Generalized sampling in U-invariant spaces

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Motivation

Let V_φ^2 be a shift-invariant subspace of $L^2(\mathbb{R})$ with a (stable) generator $\varphi \in L^2(\mathbb{R})$, i.e.,

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n\, \varphi(t-n) \,:\, \{\alpha_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

Consider s linear-time invariant systems $\mathcal{L}_j$, $j = 1, 2, \dots, s$ defined on V_φ^2

The generalized sampling problem: Recover any function $f \in V_\varphi^2$ from the sequence of samples $\{(\mathcal{L}_j f)(kr)\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ by means of a sampling formula as

$$f(t) = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} (\mathcal{L}_j f)(kr)\, S_j(t-kr)\,, \quad t \in \mathbb{R}\,,$$

where the sequence $\{S_j(t-kr)\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 (the sampling period r necessarily satisfies $r \le s$).

Generalization

Given a Hilbert space $\mathcal{H}$ and a continuous *group of unitary operators* $\{U^t\}_{t\in\mathbb{R}}$ in $\mathcal{H}$ such that $U = U^1$,

- $\{U^t\}_{t\in\mathbb{R}}$ is family of unitary operators in $\mathcal{H}$ satisfying:
  1. $U^t U^{t'} = U^{t+t'}$,
  2. $U^0 = I_{\mathcal{H}}$,
  3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.
- For a fixed $a \in \mathcal{H}$, in case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a \,:\, \{\alpha_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

- On the other hand, for $b_j \in \mathcal{H}$, $j = 1, 2\dots, s$ we consider the linear operators $x \in \mathcal{H} \mapsto \mathcal{L}_j x \in C(\mathbb{R})$ defined on $\mathbb{R}$ as

$$(\mathcal{L}_j x)(t) := \langle x, U^t b_j\rangle_{\mathcal{H}}, \quad t \in \mathbb{R}\,.$$

The generalized U-sampling problem: Given $b_j \in \mathcal{A}_a$, $j = 1, 2\dots, s$, our aim is to recover any $x \in \mathcal{A}_a$, in a stable way, by means of the sequence of generalized samples

$$\big\{(\mathcal{L}_j x)(kr)\big\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}\,.$$

In other words, to obtain a sampling formula like

$$x = \sum_{k\in\mathbb{Z}} \sum_{j=1}^{s} (\mathcal{L}_j x)(kr)\, U^{kr} c_j \quad \text{in } \mathcal{H}\,.$$

where $\{U^{kr} c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$. (Here r denotes a fixed number in $\mathbb{N}$).

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{ik\theta} d\mu_a(\theta)\,, \qquad k \in \mathbb{Z}\,.$$

- The positive Borel spectral measure μ_a can be decomposed as

$$d\mu_a(\theta) = \phi_a(\theta) d\theta + d\mu_a^s(\theta).$$

- The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

The study of the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$

If for every $j = 1, 2, \dots s$ the spectral measure in the integral representation of the (cross)-covariance function of the sequences $\{U^k a\}_{k\in\mathbb{Z}}$, $\{U^k b_j\}_{k\in\mathbb{Z}}$ has no singular part, we have the following representation

$$\langle U^k a, U^{nr} b_j\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta.$$

where ϕ_{a,b_j} stands for the cross spectral density of the stationary correlated sequences $\{U^k a\}_{k\in\mathbb{Z}}$ and $\{U^k b_j\}_{k\in\mathbb{Z}}$. Consider the $s \times 1$ matrices of functions defined on the torus $\mathbb{T} := \{e^{i\theta} : \theta \in [-\pi, \pi)\}$

$$\mathbf{\Phi}_{a,b}(e^{i\theta}) := \Big(\phi_{a,b_1}(e^{i\theta}), \phi_{a,b_2}(e^{i\theta}), \cdots, \phi_{a,b_s}(e^{i\theta})\Big)^\top,$$

and

$$\mathbf{\Psi}^{l}_{a,b}(e^{i\theta}) := (D_r S^{-l} \mathbf{\Phi}_{a,b})(e^{i\theta})\,, \quad l = 0, 1, \dots, r-1\,,$$

where $D_r : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the decimation operator

