From Shannon’s sampling theory to regular and irregular $U$-invariant sampling

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Abstract

The classical Whittaker-Shannon-Kotel’nikov theorem states that any function with compact supported Fourier transforms is completely determined by its ordinates at a series of equally spaced points. This revolutionary result has had an enormous impact due to its applications in many branches of applied mathematics. Regularized signals are assumed to belong to some shift-invariant subspaces of $L^2(\mathbb{R})$. Besides, in many common situations the available data of a signal are samples of some filtered version of the signal itself. This leads to the problem of generalized sampling in shift-invariant spaces, i.e., to recover any function in this subspace by means of its samples. A more general problem is to consider subspaces of a Hilbert space generated by an orthonormal operator $U$. The goal of this work is to give a survey on the history of the WSK theorem and conclude with some results in shift-invariant and $U$-invariant sampling.

Whittaker-Shannon-Kotel’nikov theorem

Shannon’s sampling theorem: If a function of time is limited to the band from $0$ to $W$ cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/(2W)$ seconds apart in the manner indicated by the following result. If $f(t)$ has no frequencies over $1/W$ cycles per second, then

$$f(t) = \sum_{n=-\infty}^{\infty} f(nW) \frac{\sin \pi (t-nW)}{\pi (t-nW)}.$$


Whittaker-Shannon-Kotel’nikov theorem: If $f(t)$ is a signal (function) band-limited to $[−W,W]$ i.e.,

$$|\hat{f}(\omega)| \leq \frac{a}{|\omega|} \text{ for } \omega \neq 0,$$

for some $a \in (0,\infty)$, then it can be reconstructed from its samples at the points $t_k = kT = k/T_0 \in \mathbb{R}$, via the formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT_0) \frac{\sin \pi (t-t_0)}{\pi (t-t_0)},$$

with the series being absolutely and uniformly convergent on compact sets.

Drawbacks in WSK

• it relies on the use of low-pass ideal filters.
• the band-limit hypothesis is in contradiction with the idea of a finite duration signal.
• the band-limiting operation generates Gibbs oscillations.
• the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient.
• the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
• in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval.

For this reason, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces.

Frames and Riesz bases.

A sequence $(x_n)_{n \in \mathbb{Z}}$ in a separable Hilbert space $H$ is called a Riesz sequence if there exists constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq \left\| \sum_{n \in \mathbb{Z}} x_n \phi_n \right\|^2 \leq \beta \left\| \sum_{n \in \mathbb{Z}} x_n \phi_n \right\|^2 \text{ for all } x \in H.$$

A Riesz sequence $(x_n)_{n \in \mathbb{Z}}$ in $H$ is a frame for a separable Hilbert space $H$ if there exist constants $A, B > 0$ (frame bounds) such that

$$A \|x\|^2 \leq \left\| \sum_{n \in \mathbb{Z}} \langle x, \phi_n \rangle \phi_n \right\|^2 \leq B \|x\|^2 \text{ for all } x \in H.$$

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $f \in L^2(\mathbb{R})$ if the sequence $(\hat{f}(n))_{n \in \mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space $V_2^f$

$$V_2^f = \{\sum_{n \in \mathbb{Z}} a_n \phi_{n-k} : (a_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})\}.$$

In order to have better reconstruction properties one may choose $\phi$ as the $D$-spline of order $m - 1$, i.e.,

$$\phi_n(t) = \frac{1}{m!} \left(\frac{\sin \pi t}{\pi t}\right)^m.$$

• $\mathcal{L}_f : f \mapsto h$, $j = 1, 2, \ldots$ are convolutions systems (linear time-invariant systems) defined on $V_2^f$.

• The sequence of samples $(\mathcal{L}_f \{f(nm)\}_{m \in \mathbb{Z}})_{|m| \geq 1, j \geq 1, n \in \mathbb{N}}$ is available for any $f \in V_2^f$.

Consider the linear time-invariant systems $\mathcal{L}_f$, $j = 1, 2, \ldots$ defined on $V_2^f$

Generalized sampling problem in shift-invariant subspaces.

The generalized sampling problem is to obtain sampling formulas in $V_2^f$ having the form

$$f(t) = \sum_{n \in \mathbb{Z}} \mathcal{L}_f \{ f(nm) \} \phi_{n-k} : t \in \mathbb{R},$$

where the reconstruction sequence $(\mathcal{L}_f \{ f(nm) \} \phi_{n-k})_{|m| \geq 1, j \geq 1, n \in \mathbb{N}}$ is a frame for $V_2^f$.

The sampling period $r$ necessarily satisfies $r \leq s$. The results concerning this problem can be found in the references [3] and [4] below.