Sampling, reconstruction and approximation in $L^2(\mathbb{R}^d)$ shift-invariant subspaces

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Contents

Claude E. Shannon (1916–2001)

- B. S. Engineering Mathematics, University of Michigan, 1936
  Ph. D. Mathematics, MIT, 1940
- Research Mathematician, Bell Labs, 1941–1972; MIT Faculty Member, 1956–1978; Donner Professor of Science 1958
- Major publication: **A Mathematical Theory of Communication**, Bell System Technical Report, 1948
- Honorary Degree, Univ. of Michigan, 1961; The National Medal of Science, 1966; The Audio Engineering Society Gold Medal, 1985; The Kyoto Prize, 1985

. . . *the american mathematician, computer scientist, communication engineer, and the founder of the field of* **Information Theory**, *whose work has laid the foundation for the telecommunication networks that lace the globe*

Classical Shannon's sampling theorem

THEOREM 1: *If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart.*

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Shannon sampling theorem

Any function f **band-limited** to the interval $[-1/2, 1/2]$, that is, $f(t) = \int_{-1/2}^{1/2} \widehat{f}(w) e^{2\pi i t w} dw$ for each $t \in \mathbb{R}$, may be reconstructed from the sequence of samples $\{f(n)\}_{n\in\mathbb{Z}}$ as

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In the original Shannon's theorem f is band-limited to $[-W, W]$

An easy proof due to Hardy

- For any f in the **Paley-Wiener space** $PW_{1/2}$, which is **reproducing kernel Hilbert space**, we have

$$f(t) = \langle \widehat{f}, e^{-2\pi i t w} \rangle_{L^2[-1/2,1/2]} \quad \left(\Rightarrow |f(t)| \leq \|f\|\,, \; t \in \mathbb{R} \right)$$

- Expanding $\widehat{f} \in L^2[-1/2, 1/2]$ with respect to the orthonormal basis $\{e^{-2\pi i n w}\}_{n \in \mathbb{Z}}$

$$\widehat{f} = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, e^{-2\pi i n w} \rangle_{L^2[-1/2,1/2]} e^{-2\pi i n w} = \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i n w}$$

- Applying the inverse Fourier transform $\mathcal{F}^{-1}$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \mathcal{F}^{-1}\left[e^{-2\pi i n w} \chi_{[-1/2,1/2]}(w) \right](t)$$

$$= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}\,, \quad t \in \mathbb{R}$$

The d-dimensional Shannon's formula

Any function f band-limited to the d-dimensional cube $[-1/2, 1/2]^d$, i.e.,

$$f(t) = \int_{[-1/2,1/2]^d} \widehat{f}(x) \mathrm{e}^{2\pi i x^\top t} dx \,, \quad t \in \mathbb{R}^d \,,$$

may be reconstructed from the sequence of samples $\{f(\alpha)\}_{\alpha \in \mathbb{Z}^d}$ as

$$f(t) = \sum_{\alpha \in \mathbb{Z}^d} f(\alpha) \frac{\sin \pi(t_1 - \alpha_1)}{\pi(t_1 - \alpha_1)} \cdots \frac{\sin \pi(t_d - \alpha_d)}{\pi(t_d - \alpha_d)} \,, \quad t \in \mathbb{R}^d \,.$$

A historical remark

- **E. T. Whittaker** (1915) and **J. M. Whittaker** (1935): Study of cardinal series
- **K. Ogura** (1920): *On a certain transcendental integral function in the theory of interpolation*
- **H. Nyquist** (1928): *Certain topics in telegraph transmission theory*
- **V. Kotel'nikov** (1933): *On the carrying capacity of the "ether" and wire in telecommunications*
- **H. Raabe** (1939): *Untersuchungen an der wechselzeitigen Mehrfachübertragung (Multiplexübertragung)*
- **C. E. Shannon** (1949): *Communication in the presence of noise*
- **I. Someya** (1949): *Waveform Transmission*

(See **Ferreira & Higgins** in *Notices of the AMS* (november 2011))

Drawbacks in Shannon's theory

- It relies on the use of **low-pass ideal filters**
- The band-limited hypothesis is in contradiction with the idea of a **finite duration signal**
- The band-limiting operation generates **Gibbs oscillations**
- The **sinc function** has a very **slow decay** at infinity which makes computation in the signal domain very inefficient
- The sinc function is well-localized in the frecuency domain but it is **bad-localized** in the time domain
- In several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

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Multiplying by the function $\chi_{[-1/2,1/2]}$ in the Fourier-domain is equivalent to convolve in the time-domain with the sinc function. Since this function does not vanish at $(-\infty, 0)$, the corresponding filter is not causal:

$$(f * \mathrm{sinc})(t) = \int_{-\infty}^{\infty} f(t-x)\,\mathrm{sinc}\,x\,dx\,, \quad t \in \mathbb{R}$$

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Drawbacks in Shannon's theory

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Every bandlimited function $f \in PW_{1/2}$ can be extended to the entire function

$$f(z) = \int_{-1/2}^{1/2} \widehat{f}(w) e^{2\pi i z w} dw\,, \quad z \in \mathbb{C}$$

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The numerical calculation of $f(1/2)$ from a noisy sequence of samples $\{f(n)+\varepsilon_n\}_{n\in\mathbb{Z}}$ gives an error $\left|\sum_{n=-\infty}^{\infty}\frac{(-1)^n\varepsilon_n}{\pi(\frac{1}{2}-n)}\right|$ which could be infinity even when $|\varepsilon_n|\leq\varepsilon$ for all $n\in\mathbb{Z}$

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$$\int_{-\infty}^{\infty} t^2 |\operatorname{sinc} t|^2 dt = +\infty$$

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Oversampling technique

Some one-dimensional drawbacks can be solved by using the **oversampling technique**: Assume that $f \in PW_{1/2}$ has

$$\operatorname{supp} \widehat{f} \subseteq [-\sigma/2, \sigma/2] \subset [-1/2, 1/2] \quad \text{for } \sigma < 1$$

