Sampling Theory in Shift-Invariant Spaces: Generalizations

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Outline

Introduction to sampling theory

Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

Uniform average sampling in frame generated weighted shift-invariant spaces

Sampling theory in $U$-invariant spaces
Introduction


Shannon’s sampling theorem.

If a function of time is limited to the band from 0 to $W$ cycles per second, it is completely determined by giving its ordinates at a series of discrete points spaced $1/2W$ seconds apart in the manner indicated by the following result: If $f(t)$ has no frequencies over $W$ cycles per second, then

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}$$

Whittaker-Shannon-Kotel’nikov theorem.

If \( f(t) \) is a signal (function) band-limited to \([-\sigma, \sigma]\), i.e.,

\[
f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} F(w)e^{itw} \, dw, \quad t \in \mathbb{R}
\]

for some \( F \in L^2(-\sigma, \sigma) \), then it can be reconstructed from its samples values at the points \( t_k = k\pi/\sigma, \, k \in \mathbb{Z} \), via the formula

\[
f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\sin \sigma(t-t_k)}{\sigma(t-t_k)}, \quad t \in \mathbb{R}
\]

with the series being absolutely and uniformly convergent on compact sets.

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Drawbacks in WSK theorem

- it relies on the use of **low-pass ideal filters**.
- the **band-limited hypothesis** is in contradiction with the idea of a **finite duration signal**.
- the band-limiting operation generates Gibbs oscillations.
- the sinc function has a **very slow decay at infinity** which makes computation in the signal domain very inefficient.
- the sinc function is well-localized in the frequency domain but it is bad-localized in the time domain.
- in several dimensions it is also inefficient to assume that a multidimensional signal is band-limited to a $d$-dimensional interval.
Chapter 1: Motivation

Paley-Wiener space

The space of band limited functions to the interval \([-\pi, \pi]\) can be written as

\[ PW_{\pi} = \left\{ \sum_{n \in \mathbb{Z}} a_n \text{sinc}(t - n) : \{a_n\} \in l^2(\mathbb{Z}) \right\} \]

Furthermore, the coefficients \(\{a_n\}_{n \in \mathbb{Z}}\) of \(f \in PW_{\pi}\) are precisely the samples of the function at the integers numbers \(\{f(n)\}_{n \in \mathbb{Z}}\).
Chapter 1: Motivation

Paley-Wiener space

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Furthermore, the coefficients $\{a_n\}_{n \in \mathbb{Z}}$ of $f \in PW_{\pi}$ are precisely the samples of the function at the integers numbers $\{f(n)\}_{n \in \mathbb{Z}}$.

Shift-invariant space

$$V^2_{\varphi} = \left\{ \sum_{n \in \mathbb{Z}} a_n \varphi(t - n) : \{a_n\} \in l^2(\mathbb{Z}) \right\}$$
Generalized sampling in $L^2(\mathbb{R}^d)$ shift-invariant subspaces with multiple stable generators

On the separable Hilbert $L^2(\mathbb{R}^d)$ we can define the shift-invariant subspaces $V^2_\Phi$ in the following way

$$V^2_\Phi := \text{span} L^2(\mathbb{R}^d) \{ \phi_k(t - \alpha) : k = 1, 2, ..., r \text{ and } \alpha \in \mathbb{Z}^d \},$$

where the functions in $\Phi := \{ \phi_1, ..., \phi_r \}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for $V^2_\Phi$.

Assuming that the sequence $\{ \phi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d; k = 1, 2, ...}$ is a Riesz sequence, i.e. a Riesz basis for its span, this space can be described as

$$V^2_\Phi = \{ \sum_{\alpha \in \mathbb{Z}^d} d_k(\alpha) \phi_k(t - \alpha) : d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2, ..., r \}.$$
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On the separable Hilbert $L^2(\mathbb{R}^d)$ we can define the shift-invariant subspaces $V^2_\Phi$ in the following way

$$V^2_\Phi := \text{span}_{L^2(\mathbb{R}^d)} \{ \varphi_k(t - \alpha) : k = 1, 2, \ldots, r \text{ and } \alpha \in \mathbb{Z}^d \},$$

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where the functions in $\Phi := \{ \varphi_1, \ldots, \varphi_r \}$ in $L^2(\mathbb{R}^d)$ are called a set of generators for $V^2_\Phi$. Assuming that the sequence $\{ \varphi_k(t - \alpha) \}_{\alpha \in \mathbb{Z}^d ; k = 1, 2, \ldots, r}$ is a Riesz sequence, i.e. a Riesz basis for its span, this space can be described as

$$V^2_\Phi = \left\{ \sum_{\alpha \in \mathbb{Z}^d} \sum_{k=1}^r d_k(\alpha) \varphi_k(t - \alpha) : d_k \in \ell^2(\mathbb{Z}^d), k = 1, 2 \ldots, r \right\}.$$
Generalized sampling problem
Generalized sampling problem