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \overset{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta}$$

and $S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the (left) shift operator

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \overset{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta}\,.$$

Finally, defining the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\mathbf{\Psi}_{a,b}(e^{i\theta}) := \big(\Psi^0_{a,b}(e^{i\theta})\ \Psi^1_{a,b}(e^{i\theta})\ \dots \Psi^{r-1}_{a,b}(e^{i\theta})\big)\,,$$

and its related constants,

$$\boldsymbol{A}_{\mathbf{\Psi}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \lambda_{\min}\big[\mathbf{\Psi}_{a,b}{}^*(\zeta) \mathbf{\Psi}_{a,b}(\zeta)\big];$$
$$\boldsymbol{B}_{\mathbf{\Psi}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \lambda_{\max}\big[\mathbf{\Psi}_{a,b}{}^*(\zeta) \mathbf{\Psi}_{a,b}(\zeta)\big].$$

Two problems to solve:

- Characterize the sequence $\{U^{kr} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ in $\mathcal{A}_a$

Theorem. Let b_j be in $\mathcal{A}_a$ for $j = 1, 2, \dots, s$, $\mathbf{\Psi}_{a,b}$ be the associated matrix and $\boldsymbol{A}_{\mathbf{\Psi}}$, $\boldsymbol{B}_{\mathbf{\Psi}}$ its related constants. Then, the following results hold:

i) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a complete system in $\mathcal{A}_a$ if and only if the rank of the matrix $\mathbf{\Psi}_{a,b}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.

ii) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only if the constant $\boldsymbol{B}_{\mathbf{\Psi}} < \infty$.

iii) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$ if and only if constants $\boldsymbol{A}_{\mathbf{\Psi}}$ and $\boldsymbol{B}_{\mathbf{\Psi}}$ satisfy $0 < \boldsymbol{A}_{\mathbf{\Psi}} \le \boldsymbol{B}_{\mathbf{\Psi}} < \infty$. In this case, $\boldsymbol{A}_{\mathbf{\Psi}}$ and $\boldsymbol{B}_{\mathbf{\Psi}}$ are the optimal frame bounds for $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$.

iv) The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

- To find its dual frames.
  Define the $r \times s$ matrix Γ of functions on $\mathbb{T}$ as

$$\Gamma(e^{i\theta}) := \sum_{k\in\mathbb{Z}} \Gamma_k e^{ik\theta} = [\mathbf{\Psi}_{a,b}{}^*(e^{i\theta}) \mathbf{\Psi}_{a,b}(e^{i\theta})]^{-1} \mathbf{\Psi}_{a,b}{}^*(e^{i\theta}).$$

Note that $\mathbf{\Psi}_{a,b}{}^\dagger(e^{i\theta}) := [\mathbf{\Psi}_{a,b}{}^*(e^{i\theta}) \mathbf{\Psi}_{a,b}(e^{i\theta})]^{-1} \mathbf{\Psi}_{a,b}{}^*(e^{i\theta})$ stands for the Moore-Penrose left-inverse. In the frame case we can define,

$$\widetilde{a}_n := \Big(U^{nr}a, U^{nr+1}a, \cdots, U^{nr+r-1}a\Big)^\top$$

and

$$(c_1, c_2, \dots, c_s)^\top := \sum_{k\in\mathbb{Z}} \Gamma_k^\top \widetilde{a}_k\,.$$

Note that, under condition iii) in the Theorem, the matrix $\Gamma(e^{i\theta})$ has entries in $L^\infty(\mathbb{T})$.
Then, the sequences $\{U^{kr} c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ and $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ are a pair of dual frames for $\mathcal{A}_a$. Hence we obtain the following recovery formula in $\mathcal{A}_a$:

For any $x \in \mathcal{A}_a$, the following sampling expansion holds

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{kr} b_j\rangle U^{kr} c_j \quad \text{in } \mathcal{H}.$$