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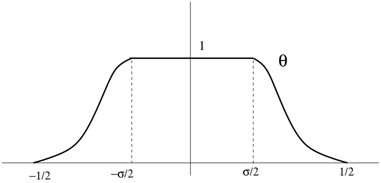
Let θ be an enough smooth function such that

$$\theta(w) := \begin{cases} 1 & w \in [-\sigma/2, \sigma/2] \\ 0 & w \notin [-1/2, 1/2] \end{cases}$$



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Then,

$$\widehat{f}(w) = \sum_n f(n)\theta(w)e^{-2\pi i n w} \quad \text{in } L^2[-1/2, 1/2]$$

$$\Downarrow$$

$$f(t) = \sum_n f(n) S(t-n)\,, \quad t \in \mathbb{R}$$

where $\quad S(t-n) = \mathcal{F}^{-1}[\theta(w)e^{-2\pi i n w}](t), \quad t \in \mathbb{R}$

A general solution:

To investigate sampling and reconstruction problems in spline spaces, wavelets spaces, and general shift-invariant spaces

$$V_\varphi^2 := \overline{\text{span}}_{L^2(\mathbb{R}^d)}\{\varphi(t-\alpha) \,:\, \alpha \in \mathbb{Z}^d\} = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} a_\alpha\, \varphi(t-\alpha) \,:\, \{a_\alpha\} \in \ell^2(\mathbb{Z}^d) \Big\} \subset L^2(\mathbb{R}^d)\,.$$

where

- The function $\varphi \in L^2(\mathbb{R}^d)$ is the generator of V_φ^2
- It is assumed that the sequence $\{\varphi(t-\alpha)\}_{\alpha\in\mathbb{Z}^d}$ is a Riesz basis for V_φ^2

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$$\begin{aligned} V_\varphi^2 :=& \overline{\operatorname{span}}_{L^2(\mathbb{R}^d)}\big\{\varphi(t-\alpha)\ :\ \alpha\in\mathbb{Z}^d\big\} \\ =& \Big\{\sum_{\alpha\in\mathbb{Z}^d} a_\alpha\,\varphi(t-\alpha)\ :\ \{a_\alpha\}\in\ell^2(\mathbb{Z}^d)\Big\}\subset L^2(\mathbb{R}^d)\,. \end{aligned}$$

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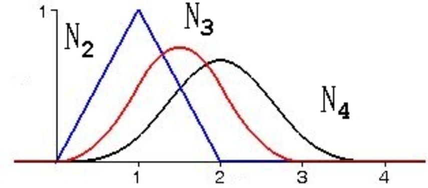
A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator

An important example

B-splines

Consider $\varphi = N_m$ where N_m is the B-**spline of order** $m-1$, i.e., $N_m := N_1 * N_1 * \cdots * N_1$ (m times) where $N_1 := \chi_{[0,1]}$ denotes the characteristic function of the interval $[0,1]$



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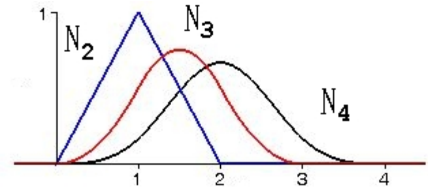

For any B-spline φ, the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a **Riesz sequence** in $L^2(\mathbb{R})$, i.e., a Riesz basis for V_φ^2

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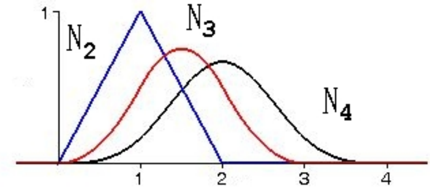

A sequence $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2 if and only if $0 < \|\Phi\|_0 \leq \|\Phi\|_\infty < \infty$, where $\|\Phi\|_0$ denotes the essential infimum of the function $\Phi(w) := \sum_{k\in\mathbb{Z}} |\widehat{\varphi}(w+k)|^2$ en $(0,1)$, and $\|\Phi\|_\infty$ its essential supremum

The shift-invariant space V_φ^2

We assume the following hypotheses on the generator φ :

- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2
- φ is a continuous function on $\mathbb{R}$
- The series $\sum_{n=-\infty}^{\infty} |\varphi(t-n)|^2$ is uniformly bounded on $\mathbb{R}$

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Thus, any $f\in V_\varphi^2$ is a continuous function on $\mathbb{R}$ given by the pointwise sum $f(t)=\sum_{n\in\mathbb{Z}} a_n\varphi(t-n), \quad t\in\mathbb{R}$.

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The shift-invariant space V_φ^2 is a **reproducing kernel Hilbert space**: For each fixed $t\in\mathbb{R}$ we have

$$|f(t)|^2 \leq \frac{\|f\|^2}{\|\Phi\|_0}\sum_{n\in\mathbb{Z}}|\varphi(t-n)|^2\,, \quad f\in V_\varphi^2\,.$$

Convergence in the L^2-norm in the space V_φ^2 implies pointwise convergence, which is uniform on $\mathbb{R}$

The isomorphism $\mathcal{T}_\varphi$

The space V_φ^2 is the image of the space $L^2(0,1)$ by means of the isomorphism

$$\mathcal{T}_\Phi : L^2(0,1) \longrightarrow V_\varphi^2$$
$$\{e^{-2\pi i n w}\}_{n\in\mathbb{Z}} \longmapsto \{\varphi(t-n)\}_{n\in\mathbb{Z}}$$

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The coefficients of any function $f \in V_\varphi^2$ coincides with the standard Fourier coefficients of the function $F = \mathcal{T}_\varphi^{-1} f \in L^2(0,1)$

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- For any $f \in V_\varphi^2$ we have

$$f(t) = \langle F, K_t \rangle_{L^2(0,1)}\,, \quad t \in \mathbb{R}$$

where $F = \mathcal{T}_\varphi^{-1} f$ and

$$K_t(w) = \sum_{n=-\infty}^{\infty} \overline{\varphi(t-n)} e^{-2\pi inw} = \overline{Z\varphi(t,w)}$$

($Z\varphi$ denotes the **Zak transform** of φ)

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Some properties:

- $K_{t+m}(w) = e^{-2\pi i m w} K_t(w)\,,\ t\in\mathbb{R}\,,\ \ m\in\mathbb{Z}$
- $\mathcal{T}_\varphi\left[e^{-2\pi i m w}F(w)\right] = f(t-m)$ where $f = \mathcal{T}_\varphi(F)$
- $f(a+m) = \langle F, K_{a+m}\rangle = \langle F, K_a(w)e^{-2\pi i m w}\rangle$

