Compactly supported reconstruction functions in average sampling: A matrix pencil approach

Alberto Portal
(joined work with A.G. García,
M.A. Hernández Medina
and G. Pérez Villalón)

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Sampling in a shift-invariant space

$$f \in V_\varphi^2 := \Big\{ \sum_{n\in\mathbb{Z}} a_n \varphi(t-n) : \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R}) \,.$$

↓

Riesz basis

$$\mathcal{L}f := f * h$$

$$\{\mathcal{L}f(n)\}_{n\in\mathbb{Z}} \longrightarrow f(t) = \sum_{n\in\mathbb{Z}} \mathcal{L}f(n) S_{\mathcal{L}}(t-n)$$

Oversampling

$$\{\mathcal{L}f(n)\}_{n\in\mathbb{Z}} \xrightarrow{\;r<s\;} \left\{\mathcal{L}f\left(\frac{r}{s}n\right)\right\}_{n\in\mathbb{Z}}$$

$$\mathcal{L}_j f(t) := \mathcal{L}f\left(t + \tfrac{r}{s}(j-1)\right) \qquad (1 \le j \le s)\,.$$

$$f(t) = \sum_{j=1}^{s} \sum_{n\in\mathbb{Z}} \mathcal{L}_j f(rn) S_j(t - rn)$$

Hypotheses

$\varphi, \mathcal{L}\varphi$ compactly supported.

$$g_j(\omega) := \sum_{n\in\mathbb{Z}} \mathcal{L}_j\phi(n)e^{-2\pi i n\omega} \in L^\infty(0,1)\,, \quad j = 1, 2, \ldots, s$$

$$\mathbf{G}(\omega) := \begin{pmatrix} g_1(\omega) & g_1(\omega+\frac{1}{s}) & \cdots & g_1(\omega+\frac{r-1}{s}) \\ \vdots & \vdots & \ddots & \vdots \\ g_s(\omega) & g_s(\omega+\frac{1}{s}) & \cdots & g_s(\omega+\frac{r-1}{s}) \end{pmatrix}$$

$s \times r$ Trigonometric polynomials

An equivalent problem

Find an $r \times s$ matrix $\mathbf{A}(\omega)$ whose entries are trigonometric polynomials and such that

$$\mathbf{A}(\omega)\mathbf{G}(\omega) = \mathbf{I}_r \,, \qquad \omega \in [0,1] \,.$$

1. A. G. García and G. Pérez-Villalón. Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces. *Appl. Comput. Harmon. Anal.*, 20(3):422–433, 2006.
2. A. G. García, M. A. Hernández-Medina and G. Pérez-Villalón. Oversampling and reconstruction functions with compact support. *J. Comp. Appl. Math.*, 227:245–253, 2009.

Reason

$$\mathbf{A}_{[1,:]}(\omega) =: \big(\mathbf{a}_1(\omega), \mathbf{a}_2(\omega), \ldots, \mathbf{a}_s(\omega)\big)$$

$$S_j(t) = r \sum_{n \in \mathbb{Z}} \widehat{\mathbf{a}}_j(n) \varphi(t-n)\,, \quad (t \in \mathbb{R})\,,$$

$$\widehat{\mathbf{a}}_j(n) = \int_0^1 \mathbf{a}_j(\omega) e^{-2\pi i n \omega} d\omega\,, \quad (1 \le j \le s).$$

Existence of $\mathbf{A}(\omega)$

There exists a solution

$$\Updownarrow$$

$$\operatorname{rank} \mathbf{G}(\omega) = r \quad \forall \omega \in \mathbb{R}$$