If we consider

- $\mathcal{L}_j f := f \ast h_j, \quad j = 1, 2, \ldots s$ are **convolutions systems** (linear time-invariant systems) defined on $V_\varphi^2$.
Generalized sampling problem

If we consider

- \( L_j f := f \ast h_j, \ j = 1, 2, \ldots s \) are **convolutions systems** (linear time-invariant systems) defined on \( V_2^\varphi \).
- The samples \( \{ L_j f(M\alpha) \}_{\alpha \in \mathbb{Z}^d; j = 1, 2, \ldots s} \) are taken at a lattice

\[ \Lambda_M := \{ M\alpha : \alpha \in \mathbb{Z}^d \} \subset \mathbb{Z}^d. \]

where \( M \) is a nonsingular matrix with integer entries.
Generalized sampling problem

If we consider

- $\mathcal{L}_j f := f \ast h_j$, $j = 1, 2, \ldots, s$ are convolutions systems (linear time-invariant systems) defined on $V^2_{\varphi}$.
- The samples $\{\mathcal{L}_j f(M\alpha)\}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}$ are taken at a lattice

$$\Lambda_M := \{M\alpha : \alpha \in \mathbb{Z}^d\} \subset \mathbb{Z}^d.$$ where $M$ is a nonsingular matrix with integer entries.

The generalized sampling problem is to obtain sampling formulas in $V^2_{\varphi}$ having the form

$$f = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_j(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d),$$

where the reconstruction sequence of functions $\{S_j(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}$ is a frame for $V^2_{\varphi}$. 
Definition

A sequence \( \{f_k\}_{k \in \mathbb{Z}} \) is a **frame** for a separable Hilbert space \( \mathcal{H} \) if there exist constants \( A, B > 0 \) (frame bounds) such that

\[
A \|f\|^2 \leq \sum_{k \in \mathbb{Z}} |\langle f, f_k \rangle|^2 \leq B \|f\|^2 \quad \text{for all } f \in \mathcal{H}
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**Definition**

Two frames sequences \( \{f_k\}_{k \in \mathbb{Z}} \) and \( \{g_k\}_{k \in \mathbb{Z}} \) which satisfy

\[
f = \sum_{k \in \mathbb{Z}} \langle f, g_k \rangle f_k = \sum_{k \in \mathbb{Z}} \langle f, f_k \rangle g_k, \quad \text{for all } f \in \mathcal{H}.
\]

are said to be a pair of **dual frames**.
Our general technique

- Consider an isomorphism $\mathcal{T}_\Phi : L^2 \to V^2_\Phi$ such that
  
  $$Samp(f)_k = \langle F, g_k \rangle_{L^2}, \quad f = \mathcal{T}_\Phi F$$

- Characterize $\{g_k\}$ as a frame for $L^2$

- Find a dual frame $\{h_k\}$

  $$F = \sum_k \langle F, g_k \rangle_{L^2} h_k$$

- Apply $\mathcal{T}_\Phi$ to get the sampling formula

  $$f = \sum_k Samp(f)_k \mathcal{T}_\Phi h_k$$
Sketch of the procedure

(i) Consider the isomorphism $T_\Phi: L^2([0,1)_d) \rightarrow V^2 \Phi \{ e^{-2\pi i \alpha^\top w}e^k \} \mapsto \{ \phi_k(t-\alpha) \}$

Verify the shifting property for $F \in L^2([0,1)_d)$ and $\alpha \in \mathbb{Z}^d$

$T_\Phi [F(\cdot)e^{-2\pi i \alpha^\top \cdot}] (t) = T_\Phi F(t-\alpha), t \in \mathbb{R}^d$.

(ii) Deduce the expression for the samples $(L_j f)(M_\alpha) = \langle F, g_j(\cdot)e^{-2\pi i \alpha^\top M^\top \cdot} \rangle_{L^2([0,1)_d)}$, where $F = T^{-1}_\Phi f \in L^2([0,1)_d)$.

(iii) Characterize the sequence $\{ g_j(x)e^{-2\pi i \alpha^\top M^\top x} \}_{\alpha \in \mathbb{Z}^d; j = 1, 2, \ldots, s}$ as a frame in $L^2([0,1)_d)$.
Sketch of the procedure

(i) Consider the isomorphism

\[ T_{\Phi} : \quad L^2_r[0, 1)^d \quad \rightarrow \quad V^2_{\Phi} \]

\[ \{ e^{-2\pi i \alpha^\top w} e_k \} \quad \mapsto \quad \{ \varphi_k(t - \alpha) \} \]

Verify the shifting property for \( F \in L^2_r[0, 1)^d \) and \( \alpha \in \mathbb{Z}^d \)

\[ T_{\Phi} [F(\cdot)e^{-2\pi i \alpha^\top \cdot}](t) = T_{\Phi} F(t - \alpha), \quad t \in \mathbb{R}^d. \]
**Sketch of the procedure**