A sampling theorem in V_φ^2

The sequence $\left\{K_{a+m}(w) = e^{-2\pi imw}K_a(w)\right\}_{m\in\mathbb{Z}}$ is **Riesz basis** for $L^2(0,1)$ if and only if $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$, where

$$\|K_a\|_0 := \operatorname*{ess\,inf}_{w\in(0,1)} |K_a(w)|\,; \quad \|K_a\|_\infty := \operatorname*{ess\,sup}_{w\in(0,1)} |K_a(w)|$$

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Sampling theorem in V_φ^2 (Walter 1992)

Assume that $0 < \|K_a\|_0 \le \|K_a\|_\infty < \infty$. Then, for any $f \in V_\varphi^2$ we get

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n) S_a(t-n)\,, \quad t \in \mathbb{R}$$

where the reconstruction function is given by $S_a = \mathcal{T}_\varphi\big(1/\overline{K_a}\big)$. The convergence of the series is absolute and uniform on $\mathbb{R}$

Gilbert Walter

Born in 1930 in the bucolic little town of Ottawa, Illinois of German immigrant parents, I soon showed promise of things to come. At the age of five I tried to catch a ball thrown by my father and dropped it. Later when he tried teaching me to ride a bicycle, I kept falling off.
Later, my parents moved to the suburbs of Chicago where I eventually attended Riverside-Brookfield High School. When I failed to make the track team (or any other team), I realized my dream of athletic greatness was not to be realized. It was then that I turned to mathematics which was considered a sissy subject suitable mainly for girls. Since many attractive girls took the subject and looked to me for help, I decided mathematics was something to pursue. Thus a career was born.

Some examples in B-spline spaces

1. In the space $V_{N_2}^2$ the following sampling formula holds
$$f(t)=\sum_{n=-\infty}^{\infty} f(n)\,N_2(t+1-n),\quad t\in\mathbb{R}$$
2. In the space $V_{N_3}^2$ the following sampling formula holds
$$f(t)=\sum_{n=-\infty}^{\infty} f(n+1/2)\,S_{1/2}(t-n),\quad t\in\mathbb{R}$$
where $S_{1/2}(t)=4\sqrt{2}\sum_{n=-\infty}^{\infty}(2\sqrt{2}-3)^{|n+1|}\ N_3(t-n)$
3. In the space $V_{N_4}^2$ the following sampling formula holds
$$f(t)=\sum_{n=-\infty}^{\infty} f(n)\,S(t-n),\quad t\in\mathbb{R}$$
where $S(t)=\sqrt{3}\sum_{n=-\infty}^{\infty}(-1)^n(2-\sqrt{3})^{|n|}\ N_4(t-n+2)$

Contents

Statement of the problem

Consider a shift-invariant subspace V_Φ^2 in $L^2(\mathbb{R}^d)$ with r stable generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$ in $L^2(\mathbb{R}^d)$, that is,

$$V_\Phi^2 = \Big\{ \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha) \,:\, d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2 \dots, r \Big\}$$

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- Given s convolution systems $\mathcal{L}_j f = f * \mathrm{h}_j$ defined on V_Φ^2 where $\mathrm{h}_j \in L^2(\mathbb{R}^d)$ (**average sampling**) or linear combination of Dirac deltas (**usual sampling**)
- Taking samples at a lattice $M\mathbb{Z}^d$ of $\mathbb{Z}^d$ where M denotes a $d \times d$ matrix with integer entries and $\det M > 0$.

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The generalized regular sampling problem consists of the stable recovery of any function $f \in V_\Phi^2$ from the sequence of samples

$$\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$$

- Intuitively, sampling at $M\mathbb{Z}^d$ we are using the **sampling rate** $1/r(\det M)$, so we will need $s \geq r(\det M)$ convolution systems
- We should assume a **stability condition** as: There exist two positive constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \leq B\|f\|^2 \quad \text{for all } \ f \in V_\Phi^2 .$$

- The recovery will be done by means of a **sampling formula**

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(t - M\alpha) , \quad t \in \mathbb{R}^d ,$$

where the reconstruction functions $\{S_j(\cdot - M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **frame** for the shift-invariant space V_Φ^2

- Intuitively, sampling at $M\mathbb{Z}^d$ we are using the **sampling rate** $1/r(\det M)$, so we will need $s \geq r(\det M)$ convolution systems
- We should assume a **stability condition** as: There exist two positive constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} |\mathcal{L}_j f(M\alpha)|^2 \leq B\|f\|^2 \quad \text{for all } \; f \in V_\Phi^2 .$$

- The recovery will be done by means of a **sampling formula**

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The strategy to follow is:

- To identify the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ as frame coefficients
- To search for the corresponding dual frames

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A sequence $\{f_k\}_{k=1}^{\infty}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$ (frame bounds) such that

$$A\|f\|^2 \le \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \le B\|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

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- To search for the corresponding dual frames

The frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ for $\mathcal{H}$ are **dual frames** if the equivalent expansions in $\mathcal{H}$ hold

$$f=\sum_{k=1}^{\infty}\langle f,f_k\rangle g_k=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k\,,\quad f\in\mathcal{H}$$

The shift-invariant space V_Φ^2

We assume the following hypotheses on the set of generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$:

- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d, k=1,2\dots r}$ is a Riesz basis for V_Φ^2
- The functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$

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$\iff$ there exist constants $0 < c \le C$ such that

$$c\,\mathbb{I}_r \le G_\Phi(w) \le C\,\mathbb{I}_r \quad \text{a.e } w \in [0,1)^d\,,$$

where $G_\Phi(w)$ denotes the **Gramian** of Φ given by

$$G_\Phi(w) := \sum_{\alpha\in\mathbb{Z}^d} \widehat{\Phi}(w+\alpha)\overline{\widehat{\Phi}(w+\alpha)}^\top$$

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  This is equivalent to say that the set of generators Φ are **continuous** on $\mathbb{R}^d$ and the series $\sum_{\alpha\in\mathbb{Z}^d} |\Phi(t-\alpha)|^2$ is **uniformly bounded** on $\mathbb{R}^d$.

