Sampling theory in U-invariant spaces

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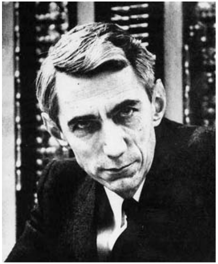
Outline



Claude Elwood Shannon 1916-2001

Shannon's sampling theorem.

If a function of time is limited to the band from 0 to W cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/2W$ seconds apart in the manner indicated by the following result: If $f(t)$ has no frequencies over W cycles per second, then

$$f(t)=\sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}.$$

A mathematical theory of comunication, *Bell System Tech. J., 27(1948), 379-423*.

Edmund T. Whittaker
1873-1956

Vladimir A. Kotelnikov
1908-2005

Whittaker-Shannon-Kotel'nikov theorem.

If $f(t)$ is a signal (function) band-limited to $[-\sigma, \sigma]$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(w) \mathrm{e}^{\mathrm{itw}} \mathrm{dw},$$

for some $F \in L^2(-\sigma, \sigma)$, then it can be reconstructed from its samples values at the points $t_k = k\pi/\sigma, k \in \mathbb{Z}$, via the formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}.$$

with the series being absolutely and uniformly convergent on compact sets

Augustin L. Cauchy 1789-1857

Siméon D. Poisson 1781-1840

Emile Borel 1871-1956

Jacques Hadamard 1865-1963

Charles de la Vallée Poussin 1866-1962

John M. Whittaker 1905-1984

Drawbacks in WSK

- it relies on the use of low-pass ideal filters.
- the band-limited hypothesis is in contradiction with the idea of a finite duration signal.
- the band-limiting operation generates Gibbs oscillations.
- the **sinc** function has a very slow decay at infinity which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in **several dimensions** it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \; : \; \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

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A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0 < c \le C < \infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \le \Big\| \sum_{n\in\mathbb{Z}} a_n x_n \Big\|^2 \le C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big)$$

for all $\{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$.

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A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator

If we consider

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \ldots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.

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The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} (\mathcal{L}_j f)(rm)\, S_j(t - rm)\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction sequence of functions $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 .

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A sequence $\{f_k\}_{k=1}^{\infty}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants A, $B > 0$ (frame bounds) such that

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}$$

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Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$$\big(T_a f\big)(t) = f(t-a) \qquad\qquad \big(M_a f\big)(t) = f(t)e^{\text{iat}}$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k\in\mathbb{Z}\,,$$

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- The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

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- Characterize the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ as a **frame (Riesz basis)** in $\mathcal{A}_a$.

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- Characterize the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ as a **frame (Riesz basis)** in $\mathcal{A}_a$.
- Look for those **dual frames** having the same form $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ for some $c_j \in \mathcal{A}_a$, so that, for any $x \in \mathcal{A}_a$ the **expansion**

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk}b_j\rangle\, U^{rk}c_j \quad \text{in } \mathcal{H}$$

holds.

Remark

In the shift-invariant case, U is defined as the shift operator $U : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ and we have

$$\langle f, U^{rk} b\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} f(u)\overline{b(u-rk)}du = (f * h)(rk)\,, \quad u \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$.

For every $j = 1, 2, \ldots s$ we have the following representation

$$\langle U^k a, U^{nr} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta.$$

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Consider the $s \times 1$ matrices of functions defined on the torus $\mathbb{T} := \{e^{i\theta} \, : \, \theta \in [-\pi, \pi)\}$

$$\Phi_{a,b}(e^{i\theta}) := \begin{pmatrix} \phi_{a,b_1}(e^{i\theta}) \\ \phi_{a,b_2}(e^{i\theta}) \\ \vdots \\ \phi_{a,b_s}(e^{i\theta}) \end{pmatrix},$$

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and

$$\Psi^l_{a,b}(e^{i\theta}) := (D_r S^{-l} \Phi_{a,b})(e^{i\theta}), \quad l = 0, 1, \ldots, r-1.$$

Where $D_r, S : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denote the *decimation operator* and the *(left) shift operator* respectively

$$\sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{D_r}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{rk} e^{ik\theta} \qquad \sum_{k\in\mathbb{Z}} a_k e^{ik\theta} \stackrel{S}{\longmapsto} \sum_{k\in\mathbb{Z}} a_{k+1} e^{ik\theta}\,.$$

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Finally, defining the $s \times r$ matrix of functions on the torus $\mathbb{T}$

$$\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(e^{i\theta}) := \left(\Psi^0_{a,b}(e^{i\theta})\ \Psi^1_{a,b}(e^{i\theta})\ \ldots\ \Psi^{r-1}_{a,b}(e^{i\theta})\right),$$

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and its related constants,

$$A_{\mathbf{\Psi}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \lambda_{\min}\left[\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}{}^*(\zeta)\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(\zeta)\right];$$

$$B_{\mathbf{\Psi}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \lambda_{\max}\left[\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}{}^*(\zeta)\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}(\zeta)\right]$$

Theorem.