(i) Consider the isomorphism

$$
\mathcal{T}_\Phi : \ L^2_r[0, 1)^d \quad \longrightarrow \quad V^2_{\Phi} \\
\{ e^{-2\pi i \alpha^\top w} e_k \} \quad \longmapsto \quad \{ \varphi_k(t - \alpha) \}
$$

Verify the **shifting property** for $F \in L^2_r[0, 1)^d$ and $\alpha \in \mathbb{Z}^d$

$$
\mathcal{T}_\Phi [F(\cdot) e^{-2\pi i \alpha^\top \cdot}] (t) = \mathcal{T}_\Phi F(t - \alpha), \quad t \in \mathbb{R}^d.
$$

(ii) Deduce the expression for the samples

$$
(L_j f)(M\alpha) = \langle F, \overline{g_j(\cdot) e^{-2\pi i \alpha^\top M^\top}} \rangle_{L^2_r[0,1)^d},
$$

where $F = \mathcal{T}_\Phi^{-1} f \in L^2_r[0, 1)^d$.
Sketch of the procedure

(i) Consider the isomorphism

$$\mathcal{T}_\Phi : \quad L^2_r[0, 1)^d \rightarrow V^2_{\Phi}$$
$$\{ e^{-2\pi i \alpha^T w} e_k \} \mapsto \{ \varphi_k(t - \alpha) \}$$

Verify the shifting property for $F \in L^2_r[0, 1)^d$ and $\alpha \in \mathbb{Z}^d$

$$\mathcal{T}_\Phi [F(\cdot)e^{-2\pi i \alpha^T \cdot}](t) = \mathcal{T}_\Phi F(t - \alpha), \quad t \in \mathbb{R}^d.$$  

(ii) Deduce the expression for the samples

$$\left( L_j f \right)(M\alpha) = \langle F, \overline{g_j(\cdot)e^{-2\pi i \alpha^T M^T \cdot}} \rangle_{L^2_r[0,1)^d},$$

where $F = \mathcal{T}_\Phi^{-1} f \in L^2_r[0, 1)^d$.

(iii) Characterize the sequence

$$\{ g_j(x)e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}$$

as a frame in $L^2_r[0, 1)^d$.
Find a dual frame of the form
\[ \{ (\det M) a_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}, \]
wich implies

\[ F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \langle F, g_j(\cdot e^{-2\pi i \alpha^T M^T \cdot}) \rangle a_j(x) e^{-2\pi i \alpha^T M^T x} \]
in \( L^2_r[0, 1)^d \).
(iv) Find a dual frame of the form
\[ \{ (\det M)a_j(x)e^{-2\pi i\alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \]
wich implies
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F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \langle F, g_j(\cdot)e^{-2\pi i\alpha^T M^T \cdot} \rangle a_j(x)e^{-2\pi i\alpha^T M^T x}
\]
in \( L^2_\mathbb{R}[0,1]^d \).

(v) Applying the isomorphism \( \mathcal{T}_\Phi \) to the expansion we get the desired

\[
f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (L_j f)(M\alpha) S_{j,a}(\cdot - M\alpha)
\]
in \( L^2(\mathbb{R}^d) \),

where \( S_{j,a} := \mathcal{T}_\Phi a_j \) for \( j = 1,2,\ldots,s \).
Find a **dual frame** of the form 
\[ \{ (\det M) a_j(x) e^{-2\pi i \alpha^T M^T x} \} \quad \alpha \in \mathbb{Z}^d; j=1,2,\ldots,s, \] which implies

\[ F(x) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} \langle F, g_j(\cdot)e^{-2\pi i \alpha^T M^T \cdot} \rangle a_j(x) e^{-2\pi i \alpha^T M^T x} \]

in \( L^2_r[0,1]^d \).

**Applying the isomorphism** \( \mathcal{T}_\Phi \) to the expansion we get the desired

\[ f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (L_j f)(M\alpha) S_j,a(\cdot - M\alpha) \quad \text{in} \quad L^2(\mathbb{R}^d), \]

where \( S_j,a := \mathcal{T}_\Phi a_j \) for \( j = 1, 2, \ldots, s \).

**As \( V_\Phi^2 \) is a RKHS**, we have

\[ f(t) = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (L_j f)(M\alpha) S_j,a(t - M\alpha) \quad t \in \mathbb{R}^d. \]
Details of the procedure

**Lemma**

Let \( \mathcal{L} \) be a convolution system. Then, for each \( f \in V_\Phi^2 \) we have

\[
(\mathcal{L}f)(t) = \langle F, (Z\mathcal{L}\Phi)(t, \cdot) \rangle_{L^2[0,1]^d}, \quad t \in \mathbb{R}^d,
\]

where \( F = \mathcal{T}_\Phi^{-1}f. \)
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**Lemma**

Let \( \mathcal{L} \) be a convolution system. Then, for each \( f \in \mathcal{V}_\Phi^2 \) we have

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\]

where \( F = T^{-1}_\Phi f \).