  Thus, any $f \in V_\Phi^2$ is defined on $\mathbb{R}^d$ as the pointwise sum

$$f(t) = \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} d_k(\alpha)\, \varphi_k(t-\alpha),\ t \in \mathbb{R}^d$$

The shift-invariant space V_Φ^2

We assume the following hypotheses on the set of generators $\Phi := \{\varphi_1, \dots, \varphi_r\}$:

- The sequence $\{\varphi_k(t-\alpha)\}_{\alpha \in \mathbb{Z}^d, k=1,2\dots r}$ is a Riesz basis for V_Φ^2
- The functions in the shift-invariant space V_Φ^2 are continuous on $\mathbb{R}^d$

Thus, the space V_Φ^2 becomes a reproducing kernel Hilbert space

$$\Downarrow$$

Convergence in V_Φ^2 in the L^2-sense implies pointwise convergence which is uniform on $\mathbb{R}^d$

The isomorphism $\mathcal{T}_\Phi$

The space V_Φ^2 is the image of $L_r^2[0,1)^d$ by means of the isomorphism

$$\mathcal{T}_\Phi : L_r^2[0,1)^d \longrightarrow V_\Phi^2$$
$$\{e^{-2\pi i\alpha^\top w}\mathbf{e}_k\}_{\alpha\in\mathbb{Z}^d,k=1,2,\dots,r} \longmapsto \{\varphi_k(t-\alpha)\}_{\alpha\in\mathbb{Z}^d,k=1,2,\dots,r}$$

where $\{\mathbf{e}_1,\dots,\mathbf{e}_r\}$ denotes the canonical basis of $\mathbb{R}^r$. Then, for any $\mathbf{F}=(F_1,\dots,F_r)^\top\in L_r^2[0,1)^d$ we have

$$f(t)=\mathcal{T}_\Phi\mathbf{F}(t)=\langle\mathbf{F},\mathbf{K}_t\rangle_{L_r^2[0,1)^d}\,,\quad t\in\mathbb{R}^d$$

where $\mathbf{K}_t(x):=\overline{\mathbf{Z}\Phi}(t,x)$, and $\mathbf{Z}\Phi$ denotes the **Zak transform** of Φ, i.e.,

$$(\mathbf{Z}\Phi)(t,w):=\sum_{\alpha\in\mathbb{Z}^d}\Phi(t+\alpha)\mathrm{e}^{-2\pi i\alpha^\top w}\,.$$

An expression for the samples

For any $f \in V_\Phi^2$ we have

$$\big(\mathcal{L}_j f\big)(t) = \langle \mathbf{F}, \big(\overline{\mathbf{Z}\mathcal{L}_j\Phi}\big)(t,\cdot)\rangle_{L_r^2[0,1)^d}\,, \quad \text{where } \mathbf{F} = \mathcal{T}_\Phi^{-1} f \in L_r^2[0,1)^d$$

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In particular:

$$\begin{aligned}\big(\mathcal{L}_j f\big)(M\alpha) &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(M\alpha,\cdot)\rangle_{L_r^2[0,1)^d} \\ &= \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle_{L_r^2[0,1)^d}\end{aligned}$$

An expression for the samples

For any $f \in V_\Phi^2$ we have

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As a consequence,

We have to study when the sequence

$$\big\{\overline{\mathbf{g}_j(x)}\mathrm{e}^{-2\pi i \alpha^\top M^\top x}\big\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}\,,$$

where $\mathbf{g}_j(x) := \mathbf{Z}\mathcal{L}_j\Phi(0,x)$, $j = 1,2,\dots,s$, is a **frame** for $L_r^2[0,1)^d$

By introducing the $s \times r(\det M)$ matrix of functions

$$\mathbb{G}(x) := \begin{bmatrix} \mathbf{g}_1^\top(x) & \mathbf{g}_1^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_1^\top(x + M^{-\top} i_{\det M}) \\ \mathbf{g}_2^\top(x) & \mathbf{g}_2^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_2^\top(x + M^{-\top} i_{\det M}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g}_s^\top(x) & \mathbf{g}_s^\top(x + M^{-\top} i_2) & \cdots & \mathbf{g}_s^\top(x + M^{-\top} i_{\det M}) \end{bmatrix},$$

and its related constants

$$A_{\mathbb{G}} := \operatorname*{ess\,inf}_{x \in [0,1)^d} \lambda_{\text{mín}}[\mathbb{G}^*(x)\mathbb{G}(x)], \quad B_{\mathbb{G}} := \operatorname*{ess\,sup}_{x \in [0,1)^d} \lambda_{\text{máx}}[\mathbb{G}^*(x)\mathbb{G}(x)],$$

where

$$\{i_1 = 0, i_2, \ldots, i_{\det M}\} = \mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\},$$

Having in mind that $\left\{e^{2\pi i \alpha^\top M^\top x}\right\}_{\alpha \in \mathbb{Z}^d}$ is an orthogonal basis for $L^2\left(M^{-T}[0,1)^d\right)$ we obtain the following result:

Let g_j be in $L_r^2[0,1)^d$ for $j = 1, 2, \dots, s$ and let $\mathbb{G}(x)$ be its associated matrix. Then:

(a) The sequence $\left\{\overline{\mathbf{g}_j(x)} e^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **complete system** for $L_r^2[0,1)^d$ if and only if the rank of the matrix $\mathbb{G}(x)$ is $r(\det M)$ a.e. in $[0,1)^d$.

(b) The sequence $\left\{\overline{\mathbf{g}_j(x)} e^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **Bessel sequence** for $L_r^2[0,1)^d$ if and only if $g_j \in L_r^\infty[0,1)^d$ (or equivalently $B_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $B_{\mathbb{G}}/(\det M)$.

(c) The sequence $\left\{\overline{\mathbf{g}_j(x)} e^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **frame** for $L_r^2[0,1)^d$ if and only if $0 < A_{\mathbb{G}} \le B_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $A_{\mathbb{G}}/(\det M)$ and $B_{\mathbb{G}}/(\det M)$.

(d) The sequence $\left\{\overline{\mathbf{g}_j(x)} e^{-2\pi i \alpha^\top M^\top x}\right\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ is a **Riesz basis** for $L_r^2[0,1)^d$ if and only if it is a frame and $s = r(\det M)$.