Towards a matrix pencil

We preserve the rank

Same rank

$$\mathbf{G}(\omega) \rightsquigarrow \mathsf{G}(z) \rightsquigarrow \mathbb{G}(z) \rightsquigarrow \widehat{\mathbb{G}}(z) \rightsquigarrow \widetilde{\mathbb{G}}(z) \rightsquigarrow \mathcal{A}^\top - \lambda \mathcal{B}^\top$$

$$\mathbf{G}(\omega) = \mathsf{G}(e^{-2\pi i \omega})$$

$$\widehat{\mathbb{G}}(z) = \mathbb{G}(z)\Omega_r^{-1}$$

$$\lambda = z^r$$
$$\operatorname{supp}(\mathcal{L}\varphi) \subseteq [0, N]$$
$$1 < N \le r$$

$$\mathbb{G}(z) = \mathsf{G}(z)\,\operatorname{diag}[(W^{j-1}z)^{r-1}]_j$$

$$\widetilde{\mathbb{G}}(z) = \widehat{\mathbb{G}}(z)\,\operatorname{diag}[z^{1-j}]_j$$

Remark

$$\widetilde{\mathbb{G}}(\lambda) = \begin{pmatrix} \mathbb{M}(\lambda) & \mathcal{G} \end{pmatrix}$$

$$\operatorname{rank} \mathcal{G} = r - N + 1 \quad \Longrightarrow \exists R \in \mathbb{C}^{s \times s} \mid R\mathcal{G} = \begin{pmatrix} \mathcal{G}' \\ \mathbf{0} \end{pmatrix}$$

$$R\widetilde{\mathbb{G}}(\lambda) = \begin{pmatrix} M_1 - \lambda N_1 & \mathcal{G}' \\ M_2 - \lambda N_2 & \mathbf{0} \end{pmatrix}$$

The main result

Existence of solution

Theorem

$\operatorname{rank}(\mathsf{G}(z)) = r$ for all $z \in \mathbb{C} \setminus \{0\}$ if and only if the following statements hold

1. $\operatorname{rank} \mathcal{G} = r - N + 1$.
2. The *Kronecker Canonical Form* (KCF) of the matrix pencil $M_2 - \lambda N_2$ has not right singular part and the only possible finite eigenvalue is $\mu = 0$.

Using the GUPTRI form of a matrix pencil

Advantages

1. The needed information about $M_2 - \lambda N_2$ can be retrieved.
2. We do not need to compute the KCF of the matrix pencil.
3. The GUPTRI (General UPper TRIangular) form can be stably computed. (J. Demmel and B. Kågström, 1993; P. Van Dooren, 1979 and 1981).

Looking for a solution

We use the matrix pencil

$$\widetilde{\mathbb{L}}(\lambda)(\mathcal{A}^\top - \lambda\mathcal{B}^\top) = \mathbf{I}_r \iff (\mathcal{A} - \lambda\mathcal{B})\widetilde{\mathbb{L}}(\lambda)^\top = \mathbf{I}_r$$

$$\left[\widetilde{\mathbb{L}}(\lambda)^\top\right]_j = \mathrm{L}_j^0 + \mathrm{L}_j^1\lambda^1 + \cdots + \mathrm{L}_j^\nu\lambda^\nu = \sum_{k=0}^{\nu} \mathrm{L}_j^k\lambda^k$$

$$\mathcal{A}\mathrm{L}_j^0 + \sum_{k=1}^{\nu}\left(\mathcal{A}\mathrm{L}_j^{p+k} - \mathcal{B}\mathrm{L}_j^{k-1}\right)\lambda^k - \mathcal{B}\mathrm{L}_j^\nu\lambda^{\nu+1} = I_r^j\,,$$

Looking for a solution

We should solve several tied linear systems

$$\left.\begin{array}{rcl} \mathcal{A}\mathrm{L}_j^0 & = & I_r^j \\ \mathcal{A}\mathrm{L}_j^1 & = & \mathcal{B}\mathrm{L}_j^0 \\ \vdots & \vdots & \vdots \\ \mathcal{A}\mathrm{L}_j^k & = & \mathcal{B}\mathrm{L}_j^{k-1} \\ \vdots & \vdots & \vdots \\ \mathcal{A}\mathrm{L}_j^\nu & = & \mathcal{B}\mathrm{L}_j^{\nu-1} \\ \mathcal{B}\mathrm{L}_j^\nu & = & \mathbf{0} \end{array}\right\}$$

What is the minimum ν which allows us to solve these tied systems at once?