Here \( Z\psi \) denotes the Zak transform of \( \psi \), i.e.,

\[
(Z\psi)(t, w) := \sum_{\alpha \in \mathbb{Z}^d} \psi(t + \alpha)e^{-2\pi i \alpha^\top w}.
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**Lemma**

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\]

**The expression for the samples**

\[
(\mathcal{L}_j f)(M\alpha) = \langle F, Z\mathcal{L}_j \Phi(0, \cdot) e^{-2\pi i \alpha^\top M^\top} \rangle_{L^2[0,1)^d},
\]
Denote
\[
g_j(x) := Z\mathcal{L}_j \Phi(0, x), \quad j = 1, 2, \ldots, s;
\]
The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \} \) \( \alpha \in \mathbb{Z}^d; j=1,2,\ldots,s \) in \( L_r^2 [0, 1)^d \)
The sequence \( \{g_j(x)e^{-2\pi i \alpha^\top M^\top x}\}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) in \( L^2_r[0,1)^d \)

The set
\[
\#\{\mathbb{Z}^d \cap \{M^\top x : x \in [0,1)^d\}\} = \det M
\]
from now on the elements in this set will be denoted
\[
\{i_1 = 0, i_2, \ldots, i_{\det M}\} \subset \mathbb{Z}^d.
\]
The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) in \( L^2_r[0, 1)^d \)

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from now on the elements in this set will be denoted
\[
\{ i_1 = 0, i_2, \ldots, i_{\det M} \} \subset \mathbb{Z}^d.
\]

Consider the \( s \times r(\det M) \) matrix of functions
\[
G(x) := \begin{bmatrix}
g_1^T(x) & g_1^T(x + M^{-T} i_2) & \cdots & g_1^T(x + M^{-T} i_{\det M}) \\
g_2^T(x) & g_2^T(x + M^{-T} i_2) & \cdots & g_2^T(x + M^{-T} i_{\det M}) \\
\vdots & \vdots & \ddots & \vdots \\
g_s^T(x) & g_s^T(x + M^{-T} i_2) & \cdots & g_s^T(x + M^{-T} i_{\det M})
\end{bmatrix},
\]
The sequence \( \{g_j(x)e^{-2\pi i\alpha^T M^T x}\}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s} \) in \( L^2_r[0, 1)^d \)

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\vdots & \vdots & \ddots & \vdots \\
g_s^T(x) & g_s^T(x + M^{-T} i_2) & \cdots & g_s^T(x + M^{-T} i_{\det M})
\end{bmatrix},
\]
and its related constants
\[
A_G := \text{ess inf}_{x \in [0, 1)^d} \lambda_{\min}[G^*(x)G(x)], \quad B_G := \text{ess sup}_{x \in [0, 1)^d} \lambda_{\max}[G^*(x)G(x)]
\]
Lemma

Let \( g_j \) be in \( L_r^2(0, 1)^d \) for \( j = 1, 2, \ldots, s \) and let \( G(x) \) be its associated matrix. Then,

(a) The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) is a complete system for \( L_r^2(0, 1)^d \) if and only if the rank of the matrix \( G(x) \) is \( r(\det M) \) a.e. in \([0, 1)^d\).

(b) The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) is a Bessel sequence for \( L_r^2(0, 1)^d \) if and only if \( g_j \in L_r^\infty(0, 1)^d \) (or equivalently \( B_{G} < \infty \)). In this case, the optimal Bessel bound is \( B_{G}/(\det M) \).

(c) The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) is a frame for \( L_r^2(0, 1)^d \) if and only if \( 0 < A_G \leq B_G < \infty \). In this case, the optimal frame bounds are \( A_G/(\det M) \) and \( B_G/(\det M) \).

(d) The sequence \( \{ g_j(x) e^{-2\pi i \alpha^T M^T x} \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s} \) is a Riesz basis for \( L_r^2(0, 1)^d \) if and only if it is a frame and \( s = r(\det M) \).
The sampling result

**Theorem**

Assume that the functions $g_j$ belong to $L^\infty_r[0, 1)^d$ for each $j = 1, 2, \ldots, s$. The following statements are equivalents:

(a) $A_G > 0$.

