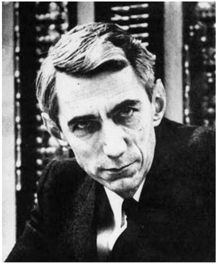
From Whittaker-Shannon-Kotelnikov theorem to shift-invariant and U-invariant sampling

Héctor R. Fernández Morales

Supervisor: **Prof. Dr. D. Antonio García García**

Departamento de Matemáticas
Universidad Carlos III de Madrid

Outline



Claude Elwood Shannon 1916-2001

Shannon's sampling theorem.

If a function of time is limited to the band from 0 to W cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/2W$ seconds apart in the manner indicated by the following result: If $f(t)$ has no frequencies over W cycles per second, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}.$$

A mathematical theory of comunication, *Bell System Tech. J., 27(1948), 379-423*.

Edmund T. Whittaker
1873-1956

Vladimir A. Kotelnikov
1908-2005

Whittaker-Shannon-Kotel'nikov theorem.

If $f(t)$ is a signal (function) band-limited to $[-\sigma, \sigma]$, i.e.,

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(w) e^{\mathrm{itw}} \mathrm{dw},$$

for some $F \in L^2(-\sigma, \sigma)$, then it can be reconstructed from its samples values at the points $t_k = k\pi/\sigma, k \in \mathbb{Z}$, via the formula

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)}.$$

with the series being absolutely and uniformly convergent on compact sets

Proof

Consider the **Paley-Wiener space** $PW_{1/2}$ of band-limited functions to $[-1/2, 1/2]$ and the Fourier transform

$$\begin{array}{rccc} \mathcal{F}: & PW_{1/2} & \longrightarrow & L^2[-1/2,1/2] \\ & f & \longmapsto & \widehat{f} \end{array}$$

Applying the inverse Fourier transform $\mathcal{F}^{-1}$ to the Fourier series $\widehat{f} = \sum_{n=-\infty}^{\infty} f(n)\, \mathrm{e}^{-2\pi i n w}$ in $L^2[-1/2,1/2]$ one gets

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} f(n)\mathcal{F}^{-1}\left[\mathrm{e}^{-2\pi i n w}\chi_{[-1/2,1/2]}(w)\right](t) \\ &= \sum_{n=-\infty}^{\infty} f(n)\, \frac{\sin \pi(t-n)}{\pi(t-n)} \quad \text{in} \quad L^2(\mathbb{R})\,. \end{aligned}$$

The pointwise convergence comes from the fact that $PW_{1/2}$ is a reproducing kernel Hilbert space where convergence in norm implies pointwise convergence (which is, in this case, uniform on $\mathbb{R}$).

Augustin L. Cauchy 1789-1857

Siméon D. Poisson 1781-1840

Emile Borel 1871-1956

Jacques Hadamard 1865-1963

Charles de la Vallée Poussin 1866-1962

John M. Whittaker 1905-1984

Drawbacks in WSK

- it relies on the use of low-pass ideal filters.
- the band-limited hypothesis is in contradiction with the idea of a finite duration signal.
- the band-limiting operation generates Gibbs oscillations.
- the **sinc** function has a very slow decay at infinity which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in **several dimensions** it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

Drawbacks in WSK

- it relies on the use of low-pass ideal filters.
- the band-limited hypothesis is in contradiction with the idea of a finite duration signal.
- the band-limiting operation generates Gibbs oscillations.
- the **sinc** function has a very slow decay at infinity which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in **several dimensions** it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

For this reason, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces.

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \;:\; \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \ : \ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

In order to have better reconstruction properties one may choose φ as the $B-$spline of order $m-1$, i.e.

$$\varphi(t) = N_m(t) := \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{m \text{ times}} .$$

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi \in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n \, \varphi(t-n) \; : \; \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \Big\}$$

In order to have better reconstruction properties one may choose φ as the $B-$spline of order $m-1$, i.e.

$$\varphi(t) = N_m(t) := \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{m \text{ times}} .$$

- $\mathcal{L}_j f := f * \mathrm{h}_j, \;\; j = 1, 2, \ldots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.
- The sequence of samples $\{(\mathcal{L}_j f)(rm)\}_{m\in\mathbb{Z};\, j=1,2,\ldots,s}$ where $r \in \mathbb{N}$, is available for any f in V_φ^2.