Moreover, assuming the existence of an $r \times s$ matrix $\mathbf{a}(x) := [\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)]$, with entries in $L^\infty[0,1)^d$, such that

$$\big[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\big]\ \mathbb{G}(x) = \big[\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}\big] \quad \text{a.e. in } [0,1)^d\,,$$

for $\mathbf{F} = \mathcal{T}_\Phi^{-1} f$ we get

$$\begin{aligned}\mathbf{F}(x) &= (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \langle \mathbf{F}, \overline{\mathbf{Z}\mathcal{L}_j\Phi}(0,\cdot)\mathrm{e}^{-2\pi i\alpha^\top M^\top \cdot}\rangle \mathbf{a}_j(x)\mathrm{e}^{-2\pi i \alpha^\top M^\top x} \\ &= (\det M) \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} \big(\mathcal{L}_j f\big)(M\alpha)\, \mathbf{a}_j(x)\mathrm{e}^{-2\pi i \alpha^\top M^\top x} \quad \text{in} \quad L_r^2[0,1)^d\end{aligned}$$

Finally, by using the isomorphism $\mathcal{T}_\Phi$ we obtain

$$
\begin{aligned}
f(t) &= (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)\mathcal{T}_\Phi\big(\mathbf{a}_j(x)\mathrm{e}^{-2\pi i\alpha^\top M^\top x}\big)(t)\\
&= (\det M)\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)(\mathcal{T}_\Phi\mathbf{a}_j)(t-M\alpha)\\
&= \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t\in\mathbb{R}^d
\end{aligned}
$$

where we have used the **shifting property**

$$
\mathcal{T}_\Phi\left[\mathbf{F}(\cdot)e^{-2\pi i\alpha^\top\cdot}\right](t) = \mathcal{T}_\Phi\mathbf{F}(t-\alpha),
$$

and that the space V_Φ^2 is a RKHS.

The main result

Assume that the functions $\mathbf{g}_j \in L_r^\infty[0,1)^d$, $j = 1, 2, \dots, s$ ($\iff B_\mathbb{G} < \infty$). The following statements are equivalents:

(a) $A_\mathbb{G} > 0$

(b) There exists an $r \times s$ matrix $\left[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\right]$ with entries $\mathbf{a}_j \in L_r^\infty[0,1)^d$ and satisfying

$$\left[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\right]\mathbb{G}(x) = [\mathbb{I}_r, \mathbb{O}_{(\det M - 1)r \times r}] \quad \text{a.e. in } [0,1)^d$$

(c) There exists a frame $\{S_j(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ for V_Φ^2 such that for any $f \in V_\Phi^2$

$$f = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d)$$

(d) There exists a frame $\{S_{j,\alpha}(\cdot)\}_{\alpha \in \mathbb{Z}^d,\ j=1,2,\dots,s}$ for V_Φ^2 such that

$$f = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\alpha}(t)\,, \quad f \in V_\Phi^2$$

Some additional comments

- All the matrices $\big[\mathbf{a}_1(x), \dots, \mathbf{a}_s(x)\big]$ in the former result correspond to the first r rows of the matrices

$$\mathbb{G}^\dagger(x) + \mathbb{U}(x)\big[\mathbb{I}_s - \mathbb{G}(x)\mathbb{G}^\dagger(x)\big]\,,$$

  where $\mathbb{U}(x)$ is any $r(\det M) \times s$ matrix with entries in $L^\infty[0,1)^d$, and $\mathbb{G}^\dagger(x) := [\mathbb{G}^*(x)\mathbb{G}(x)]^{-1}\mathbb{G}^*(x)$
- If $s = r(\det M)$ we are in the **Riesz bases** setting: Corresponding with the sequence of samples $\{(\mathcal{L}_j f)(M\alpha)\}_{\alpha\in\mathbb{Z}^d,\ j=1,2,\dots,s}$ there is a unique sampling formula having the **interpolation property**

$$(\mathcal{L}_{j'} S_{j,\mathbf{a}})(M\alpha) = \delta_{j,j'}\delta_{\alpha,0}\,,$$

  where $j, j' = 1, 2, \dots, s$ and $\alpha \in \mathbb{Z}^d$

Some additional comments

- The reconstruction functions

$$S_{j,\mathbf{a}} = (\det M)\mathcal{T}_\Phi(\mathbf{a}_j)\,, \quad j = 1, 2, \ldots, s\,,$$

are determined from the Fourier coefficients of the components of

$$\mathbf{a}_j(x) := [a_{1,j}(x), a_{2,j}(x), \ldots, a_{r,j}]^\top\,, \quad j = 1, 2, \ldots, s\,.$$

If $\widehat{a}_{k,j}(\alpha) := \int_{[0,1)^d} a_{k,j}(x)\mathrm{e}^{2\pi i\alpha^\top x}dx$,

$$S_{j,\mathbf{a}}(t) = (\det M)\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{a}_{k,j}(\alpha)\varphi_k(t-\alpha)\,, \quad t \in \mathbb{R}^d$$

Contents

A generalized sampling formula as a filter bank

$$f(t)=\sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)$$

A generalized sampling formula as a filter bank

$$f(t) = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)$$

$$S_{j,\mathbf{a}}(t) = (\det M) \sum_{\beta \in \mathbb{Z}^d} \sum_{k=1}^{r} \widehat{a}_{k,j}(\beta) \varphi_k(t - \beta)$$

A generalized sampling formula as a filter bank

$$
\begin{aligned}
f(t) &= \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha) \\
&= \sum_{k=1}^{r} \sum_{\gamma\in\mathbb{Z}^d} \left\{ \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha)\widehat{a}_{k,j}(\gamma - M\alpha) \right\} \varphi_k(t-\gamma)
\end{aligned}
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A generalized sampling formula as a filter bank

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&= \sum_{k=1}^{r} \sum_{\gamma\in\mathbb{Z}^d} d_k(\gamma)\, \varphi_k(t-\gamma)
\end{aligned}
$$

Oversampling, that is, $s > r(\det M)$, allows reconstruction functions $S_{j,\mathbf{a}}$ with prescribed properties

We introduce some complex notation:

- We denote $\mathbf{z}^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_d^{\alpha_d}$ for $\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}^d$, and the d-torus by $\mathbb{T}^d := \{\mathbf{z} \in \mathbb{C}^d : |z_1| = |z_2| = \dots = |z_d| = 1\}$
- For $1 \leq j \leq s$ and $1 \leq k \leq r$ we define

$$\mathrm{g}_{j,k}(\mathbf{z}) := \sum_{\mu \in \mathbb{Z}^d} \mathcal{L}_j \varphi_k(\mu) \mathbf{z}^{-\mu}, \quad \mathrm{g}_j^\top(\mathbf{z}) := \big(\mathrm{g}_{j,1}(\mathbf{z}), \mathrm{g}_{j,2}(\mathbf{z}), \dots, \mathrm{g}_{j,r}(\mathbf{z})\big)$$

  and the $s \times r(\det M)$ matrix

$$\mathrm{G}(\mathbf{z}) := \left[\mathrm{g}_j^\top(z_1 \mathrm{e}^{2\pi i m_1^\top i_l}, \dots, z_d \mathrm{e}^{2\pi i m_d^\top i_l})\right]_{\substack{j=1,2,\dots,s \\ k=1,2,\dots,r;\ l=1,2,\dots,\det M}}$$

  where $m_1, \dots, m_d$ denote the columns of the matrix M^{-1}, and $i_1, i_2, \dots, i_{\det M}$ in $\mathbb{Z}^d$ are the elements of $\mathcal{N}(M^\top)$
- For $x = (x_1, \dots, x_d) \in [0,1)^d$, $\mathbf{z} = (\mathrm{e}^{2\pi i x_1}, \dots, \mathrm{e}^{2\pi i x_d}) \in \mathbb{T}^d$, and we have $\mathbb{G}(x) = \mathrm{G}(\mathbf{z})$.

Reconstruction functions with compact support

Assume that the generators φ_k and the functions $\mathcal{L}_j\varphi_k$, $1 \le k \le r$ and $1 \le j \le s$, have compact support. Then, there exists an $r \times s$ matrix $\mathrm{a}(\mathbf{z})$ whose entries are **Laurent polynomials** and satisfying

$$\mathrm{a}(\mathbf{z})\mathrm{G}(\mathbf{z}) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{for all } \mathbf{z} \in \mathbb{T}^d$$

if and only if

$$\operatorname{rank} \mathrm{G}(\mathbf{z}) = r(\det M) \ \text{ for all } \ \mathbf{z} \in (\mathbb{C} \setminus \{0\})^d \,.$$

Reconstruction functions with compact support

The corresponding **reconstruction functions** $S_{j,\mathsf{a}}$ given by

$$S_{j,\mathsf{a}}(t)=(\det M)\sum_{\alpha\in\mathbb{Z}^d}\sum_{k=1}^{r}\widehat{\mathsf{a}}_{k,j}(\alpha)\varphi_k(t-\alpha)\,,\quad t\in\mathbb{R}^d\,,$$

where $\widehat{\mathsf{a}}_{k,j}(\alpha)$, $\alpha\in\mathbb{Z}^d$, are the Laurent coefficients of the functions $\mathsf{a}_{k,j}(\mathbf{z})$, i.e.,

$$\mathsf{a}_{k,j}(\mathbf{z})=\sum_{\alpha\in\mathbb{Z}^d}\widehat{\mathsf{a}}_{k,j}(\alpha)\mathbf{z}^{-\alpha}\,,$$

have **compact support**

Reconstruction functions with compact support

From one of these $r \times s$ matrices, say $\widetilde{\mathrm{a}}(\mathbf{z}) = [\widetilde{\mathrm{a}}_1(\mathbf{z}), \dots, \widetilde{\mathrm{a}}_s(\mathbf{z})]$, we can get all of them. Indeed, they are given by the r first rows of the $r(\det M) \times s$ matrices of the form

$$\mathrm{A}(\mathbf{z}) = \widetilde{\mathrm{A}}(\mathbf{z}) + \mathrm{U}(\mathbf{z})\left[\mathbb{I}_s - \mathrm{G}(\mathbf{z})\widetilde{\mathrm{A}}(\mathbf{z})\right],$$

where

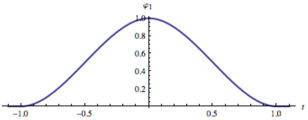
$$\widetilde{\mathrm{A}}(\mathbf{z}) := \left[\widetilde{\mathrm{a}}_j(z_1 \mathrm{e}^{2\pi i m_1^\top i_l}, \dots, z_d \mathrm{e}^{2\pi i m_d^\top i_l})\right]_{\substack{k=1,2,\dots,r;\ l=1,2,\dots,\det M \\ j=1,2,\dots,s}},$$

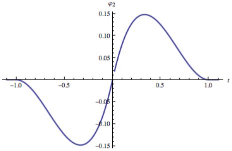
and $\mathrm{U}(\mathbf{z})$ is any $r(\det M) \times s$ matrix with Laurent polynomial entries

An illustrative example

Consider the Hermite cubic splines

$$\varphi_1(t) = \begin{cases} (t+1)^2(1-2t), & t \in [-1,0] \\ (1-t)^2(1+2t), & t \in [0,1] \\ 0, & |t| > 1 \end{cases}, \qquad \varphi_2(t) = \begin{cases} (t+1)^2 t, & t \in [-1,0] \\ (1-t)^2 t, & t \in [0,1] \\ 0, & |t| > 1 \end{cases}$$





which are stable generators for the subspace $V^2_{\varphi_1,\varphi_2} \subset L^2(\mathbb{R})$

In the shift-invariant space $V^2_{\varphi_1,\varphi_2}$ consider the sampling period $M=1$ and the convolution systems defined by

$$\mathcal{L}_1 f(t) := \int_t^{t+1/3} f(u)du\,, \quad \mathcal{L}_2 f(t) := \mathcal{L}_1 f\big(t+\frac{1}{3}\big), \quad \mathcal{L}_3 f(t) := \mathcal{L}_1 f\big(t+\frac{2}{3}\big)$$