(b) There exists an $r \times s$ matrix $a(x) := \begin{bmatrix} a_1(x), \ldots, a_s(x) \end{bmatrix}$ with columns $a_j \in L^\infty_r[0, 1)^d$, and satisfying

$$\begin{bmatrix} a_1(x), \ldots, a_s(x) \end{bmatrix}G(x) = \begin{bmatrix} I_r, 0 \end{bmatrix}(\det M^{-1})r \times r \quad a.e. \text{ in } [0, 1)^d.$$

(c) There exists a frame for $V^2_\Phi$ having the form

$$\{S_{j, a}(\cdot - M\alpha)\}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}$$

such that for any $f \in V^2_\Phi$

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j, a}(\cdot - M\alpha) \quad \text{in } L^2(\mathbb{R}^d).$$

(d) There exists a frame $\{S_{j, \alpha}(\cdot)\}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,s}$ for $V^2_\Phi$ such that

$$f = (\det M) \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j, \alpha} \quad \text{in } L^2(\mathbb{R}^d).$$
Remark

Having in mind the Moore-Penrose pseudo inverse

$$G^\dagger(x) := [G^*(x)G(x)]^{-1}G^*(x).$$

All matrices $a(x)$ with entries in $L^\infty[0, 1)^d$, and satisfying (b) in the previous theorem correspond to the first $r$ rows of the matrices of the form

$$A = G^\dagger(x) + U(x)[I_s - G(x)G^\dagger(x)],$$

where $U(x)$ is any $r(\det M) \times s$ matrix with entries in $L^\infty[0, 1)^d$. 


Reconstruction functions with prescribed properties

Theorem ▶ If the generators $\varphi_k$ and the functions $L_j \varphi_k$ have compact support, then the reconstruction functions $S_j$, $a$ have compact support if and only if $\text{rank} \ G(z) = r(\det M)$ for all $z \in (C \{0\})^d$.

▶ If the generators $\varphi_k$ and the functions $L_j \varphi_k$ have exponential decay, then the reconstruction functions $S_j$, $a$ have exponential decay if and only if $\text{rank} \ G(z) = r(\det M)$ for all $z \in T^d$.

$g_j(k)(z) := \sum_{\mu \in Z^d} L_j \varphi_k(\mu) z - \mu, g^{\top}_j(z) := (g_j, 1(z), g_j, 2(z), \ldots, g_j, r(z)),$

$G(z) := \begin{bmatrix} g^{\top}_j(z) e^{2\pi i m^{\top} 1 i l, \ldots, g^{\top}_j(z) e^{2\pi i m^{\top} d i l} \end{bmatrix}_{j=1, 2, \ldots, s; k=1, 2, \ldots, r}.$

Note also that for the values $x = (x_1, \ldots, x_d) \in [0, 1)^d$ and $z = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) \in T^d$ we have $G(x) = G(z)$.
Reconstruction functions with prescribed properties

**Theorem**

- If the generators $\varphi_k$ and the functions $L_j \varphi_k$ have **compact support**, then the reconstruction functions $S_{j,a}$ have compact support if and only if
  \[
  \text{rank } G(z) = r(\det M) \quad \text{for all } z \in (\mathbb{C} \setminus \{0\})^d.
  \]

- If the generators $\varphi_k$ and the functions $L_j \varphi_k$ have **exponential decay**, then the reconstruction functions $S_{j,a}$ have exponential decay if and only if
  \[
  \text{rank } G(z) = r(\det M) \quad \text{for all } z \in \mathbb{T}^d.
  \]

\[
g_{j,k}(z) := \sum_{\mu \in \mathbb{Z}^d} L_j \varphi_k(\mu) z^{-\mu}, \quad g_j^\top(z) := (g_{j,1}(z), g_{j,2}(z), \ldots, g_{j,r}(z)),
\]
\[
G(z) := \begin{bmatrix} g_j^\top(z_1 e^{2\pi i m_1^\top l_1}, \ldots, z_d e^{2\pi i m_d^\top l_d} ) \end{bmatrix}_{j=1,2,\ldots,s} \quad k=1,2,\ldots, r; \quad l=1,2,\ldots, \det M
\]

Note also that for the values $x = (x_1, \ldots, x_d) \in [0, 1)^d$ and $z = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}) \in \mathbb{T}^d$ we have $G(x) = G(z)$.
$L^2$-approximation properties
Consider the scaled version $\Gamma_a^h := \sigma_1/h \Gamma_a \sigma_h$, where for $h > 0$ we are using the notation $\sigma_h f(t) := f(ht)$, $t \in \mathbb{R}^d$, of the sampling operator $\Gamma_a$

$$\Gamma_a f(t) := \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (L_j f)(M\alpha) S_{j,a}(t - M\alpha), \quad t \in \mathbb{R}^d,$$
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**Theorem**

Under the following conditions:

- The set of generators $\Phi = \{\varphi_k\}_{k=1}^r$ satisfies the **Strang-Fix conditions of order** $\ell$
- The **decay condition** $\varphi_k(t) = O([1 + |t|]^{-d-\ell-\epsilon})$ for some $\epsilon > 0$,
- The impulse responses satisfy $\sum_{\alpha \in \mathbb{Z}^d} |h_j(t - \alpha)| \in L^2[0, 1)^d$
Consider the scaled version $\Gamma^h_a := \sigma_{1/h} \Gamma_a \sigma_h$, where for $h > 0$ we are using the notation $\sigma_h f(t) := f(ht)$, $t \in \mathbb{R}^d$, of the sampling operator $\Gamma_a$

$$\Gamma_a f(t) := \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha) S_{j,a}(t - M\alpha), \quad t \in \mathbb{R}^d,$$

**Theorem**

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we get

$$\|f - \Gamma^h_a f\|_2 \leq C \|f\|_{L^2,\ell} h^\ell$$

for all $f \in W^{\ell}_2(\mathbb{R}^d)$, where the constant $C$ does not depend on $h$ and $f$. 
Irregular sampling in $V_\Phi^2$: time-jitter error
Irregular sampling in $V^2_\Phi$: time-jitter error

- An error sequence $\varepsilon := \{\varepsilon_j, \alpha\}_{\alpha \in \mathbb{Z}^d; j = 1, 2, \ldots, s}$ in $\mathbb{R}^d$
Irregular sampling in $V^2_\Phi$: time-jitter error

- An error sequence $\varepsilon := \{\varepsilon_j, \alpha\}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s}$ in $\mathbb{R}^d$
- The sequence of perturbed samples $\{(L_j f)(M\alpha + \varepsilon_j, \alpha)\}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s}$
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$$(L_j f)(M\alpha + \varepsilon_j, \alpha) = \langle F, (T_{L_j \Phi})(\varepsilon_j, \alpha, \cdot) e^{-2\pi i \alpha^\top M^\top \cdot} \rangle_{L^2_t[0,1)^d}, \quad \alpha \in \mathbb{Z}^d.$$
Irregular sampling in $V^2_\Phi$: time-jitter error

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$$(L_j f)(M\alpha + \varepsilon_j, \alpha) = \langle F, (ZL_j \Phi)(\varepsilon_j, \alpha, \cdot) e^{-2\pi i \alpha^\top M^\top} \rangle_{L^2[0,1]^d}, \quad \alpha \in \mathbb{Z}^d.$$ 

Leads us to study the sequence

$$\{(ZL_j \Phi)(\varepsilon_j, \alpha, \cdot) e^{-2\pi i \alpha^\top M^\top} \}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s}$$
Irregular sampling in $V^2_\Phi$: time-jitter error

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$$(L_j f)(M \alpha + \varepsilon_j, \alpha) = \langle F, (\mathbf{ZL}_j \Phi)(\varepsilon_j, \alpha, \cdot) e^{-2\pi i \alpha^\top M^\top \cdot} \rangle_{L^2[0,1)^d}, \quad \alpha \in \mathbb{Z}^d.$$ Leads us to study the sequence

$$\{(\mathbf{ZL}_j \Phi)(\varepsilon_j, \alpha, \cdot) e^{-2\pi i \alpha^\top M^\top \cdot}\}_{\alpha \in \mathbb{Z}^d; j = 1,2,\ldots,s}$$
as a perturbation of the previous frame

$$\{(\mathbf{ZL}_j \Phi)(0, \cdot) e^{-2\pi i \alpha^\top M^\top \cdot}\}_{\alpha \in \mathbb{Z}^d; j = 1,2,\ldots,s}$$
**Theorem**

For sufficiently small errors there exists a frame\[ \{ S_{j,\alpha}^\varepsilon \}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s} \] for $V_\phi^2$ such that, for any $f \in V_\phi^2$

\[ f(t) = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S_{j,\alpha}^\varepsilon(t), \quad t \in \mathbb{R}^d, \]
### Theorem

For sufficiently small errors there exists a frame \( \{ S_{j,\alpha}^\varepsilon \}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s} \) for \( V_\Phi^2 \) such that, for any \( f \in V_\Phi^2 \)

\[
f(t) = \sum_{j=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_j f)(M\alpha + \varepsilon_{j,\alpha}) S_{j,\alpha}^\varepsilon(t), \quad t \in \mathbb{R}^d,
\]

### Remark

Notice that the frame \( \{ S_{j,\alpha}^\varepsilon \}_{\alpha \in \mathbb{Z}^d; j=1,2,...,s} \), depends on the error sequence.

