(\sqrt{z}) + G_{1/2}(\sqrt{z}), \quad B_{1/2}(z) = G_{1/2}(\sqrt{z})$$

The phase function for the space $\mathcal{H}(E_{1/2})$ is given by

$$\phi(x) = -\arctan \frac{-G_{1/2}(\sqrt{x})}{F_{1/2}(\sqrt{x}) + G_{1/2}(\sqrt{x})}$$

Sampling in $\mathcal{H}(E)$ (Example)

- For a given real number α, the sequence $\{t_n^\alpha\}$ should verify $\phi(t_n^\alpha) = \alpha \mod \pi$.

For this sequences $\{t_n^\alpha\}$, the sequence $\{\kappa(t_n^\alpha, z)\}$ is an orthogonal basis for $\mathcal{H}(E_{1/2})$ if and only if the function $e^{i\alpha}E_{1/2}(z) - e^{-i\alpha}\overline{E_{1/2}(\overline{z})}$ does not belong to $\mathcal{H}(E_{1/2})$.

This occurs for instance if $\alpha = 0$: The points $t_n^0 = n^2\pi^2 \quad n \in \mathbb{N}$, are the zeros of the function $B_{1/2}(z)$.

Then, for each f in $\mathcal{H}(E_{1/2})$ the following sampling formula holds

$$f(z) = \sum_{n=1}^{\infty} f(n^2\pi^2)\frac{\kappa(n^2\pi^2, z)}{\kappa(n^2\pi^2, n^2\pi^2)} = \sum_{n=1}^{\infty} f(n^2\pi^2)\frac{2(-1)^n n^2\pi^2 \sin\sqrt{z}}{(z - n^2\pi^2)\sqrt{z}}, \quad z \in \mathbb{C}.$$

i.e.,

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The spaces $\mathcal{H}_K$ as de Branges spaces

Theorem (Characterization of $\mathcal{H}_K$ as a de Branges space.)

A space $\mathcal{H}_K$ is a de Branges space if and only if there exists an orthogonal sampling formula

$$f(z) = \sum_{n=1}^{\infty} \frac{f(z_n)}{a_n} S_n(z), \quad z \in \mathbb{C}.$$

in $\mathcal{H}_K$ such that it can be written as a Lagrange-type interpolation formula

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The spaces $\mathcal{H}_K$ as de Branges spaces.

Example.

We consider the kernel $[K(z)](n) = P_n,\ n \in \mathbb{N}_0$ where $\{P_n\}_{n=0}^{\infty}$ is the sequence of orthonormal polynomials associated to an indeterminate Hamburger moment problem.

It is known that this kernel defines an analytic kramer kernel in $\ell^2(\mathbb{N}_0)$ and in the corresponding space

$$\mathcal{H}_K := \left\{ f(z) = \sum_{n=0}^{\infty} a_n P_n(z),\ \ z \in \mathbb{C},\ \{a_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N}_0) \right\}$$

an orthogonal sampling formula holds. As a consequence, using the above theorem, $\mathcal{H}_K$ is a de Branges space.

1. **"De Branges spaces, Analytic Kramer kernels and Lagrange-type interpolation series"**. Accepted in *Complex Variables and Elliptic Equations, 2011.*

2. **"The zero-removing property and Lagrange-type interpolation series"**. Accepted in *Num. Fun. Anal. and Optimin., 2011.*