In this case, we have a 3×2 matrix $\mathrm{G}(\mathbf{z})$ with Laurent polynomial entries; we can try to search for a 2×3 matrix $[\mathrm{a}_1(\mathbf{z}), \mathrm{a}_2(\mathbf{z}), \mathrm{a}_3(\mathbf{z})]$, with Laurent polynomial entries, such that

$$[\mathrm{a}_1(\mathbf{z}), \mathrm{a}_2(\mathbf{z}), \mathrm{a}_3(\mathbf{z})]\,\mathrm{G}(\mathbf{z}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, in the shift-invariant space $V^2_{\varphi_1,\varphi_2}$ we obtain the following sampling formula

$$f(t)=\sum_{n\in\mathbb{Z}}\sum_{j=1}^{3}\mathcal{L}_j f(n)\,S_{j,\mathbf{a}}(t-n)\,,\quad t\in\mathbb{R}\,,$$

where the sampling functions are given by

$$\begin{aligned}
S_{1,\mathbf{a}}(t) &:= \frac{85}{44}\varphi_1(t)+\frac{1}{11}\varphi_1(t-1)+\frac{85}{4}\varphi_2(t)-\varphi_2(t-1)\,,\\
S_{2,\mathbf{a}}(t) &:= \frac{-23}{44}\varphi_1(t)-\frac{23}{44}\varphi_1(t-1)-\frac{23}{4}\varphi_2(t)+\frac{23}{4}\varphi_2(t-1)\,,\\
S_{3,\mathbf{a}}(t) &:= \frac{1}{11}\varphi_1(t)+\frac{85}{44}\varphi_1(t-1)+\varphi_2(t)-\frac{85}{4}\varphi_2(t-1)\,,\quad t\in\mathbb{R}\,.
\end{aligned}$$

Reconstruction functions with exponential decay

Assume that the generators φ_k and the functions $\mathcal{L}_j\varphi_k$, $j = 1, 2, \dots, s$ and $k = 1, 2, \dots, r$, have exponential decay. Then, there exists an $r \times s$ matrix $\mathrm{a}(\mathbf{z}) = [\mathrm{a}_1(\mathbf{z}), \dots, \mathrm{a}_s(\mathbf{z})]$ with entries in the **algebra** $\mathcal{H}(\mathbb{T}^d)$ (holomorphic functions in a neighborhood of the d-torus $\mathbb{T}^d$) and satisfying

$$\mathrm{a}(\mathbf{z})\mathrm{G}(\mathbf{z}) = [\mathbb{I}_r, \mathbb{O}_{(\det M-1)r\times r}] \quad \text{for all } \mathbf{z} \in \mathbb{T}^d$$

if and only if

$$\operatorname{rank} \mathrm{G}(\mathbf{z}) = r(\det M) \quad \text{for all } \mathbf{z} \in \mathbb{T}^d$$

Reconstruction functions with exponential decay

In this case, all of such matrices $\mathsf{a}(\mathbf{z})$ are given as the first r rows of a $r(\det M) \times s$ matrix $\mathsf{A}(\mathbf{z})$ of the form

$$\mathsf{A}(\mathbf{z}) = \mathsf{G}^\dagger(\mathbf{z}) + \mathsf{U}(\mathbf{z})\left[\mathbb{I}_s - \mathsf{G}(\mathbf{z})\mathsf{G}^\dagger(\mathbf{z})\right],$$

where $\mathsf{U}(\mathbf{z})$ denotes any $r(\det M) \times s$ matrix with entries in the algebra $\mathcal{H}(\mathbb{T}^d)$ and $\mathsf{G}^\dagger(\mathbf{z}) := \left[\mathsf{G}^*(\mathbf{z})\mathsf{G}(\mathbf{z})\right]^{-1}\mathsf{G}^*(\mathbf{z})$.
The corresponding **reconstruction functions** $S_{j,\mathsf{a}}$ given by

$$S_{j,\mathsf{a}}(t) = (\det M) \sum_{\alpha\in\mathbb{Z}^d} \sum_{k=1}^{r} \widehat{\mathsf{a}}_{k,j}(\alpha)\varphi_k(t-\alpha), \quad t \in \mathbb{R}^d,$$

where $\widehat{\mathsf{a}}_{k,j}(\alpha)$, $\alpha \in \mathbb{Z}^d$, are the Laurent coefficients of the functions $\mathsf{a}_{k,j}(\mathbf{z})$, i.e., $\mathsf{a}_{k,j}(\mathbf{z}) = \sum_{\alpha\in\mathbb{Z}^d} \widehat{\mathsf{a}}_{k,j}(\alpha)\mathbf{z}^{-\alpha}$, have **exponential decay**

L^2-approximation properties

Consider the scaled version $\Gamma_{\mathbf{a}}^h := \sigma_{1/h}\Gamma_{\mathbf{a}}\sigma_h$, where for $h>0$ we are using the notation $\sigma_h f(t) := f(ht), \quad t\in\mathbb{R}^d$, of the sampling operator $\Gamma_{\mathbf{a}}$

$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s}\sum_{\alpha\in\mathbb{Z}^d}(\mathcal{L}_j f)(M\alpha)S_{j,\mathbf{a}}(t-M\alpha)\,, \quad t\in\mathbb{R}^d\,,$$

L^2-approximation properties

Consider the scaled version $\Gamma_{\mathbf{a}}^h := \sigma_{1/h}\Gamma_{\mathbf{a}}\sigma_h$, where for $h > 0$ we are using the notation $\sigma_h f(t) := f(ht)$, $\quad t \in \mathbb{R}^d$, of the sampling operator $\Gamma_{\mathbf{a}}$

$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

Under appropriate hypotheses this operator approximates, in the L^2-norm sense, as $h \to 0^+$ any function in the Sobolev space

$$W_2^\ell(\mathbb{R}^d) := \{f \;:\; \|D^\gamma f\|_2 < \infty\,,\, |\gamma| \le \ell\}$$

L^2-approximation properties

Consider the scaled version $\Gamma_{\mathbf{a}}^h := \sigma_{1/h}\Gamma_{\mathbf{a}}\sigma_h$, where for $h > 0$ we are using the notation $\sigma_h f(t) := f(ht)$, $\quad t \in \mathbb{R}^d$, of the sampling operator $\Gamma_{\mathbf{a}}$

$$\Gamma_{\mathbf{a}} f(t) := \sum_{j=1}^{s} \sum_{\alpha\in\mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,\mathbf{a}}(t - M\alpha)\,, \quad t \in \mathbb{R}^d\,,$$