We implemented a **frame algorithm** in the \( \ell^2(\mathbb{Z}^d) \) setting which approximates the sequence \( \{ a_{k\alpha} \}_{\alpha \in \mathbb{Z}^d; k=1,...,r} \in \ell^2(\mathbb{Z}^d) \) defining \( f \in V_\Phi^2 \) by a sequence \( \{ a_{k\alpha}^{(n)} \}_{\alpha \in \mathbb{Z}^d; k=1,...,r} \in \ell^2(\mathbb{Z}^d) \) in such a way that

\[
f_n(t) = \sum_{k=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} a_{k\alpha}^{(n)} \varphi_k(t - \alpha) \rightarrow f(t) = \sum_{k=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} a_{k\alpha} \varphi_k(t - \alpha)
\]
Chapter 2: Motivation
Chapter 2: Motivation

Shift-invariant space

\[ V^2_\phi := \left\{ \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} d_j(\alpha) \varphi_j(t - \alpha) : d_j \in \ell^2(\mathbb{Z}^d), k = 1, 2 \ldots, r \right\} \]
**Chapter 2: Motivation**

**Shift-invariant space**

\[ V_\Phi^2 := \left\{ \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} d_j(\alpha) \varphi_j(t - \alpha) : d_j \in \ell^2(\mathbb{Z}^d), k = 1, 2 \ldots, r \right\} \]

**Weighted shift-invariant spaces**

\[ V^p_\nu(\Phi) := \left\{ \sum_{j=1}^{r} \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha) \phi_j(t - \alpha) : a_j \in \ell^p_\nu(\mathbb{Z}^d), j = 1, 2, \ldots, r \right\} \]
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\]

Weight functions \( \nu \) control the decay or growth of the signals \( f \in V_{\nu}^p(\Phi) \).
Uniform average sampling in frame generated weighted shift-invariant spaces
Uniform average sampling in frame generated weighted shift-invariant spaces

- \( f \) belongs to \( L^p_\nu(\mathbb{R}^d) \) if \( \nu f \) belongs to \( L^p(\mathbb{R}^d) \)
- \( \| f \|_{L^p_\nu(\mathbb{R}^d)} = \| \nu f \|_{L^p(\mathbb{R}^d)} \)
- weight function \( \nu \) satisfies
  \[
  0 < \nu(x + y) \leq \nu(x)\nu(y), \quad \text{for all } x, y \in \mathbb{R}^d.
  \]
Uniform average sampling in frame generated weighted shift-invariant spaces

- $f$ belongs to $L^p_{\nu}(\mathbb{R}^d)$ if $\nu f$ belongs to $L^p(\mathbb{R}^d)$
- $\|f\|_{L^p_{\nu}(\mathbb{R}^d)} = \|\nu f\|_{L^p(\mathbb{R}^d)}$
- Weight function $\nu$ satisfies
  
  $$0 < \nu(x + y) \leq \nu(x)\nu(y), \quad \text{for all } x, y \in \mathbb{R}^d.$$  

Some typical examples

- Subexponential weight $\nu(x) = e^{\alpha|x|^{\beta}}$ with $\alpha \geq 0$, $\beta \in [0, 1]$
- Sobolev weight $\nu(x) = (1 + |x|)^{\alpha}$, with $\alpha \geq 0$.  

Wiener amalgam spaces of measurable functions

For $1 \leq p < \infty$

\[
W(L^p_\nu) := \left\{ f : \|f\|^p_{W(L^p_\nu)} := \sum_{\alpha \in \mathbb{Z}^d} \text{ess sup}_{x \in [0,1]^d} \{ |f(x + \alpha)|^p \nu(\alpha)^p \} < \infty \right\}
\]

and for $p = \infty$

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W(L^\infty_\nu) := \left\{ f : \|f\|_{W(L^\infty_\nu)} := \sup_{\alpha \in \mathbb{Z}^d} \text{ess sup}_{x \in [0,1]^d} \{ |f(x + \alpha)| \nu(\alpha) \} < \infty \right\}
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Definition

A collection $\{ \phi_j(\cdot - \alpha) \}_{\alpha \in \mathbb{Z}^d; j=1,2,\ldots,r}$ is said to be a $p$-frame for $V^p_\nu(\Phi)$ if there exists a positive constant $C$ (depending on $\Phi$, $p$ and $\nu$) such that for every $f \in V^p_\nu(\Phi)$

$$C^{-1}\|f\|_{L^p_\nu} \leq \sum_{j=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x) \phi_j(x - \alpha) \, dx \right\}_{\alpha \in \mathbb{Z}^d} \right\|_{\ell^p_\nu} \leq C\|f\|_{L^p_\nu}.$$
Weighted multiply generated shift-invariant space

Given a set of functions $\Phi := \{\phi_j\}_{j=1}^r$, the weighted multiply generated shift-invariant space $V_p^\nu(\Phi)$ is formally defined as

$$V_p^\nu(\Phi) := \left\{ \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha) \phi_j(t - \alpha) : \{a_j(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell_p^\nu(\mathbb{Z}^d) \right\}.$$
Given a set of functions $\Phi := \{\phi_j\}_{j=1}^r$, the weighted multiply generated shift-invariant space $V^p_\nu(\Phi)$ is formally defined as

$$V^p_\nu(\Phi) := \left\{ \sum_{j=1}^r \sum_{\alpha \in \mathbb{Z}^d} a_j(\alpha) \phi_j(t - \alpha) : \{a_j(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^p_\nu(\mathbb{Z}^d) \right\}.$$ 

- p-frame condition and $\Phi = \{\phi_j\}_{j=1}^r \subset W(L^1_\nu)$ assure the closedness of $V^p_\nu(\Phi)$ as a subspace of $L^p_\nu$. 