Generalized sampling problem in shift-invariant subspaces

The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \big(\mathcal{L}_j f\big)(rm)\, S_j(t-rm)\,, \quad t \in \mathbb{R}\,,$$

where the reconstruction sequence $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 .

Generalized sampling problem in shift-invariant subspaces

The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} (\mathcal{L}_j f)(rm)\, S_j(t-rm)\,, \quad t\in\mathbb{R}\,,$$

where the reconstruction sequence $\{S_j(\cdot - rm)\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for V_φ^2 .

- *A. G. García, G. Pérez-Villalón.* **Dual frames in $L^2(0,1)$ connected with generalized sampling in shift-invariant spaces.** *Appl. Comput. Harmon. Anal.*, 20(3):422–433, 2006.
- *H. R. Fernández-Morales, A. G. García, G. Pérez-Villalón.* **Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators.** *In Multiscale Signal Analysis and Modeling*, pp. 51–80, Lecture Notes in Electrical Engineering, Springer, New York, 2012.

Frames and Riesz bases

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0 < c \leq C < \infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \leq \Big\|\sum_{n\in\mathbb{Z}} a_n x_n\Big\|^2 \leq C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \quad \text{for all} \quad \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

Frames and Riesz bases

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0 < c \leq C < \infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \leq \Big\|\sum_{n\in\mathbb{Z}} a_n x_n\Big\|^2 \leq C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \quad \text{for all} \quad \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator.

Frames and Riesz bases

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0 < c \le C < \infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \le \Big\|\sum_{n\in\mathbb{Z}} a_n x_n\Big\|^2 \le C\Big(\sum_{n\in\mathbb{Z}} |a_n|^2\Big) \quad \text{for all} \quad \{a_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}).$$

A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator.

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \le \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \le B\|x\|^2 \quad \text{for all } x \in \mathcal{H}.$$

Generalize this problem in the following sense:

Consider a **continuous group of unitary operators**

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Generalize this problem in the following sense:

Consider a **continuous group of unitary operators**

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

For a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\text{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}.$$

Generalize this problem in the following sense:

Consider a **continuous group of unitary operators**

$\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying:

1. $U^t\, U^{t'} = U^{t+t'}$,
2. $U^0 = I_{\mathcal{H}}$,
3. $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

For a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\text{span}}\big\{U^n a,\ n \in \mathbb{Z}\big\}.$$

In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{\sum_{n\in\mathbb{Z}} \alpha_n\, U^n a :\ \{\alpha_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\Big\}.$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k \in \mathbb{Z}\,,$$

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k \in \mathbb{Z}\,,$$

- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$.

The sequence $\{U^n a\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k) := \langle U^k a, a\rangle_{\mathcal{H}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} d\mu_a(\theta)\,, \qquad k\in\mathbb{Z}\,,$$

- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu_a^s(\theta)$.

- The sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s \equiv 0$ and

$$0 < \operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)} \phi_a(\theta) \le \operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)} \phi_a(\theta) < \infty\,.$$

Thus, for $b \in \mathcal{H}$ we consider the linear operator $\mathcal{H} \ni x \longmapsto \mathcal{L}_b x \in C(\mathbb{R})$ such that $(\mathcal{L}_b x)(t) := \langle x, U^t b \rangle_{\mathcal{H}}$ for every $t \in \mathbb{R}$.

Thus, for $b \in \mathcal{H}$ we consider the linear operator $\mathcal{H} \ni x \longmapsto \mathcal{L}_b x \in C(\mathbb{R})$ such that $(\mathcal{L}_b x)(t) := \langle x, U^t b \rangle_{\mathcal{H}}$ for every $t \in \mathbb{R}$.