An equivalent system

$$\underbrace{\begin{pmatrix} -\mathcal{B} & & & & \\ \mathcal{A} & -\mathcal{B} & & & \\ & \mathcal{A} & \ddots & & \\ & & \ddots & -\mathcal{B} & \\ & & & \mathcal{A} & -\mathcal{B} \\ & & & & \mathcal{A} \end{pmatrix}}_{\substack{\mathsf{G}_r(\nu) \\ (\nu+1 \text{ column blocks})}} \begin{pmatrix} \mathrm{L}_j^{\nu} \\ \mathrm{L}_j^{\nu-1} \\ \vdots \\ \mathrm{L}_j^{1} \\ \mathrm{L}_j^{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ I_r^j \end{pmatrix}$$

$$j = 1, 2, \ldots, r$$

Looking for a solution

Theorem

Suppose that the $(r+1) \times r$-singular matrix pencil $\mathcal{A}^\top - \lambda \mathcal{B}^\top$ satisfies the following conditions:

- It has no finite eigenvalues.
- $\operatorname{rank}(\mathcal{A}^\top) = r$.
- $\operatorname{rank}(\mathcal{B}^\top) = N - 1$.
- $\operatorname{rank}\left[\begin{pmatrix} -\mathcal{B} & \mathbf{0} \\ \mathcal{A} & -\mathcal{B} \end{pmatrix}\right] = r + N - 1$.

Then, the $Nr \times (N-1)(r+1)$-matrix $\mathsf{G}_r(N-2)$ has maximum rank, i.e., $(N-1)(r+1)$.

Optimality problem

- We have found a matrix algebraic polynomial with $N - 1$ terms as solution of the problem.
- The number of terms of the solution is intimately related to the support of the reconstruction functions.
- What is the minimum number of nonzero terms that a solution could have? We consider two options:
    - We have some null coefficients.
    - The process can be finished for $\nu < N - 2$.

We consider Laurent polynomials

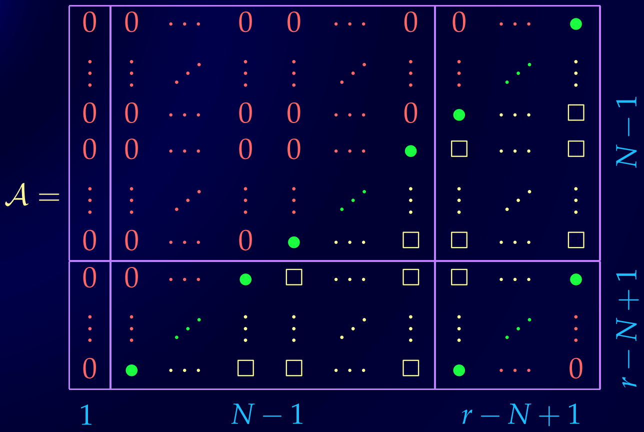
The identity matrix changes its place

$$\begin{pmatrix} -\mathcal{B} & & & & & \\ \mathcal{A} & -\mathcal{B} & & & & \\ & \mathcal{A} & \ddots & & & \\ & & \ddots & -\mathcal{B} & & \\ & & & \mathcal{A} & -\mathcal{B} \\ & & & & \mathcal{A} \end{pmatrix} \begin{pmatrix} \mathrm{L}_j^{p+\nu} \\ \mathrm{L}_j^{p+\nu-1} \\ \vdots \\ \mathrm{L}_j^{p+1} \\ \mathrm{L}_j^{p} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathcal{I}_{p+\nu}^j \\ \vdots \\ \mathcal{I}_{p+1}^j \\ \mathcal{I}_p^j \end{pmatrix}$$

$$0 \leq -p \leq \nu$$

Hypotheses that $\mathcal{A}$ verifies

Sufficient condition



Hypotheses that $\mathcal{B}$ verifies

Sufficient condition

$$\mathcal{B} = \left[\begin{array}{c|cccccc|ccc}
\square & \square & \cdots & \square & \square & \square & \bullet & 0 & \cdots & 0 \\
\square & \square & \cdots & \vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\
\square & \square & \cdots & \square & \bullet & \cdots & 0 & 0 & \cdots & 0 \\
\square & \square & \cdots & \bullet & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\
\square & \bullet & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots & \vdots & \iddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]
\begin{array}{l} N-1 \\ \\ r-N+1 \end{array}$$