The set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the **Strang-Fix conditions of order** ℓ if there exist r finitely supported sequences $b_k : \mathbb{Z}^d \to \mathbb{C}$ such that the function $\varphi(t) = \sum_{k=1}^{r} \sum_{\alpha\in\mathbb{Z}^d} b_k(\alpha)\varphi_k(t-\alpha)$ satisfies the Strang-Fix conditions of order ℓ, i.e.,

$$\widehat{\varphi}(0) \neq 0, \quad D^\beta \widehat{\varphi}(\alpha) = 0, \quad |\beta| < \ell, \quad \alpha \in \mathbb{Z}^d \setminus \{0\}\,.$$

L^2-approximation properties

The following result holds:

Assume $2\ell > d$ and that all the systems $\mathcal{L}_j$ satisfy $\mathcal{L}_j f = f * \mathrm{h}_j$ with $\mathrm{h}_j \in \mathcal{L}^2(\mathbb{R}^d)$, $j = 1, \ldots, s$
$\big(\mathrm{h} \in \mathcal{L}^2(\mathbb{R}^d)$ if and only if $\sum_{\alpha\in\mathbb{Z}^d} |\mathrm{h}(t-\alpha)| \in L^2[0,1)^d\big)$.
If the set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the **Strang-Fix conditions of order** ℓ and, for each $k = 1, 2, \ldots, r$, the **decay condition** $\varphi_k(t) = O\big([1+|t|]^{-d-\ell-\epsilon}\big)$ for some $\epsilon > 0$, then

$$\|f - \Gamma_{\mathbf{a}}^h f\|_2 \leq C\,|f|_{\ell,2}\,h^\ell \quad \text{for all } f \in W_2^\ell(\mathbb{R}^d),$$

where the constant C does not depend on h and f, and $|f|_{\ell,2} := \sum_{|\beta|=\ell} \|D^\beta f\|_2$ denotes the corresponding seminorm of $f \in W_2^\ell(\mathbb{R}^d)$.

In short ...


contradiction
operator stability Frame entries characterized
several d-dimensional derived matrix among
and/or time-invariant enormous result equivalently Although version nonsingular
inefficient sequence formula Section Moore-Smith first finite filters prove involves pointed
samples impact operation included type Riesz signal product derivatives hypothesis Sobolev Shift-invariant
denotes See sum formulas general sinc reads types
problems lattice symbol performed consider integer
shifts treat generalized assumed Given band-limited classical may span
forwe norm taken First spline systems space frame
dual approximation Reproducing consequence setting
function satisfying case uniform considering combination WSK subset becomes
paper impose linear shift-invariant
set theory data
spaces many appropriate multidimensional common Shannon Gibbs also convolution
sense obtain
sampling like
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applied forms ability state
see stable
profusely
theorem addition now
system section basis well-established means point generated schemes develop
available belong order separately
generators Thus Gramian determinant defined compact
middle generates corresponding relies wavelet sections
obtaining positive number functions
dimensions papers
scaled

References

- **Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces**. *Appl. Comput. Harmon. Anal.*, 20:422–433, 2006 (with G. Pérez-Villalón)
- **Generalized sampling in shift-invariant spaces with multiple stable generators**. *J. Math. Anal. Appl.*, 337:69–84, 2008 (with M. A. Hernández-Medina and G. Pérez-Villalón)
- **Multivariate generalized sampling in shift-invariant spaces and its approximation properties**. *J. Math. Anal. Appl.*, 355:397–413, 2009 (with G. Pérez-Villalón)
- **Sampling, approximation, and L^p shift-invariant spaces**. J. Math. Anal. Appl., 380:607–627, 2011 (with M. J. Muñoz-Bouzo and G. Pérez-Villalón)
- **Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invarinat subspaces with multiple stable generators**. In *Multiscale Signal Analysis and Modeling*, LNEE, Springer, 2012 (with H. R. Hernández-Morales and G. Pérez-Villalón)

Para finalizar, en otro orden de cosas . . .

Para finalizar, en otro orden de cosas . . .

Todavía no hace una generación, aún existía en el seno de los estudios superiores un amplio margen para la actividad libre y el pensamiento independiente. Pero hoy, la mayoría de nuestras grandes instituciones académicas están completamente automatizadas como una fábrica de laminados de acero o una red telefónica; la producción en serie de artículos de erudición, de descubrimientos, de titulaciones, de doctores, de catedráticos y de publicidad, ¡sobre todo de publicidad!, se mantiene a un nivel incomparable; y solo los que se identifican con el poder, por absurdos que sean desde una perspectiva humana, están tan bien situados para la promoción, las grandes ayudas para la investigación, el poder político y las recompensas económicas que se conceden a quienes "están en onda" con el sistema.

Para finalizar, en otro orden de cosas . . .

Entretanto, una vasta cantidad de conocimientos valiosos, a los que hay que añadir un lote de trivilidad y morralla aún mayor, se ve relegada a un gigantesco montón de desperdicios. Debido a la falta de un método, con baremos cualitativos incorporados, que se encargue de promover una evaluación y una criba constantes, y con procesos de asimilación que controlen, como en el sistema digestivo, tanto el hambre como la alimentación, la naturaleza del producto final hace de contrapeso del orden superficial que presenta el conjunto humano, dado que saber cada vez más sobre cada vez menos supone, al fin y al cabo, saber cada vez menos.

Para finalizar, en otro orden de cosas . . .

Lewis Mumford, *The Pentagon of Power: The Myth of the Machine (Vol. 2)*, Harcourt Publishing Company, San Diego CA, 1970.

- 1895 NY-1990
- History, Philosophy
- Stanford University, MIT, Penn University
- Major work: *The City in History, Technics and Civilization, The Myth of the Machine*
- Presidential Medal of Freedom, National Medal of Arts, Royal Gold Medal, National Book Award

The Pentagon of Power = *power, property, productivity, profit and publicity*

ELIMINACIÓN DEL PREMIO NOBEL DE ECONOMÍA, YA!!!

¡Gracias por vuestra atención!