Generalized sampling problem in U-invariant subspaces

Given $\mathcal{L}_j$, $j = 1, 2, \ldots, s$, corresponding to s elements $b_j \in \mathcal{H}$, i.e., $\mathcal{L}_j \equiv \mathcal{L}_{b_j}$ for each $j = 1, 2, \ldots, s$, the generalized regular sampling problem in $\mathcal{A}_a$ consists of the stable recovery of any $x \in \mathcal{A}_a$ from the sequence of the samples

$$\big\{\mathcal{L}_j x(rm)\big\}_{m \in \mathbb{Z};\, j=1,2,\ldots s} \quad \text{where} \quad r \in \mathbb{N}\,, r \geq 1\,.$$

by means of a sampling expansion like

$$x = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H},$$

where $c_j \in \mathcal{A}_a$ and $\big\{U^{rm} c_j\big\}_{m \in \mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$.

Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$$\big(T_t\, f\big)(u) = f(u - t) \qquad\qquad \big(M_t\, f\big)(u) = f(u)e^{\mathrm{itu}}$$

Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$$\big(T_t f\big)(u) = f(u-t) \qquad\qquad \big(M_t f\big)(u) = f(u)e^{\mathrm{itu}}$$

Remark

In the shift-invariant case, U is defined as the shift operator $U : f(u) \mapsto f(u-1)$ in $L^2(\mathbb{R})$ and we have

$$\langle f, U^t b\rangle_{\mathcal{H}} = \int_{-\infty}^{\infty} f(u)\overline{b(u-t)}du = (f * h)(t)\,, \quad t \in \mathbb{R}\,,$$

where $h(u) := \overline{b(-u)}$.

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\begin{array}{rccc} \mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & F=\sum_{n\in\mathbb{Z}}\alpha_n\,\mathrm{e}^{2\pi i n w} & \longmapsto & x=\sum_{n\in\mathbb{Z}}\alpha_n\,U^n a\,. \end{array}$$

We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{\mathrm{e}^{2\pi i n w}\}_{n\in\mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$, that is,

$$\begin{array}{rccc} \mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & F=\sum_{n\in\mathbb{Z}}\alpha_n\,\mathrm{e}^{2\pi i n w} & \longmapsto & x=\sum_{n\in\mathbb{Z}}\alpha_n\,U^n a\,. \end{array}$$

An expression for the samples

$$\mathcal{L}_j x(rm)=\left\langle F,\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\right\rangle_{L^2(0,1)} \quad \text{for } m\in\mathbb{Z} \text{ and } j=1,2,\dots,s\,,$$

where the function

$$g_j(w):=\sum_{k\in\mathbb{Z}}\mathcal{L}_j a(k)\,\mathrm{e}^{2\pi i k w}$$

belongs to $L^2(0,1)$ for each $j=1,2,\dots,s$.

the $s \times r$ matrix of functions in $L^2(0,1)$

$$\mathbb{G}(w) := \begin{bmatrix} g_1(w) & g_1(w+\frac{1}{r}) & \cdots & g_1(w+\frac{r-1}{r}) \\ g_2(w) & g_2(w+\frac{1}{r}) & \cdots & g_2(w+\frac{r-1}{r}) \\ \vdots & \vdots & & \vdots \\ g_s(w) & g_s(w+\frac{1}{r}) & \cdots & g_s(w+\frac{r-1}{r}) \end{bmatrix}$$

and its related constants

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)],$$

Theorem.

For the functions $g_j \in L^2(0,1)$, $j = 1, 2, \dots, s$, consider the associated matrix $\mathbb{G}(w)$. Then, the following results hold:

(a) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};\,j=1,2,\dots,s}$ is a **complete system** for $L^2(0,1)$ if and only if the rank of the matrix $\mathbb{G}(w)$ is r a.e. in $(0, 1/r)$.

(b) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};\,j=1,2,\dots,s}$ is a **Bessel sequence** for $L^2(0,1)$ if and only if $g_j \in L^\infty(0,1)$ (or equivalently $\beta_{\mathbb{G}} < \infty$). In this case, the optimal Bessel bound is $\beta_{\mathbb{G}}/r$.

(c) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};\,j=1,2,\dots,s}$ is a **frame** for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$. In this case, the optimal frame bounds are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.

(d) The sequence $\{\overline{g_j(w)}\,\mathrm{e}^{2\pi\mathrm{i}rnw}\}_{n\in\mathbb{Z};\,j=1,2,\dots,s}$ is a **Riesz basis** for $L^2(0,1)$ if and only if is a frame and $s = r$.

Theorem.

