From Shannon's sampling theory to regular and irregular U-invariant sampling

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Abstract

The classical Whittaker-Shannon-Kotel'nikov theorem states that any function with compact supported Fourier transform is completely determined by its ordinates at a series of equally spaced points. This revolutionary result has had an enormous impact due to its applications in many many branches of applied mathematics. Nowadays signals are assumed to belong to some shift-invariant subspace of $L^2(\mathbb{R})$, besides, in many common situations the available data of a signal are samples of some filtered versions of the signal itself. This leads to the problem of generalized sampling in shift-invariant spaces, i.e., to recover any function in this subspaces by means of its samples. A more general problem is to consider subspaces of a Hilbert space generated by an unitary operator U. The goal of this work is to give a survey on the history of the WSK theorem and conclude with some results in shift-invariant and U-invariant sampling.

Whittaker-Shannon-Kotel'nikov theorem

Shannon's sampling theorem: If a function of time is limited to the band from 0 to W cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/2W$ seconds apart in the manner indicated by the following result: If $f(t)$ has no frequencies over W cycles per second, then

$$f(t)=\sum_{n=-\infty}^{\infty} f\Big(\frac{n}{2W}\Big)\frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}.$$

A mathematical theory of comunication, *Bell System Tech. J., 27(1948), 379-423.*

Whittaker-Shannon-Kotel'nikov theorem: If $f(t)$ is a signal (function) band-limited to $[-\sigma,\sigma]$, i.e.,

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma} F(w)\mathrm{e}^{\mathrm{i}tw}\mathrm{d}w,$$

for some $F\in L^2(-\sigma,\sigma)$, then it can be reconstructed from its samples values at the points $t_k=k\pi/\sigma, k\in\mathbb{Z}$, via the formula

$$f(t)=\sum_{k=-\infty}^{\infty} f(t_k)\frac{\sin \sigma(t-t_k)}{\sigma(t-t_k)}.$$

with the series being absolutely and uniformly convergent on compact sets

Drawbacks in WSK

- it relies on the use of low-pass ideal filters.
- the band-limited hypothesis is in contradiction with the idea of a finite duration signal.
- the band-limiting operation generates Gibbs oscillations.
- the sinc function has a very slow decay at infinity which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a d-dimensional interval

For this reason, the sampling and reconstruction problems have been investigated in spline spaces, wavelet spaces, and general shift-invariant spaces.

Frames and Riesz bases.

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is called a **Riesz sequence** if there exists constants $0<c\le C<\infty$ such that

$$c\Big(\sum_{n\in\mathbb{Z}}|a_n|^2\Big)\le\Big\|\sum_{n\in\mathbb{Z}}a_nx_n\Big\|^2\le C\Big(\sum_{n\in\mathbb{Z}}|a_n|^2\Big)\quad\forall\{a_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z}).$$

A **Riesz basis** in a separable Hilbert space $\mathcal{H}$ is the image of an orthonormal basis by means of a bounded invertible operator.
A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants $A,B>0$ (frame bounds) such that

$$A\|x\|^2\le\sum_{n\in\mathbb{Z}}|\langle x,x_n\rangle|^2\le B\|x\|^2\quad\text{for all } x\in\mathcal{H}.$$

Generalized sampling problem in shift-invariant subspaces of $L^2(\mathbb{R})$.

Assume that $\varphi\in L^2(\mathbb{R})$; if the sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence for $L^2(\mathbb{R})$, then we can define the shift-invariant space V_φ^2

$$V_\varphi^2=\Big\{\sum_{n\in\mathbb{Z}}\alpha_n\,\varphi(t-n)\,:\,\{\alpha_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\Big\}$$

In order to have better reconstruction properties one may choose φ as the $B-$spline of order $m-1$, i.e.

$$\varphi(t)=N_m(t):=\underbrace{\chi_{[0,1]}*\chi_{[0,1]}*\cdots*\chi_{[0,1]}}_{m\ times}.$$

- $\mathcal{L}_jf:=f*\mathrm{h}_j,\quad j=1,2,\dots s$ are convolutions systems (linear time-invariant systems) defined on V_φ^2.
- The sequence of samples $\{(\mathcal{L}_jf)(rm)\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ where $r\in\mathbb{N}$, is available for any f in V_φ^2.