Let b_j be in $\mathcal{A}_a$ for $j = 1, 2, \ldots, s$ and let $\mathbf{\Psi}_{a,b}$ be the associated matrix. Then, the following results hold:

- **i)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **complete system** in $\mathcal{A}_a$ if and only if the rank of the matrix $\mathbf{\Psi}_{a,b}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.
- **ii)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **Bessel sequence** for $\mathcal{A}_a$ if and only if the constant $B_{\mathbf{\Psi}} < \infty$.
- **iii)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **frame** for $\mathcal{A}_a$ if and only if constants $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ satisfy $0 < A_{\mathbf{\Psi}} \le B_{\mathbf{\Psi}} < \infty$. In this case, $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ are the optimal frame bounds for $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$.
- **iv)** The sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a **Riesz basis** for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

The frame expansion

Taking into account the $r \times s$ matrix $\mathbf{\Gamma}_{\mathbb{U}}$ of functions on $\mathbb{T}$

$$\mathbf{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta}) := \mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) + \mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})\left[\mathbb{I}_s - \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\right],$$

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where $\mathbb{U}(e^{i\theta})$ is any $r \times s$ matrix with entries in $L^{\infty}(\mathbb{T})$, and $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}$ denotes the Moore-Penrose left-inverse of $\mathbf{\Psi}_{\boldsymbol{a},\boldsymbol{b}}$,

$$\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(e^{i\theta}) := [\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(e^{i\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(e^{i\theta})]^{-1}\mathbf{\Psi}^{*}_{\mathbf{a},\mathbf{b}}(e^{i\theta}).$$

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We can find $c_j \in \mathcal{A}_a$ such that the sequences $\{U^{kr}c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ and $\{U^{kr}b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ are a pair of **dual frames** for $\mathcal{A}_a$. Hence, for any $x \in \mathcal{A}_a$ we obtain the following recovery formula

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}\langle x, U^{kr}b_j\rangle U^{kr}c_j \quad \text{in } \mathcal{H}.$$

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In order to give sense to $U^{rk+\epsilon_{mj}} b_j$ we need to introduce a **continuous group of unitary operators** $\{U^t\}_{t\in\mathbb{R}}$, such that $U = U^1$.

A brief walk on continuous groups of unitary operators

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$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
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3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

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Classical **Stone's theorem** assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain D_T of $\mathcal{H}$.

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Notice that, whenever the self-adjoint operator T is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y\rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y\rangle\,, \quad t \in \mathbb{R}\,,$$

where $x \in D_T$ and $y \in \mathcal{H}$.

Recover $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence $\left\{\langle x, U^{rm+\epsilon_{mj}} b_j \rangle_{\mathcal{H}} \right\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$.

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Taking into account the $L^2(0,1)$ functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}\,,\ j = 1, 2, \dots, s\,,$$

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we can define the $s \times r$ matrix

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1(w) & \mathbf{g}_1(w+\frac{1}{r}) & \cdots & \mathbf{g}_1(w+\frac{r-1}{r}) \\ \mathbf{g}_2(w) & \mathbf{g}_2(w+\frac{1}{r}) & \cdots & \mathbf{g}_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s(w) & \mathbf{g}_s(w+\frac{1}{r}) & \cdots & \mathbf{g}_s(w+\frac{r-1}{r}) \end{bmatrix}$$

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$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k w}\,,\ j = 1, 2, \dots, s\,,$$

we can define the $s \times r$ matrix

$$\mathbb{G}(w) := \begin{bmatrix} \mathbf{g}_1(w) & \mathbf{g}_1(w+\frac{1}{r}) & \cdots & \mathbf{g}_1(w+\frac{r-1}{r}) \\ \mathbf{g}_2(w) & \mathbf{g}_2(w+\frac{1}{r}) & \cdots & \mathbf{g}_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ \mathbf{g}_s(w) & \mathbf{g}_s(w+\frac{1}{r}) & \cdots & \mathbf{g}_s(w+\frac{r-1}{r}) \end{bmatrix}$$

and its related the constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ given by

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

$$\beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,.$$

Theorem.

Assume that we have $0 < \alpha_{\mathbb{G}} \leq \beta_{\mathbb{G}} < \infty$, then we can find a positive number γ such that, if an error sequence $\epsilon := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ satisfies the inequality,

$$\|\epsilon\|_\infty := \max_{j=1,2,\dots,s} \sup_{m\in\mathbb{Z}} |\epsilon_{mj}| \leq \gamma$$

then there exists a **frame** $\{C^{\epsilon}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ allowing the recovery of any $x \in \mathcal{A}_a$ by means of the **sampling expansion**

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j \rangle_{\mathcal{H}}\, C^{\epsilon}_{m,j} \quad \text{in } \mathcal{H}\,.$$

THANKS.