$$\underbrace{}_{1} \qquad \underbrace{}_{N-1} \qquad \underbrace{}_{r-N+1}$$

Conclusions

- If $\nu < N - 2$, there exists no solution.
- If $\nu = N - 2$, there exists a unique solution.
- If $\nu > N - 2$, there exist several solutions.
- Moreover, if there exists any solution, it has at least $N - 1$ nonzero consecutive terms.

A toy model involving the quadratic B-spline

The generator

$$N_3(t) = \chi_{[0,1)} * \chi_{[0,1)} * \chi_{[0,1)}$$

$$\varphi(t) = N_3(t) = \frac{t^2}{2}\chi_{[0,1)}(t) + \left(-\frac{3}{2} + 3t - t^2\right)\chi_{[1,2)}(t) + \frac{1}{2}(3-t)^2\chi_{[2,3)}(t)\,.$$

$$\mathcal{L}f = f \qquad ; \qquad \operatorname{supp}\mathcal{L}\varphi \subseteq [0,3] \qquad (N = 3)$$

$$\text{Samples:} \quad f\left(\frac{4}{5}n\right) \qquad (r = 4, s = 5)$$

A toy model involving the quadratic B-spline

The functions $g_j(z)$

$$
\begin{aligned}
g_1(z) &= \tfrac{1}{2}z + \tfrac{1}{2}z^2 \\
g_2(z) &= \tfrac{8}{25} + \tfrac{33}{50}z + \tfrac{1}{50}z^2 \\
g_3(z) &= \tfrac{9}{50}z^{-1} + \tfrac{37}{50} + \tfrac{2}{25}z \\
g_4(z) &= \tfrac{2}{25}z^{-2} + \tfrac{37}{50}z^{-1} + \tfrac{9}{50} \\
g_5(z) &= \tfrac{1}{50}z^{-3} + \tfrac{33}{50}z^{-2} + \tfrac{8}{25}z^{-1}
\end{aligned}
$$

A toy model involving the quadratic B-spline

The matrix $\widehat{\mathbb{G}}(z)$

$$\widehat{\mathbb{G}}(z) = \begin{pmatrix} \frac{1}{2}z^4 & \frac{1}{2}z^5 & 0 & 0 \\ \frac{33}{50}z^4 & \frac{1}{50}z^5 & 0 & \frac{8}{25}z^3 \\ \frac{2}{25}z^4 & 0 & \frac{9}{50}z^2 & \frac{37}{50}z^3 \\ 0 & \frac{2}{25}z & \frac{37}{50}z^2 & \frac{9}{50}z^3 \\ \frac{1}{50} & \frac{33}{50}z & \frac{8}{25}z^2 & 0 \end{pmatrix}$$

A toy model involving the quadratic B-spline

The matrix $\widetilde{\mathbb{G}}(z)$

$$\widetilde{\mathbb{G}}(z) = \widehat{\mathbb{G}}(z)\,\mathrm{diag}\left[1, z^{-1}, z^{-2}, z^{-3}\right]$$

$$\widetilde{\mathbb{G}}(z) = \begin{pmatrix} \frac{1}{2}z^4 & \frac{1}{2}z^4 & 0 & 0 \\ \frac{33}{50}z^4 & \frac{1}{50}z^4 & 0 & \frac{8}{25} \\ \frac{2}{25}z^4 & 0 & \frac{9}{50} & \frac{37}{50} \\ 0 & \frac{2}{25} & \frac{37}{50} & \frac{9}{50} \\ \frac{1}{50} & \frac{33}{50} & \frac{8}{25} & 0 \end{pmatrix}$$