Consider s linear-time invariant systems $\mathcal{L}_j$, $j=1,2,\dots,s$ defined on V_φ^2

Generalized sampling problem in shift-invariant subspaces:
The generalized sampling problem is to obtain sampling formulas in V_φ^2 having the form

$$f(t)=\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}(\mathcal{L}_jf)(rm)\,S_j(t-rm),\quad t\in\mathbb{R},$$

where the reconstruction sequence $\{S_j(\cdot-rm)\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for V_φ^2.
(The sampling period r necessarily satisfies $r\le s$)

The results concerning this problem can be found in the references [3] and [4] below. See also references therein for previous and related works.

Generalized sampling problem in U-invariant spaces.

Consider a **continuous group of unitary operators** $\{U^t\}_{t\in\mathbb{R}}$ i.e. a family of unitary operators in $\mathcal{H}$ satisfying:
1. $U^tU^{t'}=U^{t+t'}$,
2. $U^0=I_{\mathcal{H}}$,
3. $\langle U^tx,y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x,y\in\mathcal{H}$.

For a fixed $a\in\mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a:=\overline{\operatorname{span}}\{U^na,\ n\in\mathbb{Z}\}.$$

In case that the sequence $\{U^na\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a=\Big\{\sum_{n\in\mathbb{Z}}\alpha_nU^na\,:\,\{\alpha_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\Big\}.$$

The sequence $\{U^na\}_{n\in\mathbb{Z}}$

- The *auto-covariance* function admits the integral representation

$$R_a(k):=\langle U^ka,a\rangle_{\mathcal{H}}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}k\theta}d\mu_a(\theta),\qquad k\in\mathbb{Z},$$

- The positive Borel spectral measure μ_a can be decomposed as $d\mu_a(\theta)=\phi_a(\theta)d\theta+d\mu_a^s(\theta)$.
- The sequence $\{U^na\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu_a^s\equiv 0$ and

$$0<\operatorname*{ess\,inf}_{\theta\in(-\pi,\pi)}\phi_a(\theta)\le\operatorname*{ess\,sup}_{\theta\in(-\pi,\pi)}\phi_a(\theta)<\infty.$$

Thus, for $b\in\mathcal{H}$ we consider the linear operator $\mathcal{H}\ni x\longmapsto\mathcal{L}_bx\in C(\mathbb{R})$ such that $\big(\mathcal{L}_bx\big)(t):=\langle x,U^tb\rangle_{\mathcal{H}}$ for every $t\in\mathbb{R}$.

Generalized sampling problem in U-invariant subspaces: Given $\mathcal{L}_j$, $j=1,2,\dots,s$, corresponding to s elements $b_j\in\mathcal{H}$, i.e., $\mathcal{L}_j\equiv\mathcal{L}_{b_j}$ for each $j=1,2,\dots,s$, the generalized regular sampling problem in $\mathcal{A}_a$ consists of the stable recovery of any $x\in\mathcal{A}_a$ from the sequence of the samples

$$\{\mathcal{L}_jx(rm)\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}\quad\text{where}\quad r\in\mathbb{N}, r\ge 1.$$

by means of a sampling expansion like

$$x=\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\mathcal{L}_jx(rm)\,U^{rm}c_j\quad\text{in }\mathcal{H},$$

where $c_j\in\mathcal{A}_a$ and $\{U^{rm}c_j\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$.

Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$$(T_tf)(u)=f(u-t)\qquad (M_tf)(u)=f(u)\mathrm{e}^{\mathrm{i}tu}$$

Remark: In the shift-invariant case, U is defined as the shift operator $U:f(u)\mapsto f(u-1)$ in $L^2(\mathbb{R})$ and we have

$$\langle f,U^tb\rangle_{\mathcal{H}}=\int_{-\infty}^{\infty}f(u)\overline{b(u-t)}du=(f*h)(t),\quad t\in\mathbb{R},$$

where $h(u):=\overline{b(-u)}$.

Time-jitter error study

The **continuous group of unitary operators** $\{U^t\}_{t\in\mathbb{R}}$, such that $U=U^1$ allows us to deal with $U^{rk+\epsilon_{mj}}b_j$, where $\{\epsilon_{mj}\}_{m\in\mathbb{Z};\,j=1,2,\dots s}$ is a sequence of errors. Then, the natural irregular sampling problem arises

Irregular sampling problem:
- Study the perturbed sequence $\big\{U^{rk+\epsilon_{kj}}b_j\big\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$.
- Recover $x\in\mathcal{A}_a$ from the perturbed sequence of samples $\big\{\langle x,U^{rk+\epsilon_{kj}}b_j\rangle\big\}_{k\in\mathbb{Z};\,j=1,2,\dots,s}$.

The results concerning the regular and irregular U-sampling problems can be found in the references [1] and [2] below.

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