Some remarks:

- The analysis done provides a whole family of dual frames; in fact, everything works if we choose a matrix of the form

$$\Gamma_{\mathbb{U}}(e^{i\theta}) := \mathbf{\Psi}_{a,b}{}^\dagger(e^{i\theta}) + \mathbb{U}(e^{i\theta}) \big[\mathbb{I}_s - \mathbf{\Psi}_{a,b}(e^{i\theta}) \mathbf{\Psi}_{a,b}{}^\dagger(e^{i\theta})\big]\,,$$

  where $\mathbb{U}(e^{i\theta})$ denotes any $r \times s$ matrix with entries in $L^\infty(\mathbb{T})$.
- Notice that if $s = r$, $\mathbf{\Psi}_{a,b}{}^\dagger = \mathbf{\Psi}_{a,b}{}^{-1}$ which implies that Γ is unique and we are in presence of a pair of dual Riesz basis.
- In the Theorem we have assumed that b_j belongs to $\mathcal{A}_a$ for each $j = 1, 2, \dots, s$ since we want the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ to be contained in $\mathcal{A}_a$. In case that some $b_j \notin \mathcal{A}_a$, the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a pseudoframe for $\mathcal{A}_a$ and the Theorem can be reformulated in terms of $\{P_{\mathcal{A}_a}(U^{rk} b_j)\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$, the orthogonal projection onto $\mathcal{A}_a$.

The study of the time-jitter error

Next, we deal with the problem of the recovery of any $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence

$$\{(\mathcal{L}_j x)(kr + \epsilon_{kj})\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

where $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ denotes a sequence of real errors.
Taking into account the $L^2(0,1)$ functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi ikw}\,,\ j = 1, 2, \dots, s\,,$$

we can define the $s \times r$ matrix

$$\mathbb{G}(w) := \Big[g_j\Big(w + \frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}}$$

and its related constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ are given by

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$
$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

The sequence $\{\overline{g_j(w)}\, \mathrm{e}^{2\pi irnw}\}_{n\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$.
The idea is to consider the sequence $\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi irmw}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ as a perturbation of the above frame in $L^2(0,1)$, where

$$g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi ikw}\,,\ j = 1, 2, \dots, s\,.$$

For $|\gamma| < 1/2$, define the functions,

$$M_{a,b_j}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{k+t} b_j\rangle - \langle a, U^k b_j\rangle|\,,$$

and

$$N_{a,b_j}(\gamma) := \max_{k=0,1,\dots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{rm+k+t} b_j\rangle - \langle a, U^{rm+k} b_j\rangle|\,.$$

Notice that $N_{a,b_j}(\gamma) \le M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds.

Consider now:

- the continuous functions $\varphi_j(t) := \langle a, U^t b_j\rangle$, $j = 1, 2, \dots, s$, satisfy a decay condition as $\varphi_j(t) = O(|t|^{-(1+\eta_j)})$ when $|t| \to \infty$ for some $\eta_j > 0$ (this implies the continuity of $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ at 0).
- the functions g_j, satisfy $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$.
- for an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$.

Then we have the following perturbation result:

Theorem. The condition $\sum_{j=1}^{s} M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$ implies that there exists a frame $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x \in \mathcal{A}_a$, the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, C^{\boldsymbol{\epsilon}}_{m,j} \quad \text{in } \mathcal{H}\,,$$

holds. Moreover, when $r = s$ the sequence $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $\langle C^{\boldsymbol{\epsilon}}_{n,j}, U^{rm+\epsilon_{ml}} b_l\rangle_{\mathcal{H}} = \delta_{j,l}\, \delta_{n,m}$ holds.

The preceding sampling formula is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$. As a consequence, in order to recover $x \in \mathcal{A}_a$ from the sequence of inner products $\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we could implement a frame algorithm in $\ell^2(\mathbb{Z})$.

The proofs of the results above exhibited can be found in the references below. See also references therein for previous and related works.

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