Let b_j be in $\mathcal{H}$ and let $\mathcal{L}_j$ be its associated U-system for $j = 1, 2, \dots, s$. Assume that the function g_j, $j = 1, 2, \dots, s$ belongs to $L^\infty(0, 1)$; or equivalently, $\beta_{\mathbb{G}} < \infty$ for the associated $s \times r$ matrix $\mathbb{G}(w)$.
The following statements are equivalent:

(a) $\alpha_{\mathbb{G}} > 0$.

(b) There exists a vector $\big[h_1(w), h_2(w), \dots, h_s(w)\big]$ with entries in $L^\infty(0, 1)$ satisfying

$$\big[h_1(w), h_2(w), \dots, h_s(w)\big]\mathbb{G}(w) = [1, 0, \dots, 0] \quad \text{a.e. in } (0, 1)\,.$$

(c) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, such that the sequence $\big\{U^{rm}c_j\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \mathcal{L}_j x(rm)\, U^{rm} c_j \quad \text{in } \mathcal{H}\,,$$

holds.

Comments and further work

- All the possible vectors $\big[h_1(w), h_2(w), \ldots, h_s(w)\big]$ satisfying the previous equation are given by the first row of the $r \times s$ matrices

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^{\dagger}(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^{\dagger}(w)\big]\,,$$

where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^{\infty}(0,1)$, and $\mathbb{G}^{\dagger}(w)$ is the Moore-Penrose left-inverse of $\mathbb{G}(w)$, $\mathbb{G}^{\dagger}(w) := \big[\mathbb{G}^{*}(w)\,\mathbb{G}(w)\big]^{-1}\,\mathbb{G}^{*}(w)\,.$

Comments and further work

- All the possible vectors $\big[h_1(w), h_2(w), \dots, h_s(w)\big]$ satisfying the previous equation are given by the first row of the $r \times s$ matrices

$$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^{\dagger}(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^{\dagger}(w)\big]\,,$$

  where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^{\infty}(0,1)$, and $\mathbb{G}^{\dagger}(w)$ is the Moore-Penrose left-inverse of $\mathbb{G}(w)$, $\mathbb{G}^{\dagger}(w) := \big[\mathbb{G}^{*}(w)\,\mathbb{G}(w)\big]^{-1}\,\mathbb{G}^{*}(w)\,.$
- Whenever the sampling period r equals the number of U-systems s we are in the presence of Riesz bases, and there exists a unique sampling expansion.

Comments and further work

- All the possible vectors $\big[h_1(w), h_2(w), \dots, h_s(w)\big]$ satisfying the previous equation are given by the first row of the $r \times s$ matrices

  $$\mathbb{H}_{\mathbb{U}}(w) := \mathbb{G}^{\dagger}(w) + \mathbb{U}(w)\big[\mathbb{I}_s - \mathbb{G}(w)\mathbb{G}^{\dagger}(w)\big]\,,$$

  where $\mathbb{U}(w)$ denotes any $r \times s$ matrix with entries in $L^{\infty}(0,1)$, and $\mathbb{G}^{\dagger}(w)$ is the Moore-Penrose left-inverse of $\mathbb{G}(w)$, $\mathbb{G}^{\dagger}(w) := \big[\mathbb{G}^{*}(w)\,\mathbb{G}(w)\big]^{-1}\,\mathbb{G}^{*}(w)\,.$
- Whenever the sampling period r equals the number of U-systems s we are in the presence of Riesz bases, and there exists a unique sampling expansion.
- The group of unitary operator allows us to work with non integer powers of U, so we can attack the irregular sampling problem if we dispose of a perturbed sequence of samples $\{(\mathcal{L}_j x)(rm + \epsilon_{mj})\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, with errors $\epsilon_{mj} \in \mathbb{R}$.

- *H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo.* **Generalized sampling: from shift-invariant to U-invariant spaces.** Submitted 2013.
- *H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo.* **On some sampling-related frames in U-invariant spaces.** Submitted 2013.

- *H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo.* **Generalized sampling: from shift-invariant to U-invariant spaces.** Submitted 2013.
- *H. R. Fernández-Morales, A. G. García, M. A. Hernández-Medina and M. J. Muñoz-Bouzo.* **On some sampling-related frames in U-invariant spaces.** Submitted 2013.

THANKS.