A toy model involving the quadratic B-spline

The matrix pencil

$$\lambda = z^4$$

$$\mathcal{A}^\top - \lambda \mathcal{B}^\top = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{8}{25} \\ 0 & 0 & \frac{9}{50} & \frac{37}{50} \\ 0 & \frac{2}{25} & \frac{37}{50} & \frac{9}{50} \\ \frac{1}{50} & \frac{33}{50} & \frac{8}{25} & 0 \end{pmatrix} - \lambda \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{-33}{50} & \frac{-1}{50} & 0 & 0 \\ \frac{-2}{25} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A toy model involving the quadratic B-spline

Solving the linear systems

$$\begin{pmatrix} -\mathcal{B} & \mathbf{0} \\ \mathcal{A} & -\mathcal{B} \\ \mathbf{0} & \mathcal{A} \end{pmatrix} \begin{pmatrix} \ell_j^1 \\ \ell_j^0 \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{I}_4^j \end{pmatrix}, \quad j = 1, 2, 3, 4$$

A toy model involving the quadratic B-spline

The left-inverse matrix

$$\widetilde{\mathbb{L}}(\lambda)^\top = \begin{pmatrix} \frac{123034939}{27456} & \frac{-125425}{36} & \frac{28925}{18} & \frac{-825}{2} & 50 \\ \frac{-3949115}{27456} & \frac{4025}{36} & \frac{-925}{18} & \frac{25}{2} & 0 \\ \frac{227683}{13728} & \frac{-925}{72} & \frac{50}{9} & 0 & 0 \\ \frac{-19483}{4576} & \frac{25}{8} & 0 & 0 & 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} \frac{-18911}{9152} & \frac{472775}{9152} & \frac{-472775}{1144} & \frac{16547125}{10296} & \frac{-286974425}{82368} \\ \frac{607}{9152} & \frac{-15175}{9152} & \frac{15175}{1144} & \frac{-531125}{10296} & \frac{9211225}{82368} \\ \frac{-35}{4576} & \frac{875}{4576} & \frac{-875}{572} & \frac{30625}{5148} & \frac{-531125}{41184} \\ \frac{9}{4576} & \frac{-225}{4576} & \frac{225}{572} & \frac{-875}{572} & \frac{15175}{4576} \end{pmatrix}$$

A toy model involving the quadratic B-spline

Finally, the compact supported reconstruction functions

$$S_1(t) := \frac{123034939}{27456}\varphi(t) - \frac{3949115}{27456}\varphi(t+1) + \frac{227683}{13728}\varphi(t+2) - \frac{19483}{4576}\varphi(t+3) - \frac{18911}{9152}\varphi(t-4) + \frac{607}{9152}\varphi(t-3) - \frac{35}{4576}\varphi(t-2) + \frac{9}{4576}\varphi(t-1)$$

$$S_2(t) := -\frac{125425}{36}\varphi(t) + \frac{4025}{36}\varphi(t+1) - \frac{925}{72}\varphi(t+2) + \frac{25}{8}\varphi(t+3) + \frac{472775}{9152}\varphi(t-4) - \frac{15175}{9152}\varphi(t-3) + \frac{875}{4576}\varphi(t-2) - \frac{225}{4576}\varphi(t-1)$$

$$S_3(t) := \frac{28925}{18}\varphi(t) - \frac{925}{18}\varphi(t+1) + \frac{50}{9}\varphi(t+2) - \frac{472775}{1144}\varphi(t-4) + \frac{15175}{1144}\varphi(t-3) - \frac{875}{572}\varphi(t-2) + \frac{225}{572}\varphi(t-1)$$

$$S_4(t) := -\frac{825}{2}\varphi(t) + \frac{25}{2}\varphi(t+1) - \frac{16547125}{10296}\varphi(t-4) - \frac{531125}{10296}\varphi(t-3) + \frac{30625}{5148}\varphi(t-2) - \frac{875}{572}\varphi(t-1)$$

$$S_5(t) := 50\varphi(t) - \frac{286974425}{82368}\varphi(t-4) + \frac{9211225}{82368}\varphi(t-3) - \frac{531125}{41184}\varphi(t-2) + \frac{15175}{4576}\varphi(t-1)$$