Weighted multiply generated shift-invariant space
Theorem

There exist functions $S_{l,d}$ such that for any $f \in V^p_\nu(\Phi)$, $1 \leq p \leq \infty$, the sampling formula

$$f = \sum_{l=1}^{s} \sum_{\alpha \in \mathbb{Z}^d} (\mathcal{L}_lf)(M\alpha) S_{l,d}(\cdot - M\alpha),$$

holds in the $L^p_\nu$-sense. The convergence is also uniform on $\mathbb{R}^d$. 
**Theorem**

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- The $s$ systems $\mathcal{L}_l$ are obtained by convolution with functions $h_l \in W(L^1_\nu)$.
- The weight function must satisfy the GRS-condition:

$$\lim_{n \to \infty} \nu(n\alpha)^{1/n} = 1$$
**Theorem**

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**Wiener’s lemma**

If $f \in \mathcal{A}_\nu$ and $f(x) \neq 0$ for every $x \in \mathbb{R}^d$, the function $1/f$ is also in $\mathcal{A}_\nu$, where $\mathcal{A}_\nu$ denotes the weighted Wiener algebra of the functions

$$f(x) = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha) e^{2\pi i \alpha^T x}, \quad \{a(\alpha)\}_{\alpha \in \mathbb{Z}^d} \in \ell^1_\nu(\mathbb{Z}^d)$$
Chapter 3: Motivation
Chapter 3: Motivation

Shift-invariant space

\[ V_\varphi^2 = \left\{ \sum_{n \in \mathbb{Z}} a_n \varphi(t - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\}. \]
### Shift-invariant space

$$V_{\varphi}^2 = \left\{ \sum_{n \in \mathbb{Z}} a_n \varphi(t - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\}.$$ 

### Shift-invariant space

$$V_{\varphi}^2 = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \varphi(t) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\}.$$ 

where $T : f(t) \mapsto f(t - 1)$ in $L^2(\mathbb{R})$ is the shift operator.
Chapter 3: Motivation

**Shift-invariant space**

\[ V^2_\varphi = \left\{ \sum_{n \in \mathbb{Z}} a_n \varphi(t - n) : \{a_n\} \in \ell^2(\mathbb{Z}) \right\}. \]

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**\( U \)-invariant space**

\[ \mathcal{A}_a = \left\{ \sum_{n \in \mathbb{Z}} \alpha_n U^n a : \{\alpha_n\}_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \right\}. \]

where \( U \) is a unitary operator and \( a \) is a fixed vector on a separable Hilbert space \( \mathcal{H} \).
Generalizing the samples

The samples in the shift-invariant case were obtained by means of convolution systems

\[ L_j f := f \ast h_j \]

The \( U \)-samples

For a fixed \( b \in H \) and a sampling period \( r \in \mathbb{N} \) the \( U \)-samples are given by

\[ L_b x(r_k) := \langle x, U r_k b \rangle_{H}, k \in \mathbb{Z}. \]

In the shift-invariant case, \( U \) is defined as the shift operator \( f(u) \mapsto f(u-1) \) in \( L^2(\mathbb{R}^d) \) and we have

\[ \langle f, U r_k b \rangle_{H} = \int_{\mathbb{R}} f(u) b(u-rk) \, du = (f \ast h)(rk), \quad u \in \mathbb{R}, \]

where \( h(u) := b(-u) \).
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The samples in the shift-invariant case were obtained by means of convolution systems $\mathcal{L}_j f := f * h_j$
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The $U$-samples

For a fixed $b \in \mathcal{H}$ and a sampling period $r \in \mathbb{N}$ the $U$-samples are given by

$$\mathcal{L}_b x (rk) := \langle x, U^{rk} b \rangle_{\mathcal{H}}, \quad k \in \mathbb{Z}.$$
Generalizing the samples

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\[
\langle f, U^{rk} b \rangle \mathcal{H} = \int_{-\infty}^{\infty} f(u) b(u - rk) du = (f \ast h)(rk), \quad u \in \mathbb{R},
\]

where \( h(u) := b(-u) \).
Sampling theory in $U$-invariant spaces

Let $U$ be an unitary operator in a separable Hilbert space $H$; for a fixed $a \in H$, consider the closed subspace given by $A_a = \text{span}\{U^n a | n \in \mathbb{Z}\}$.

In case that the sequence $\{U^n a | n \in \mathbb{Z}\}$ is a Riesz sequence in $H$ we have $A_a = \{\sum_{n \in \mathbb{Z}} \alpha_n U^n a : \{\alpha_n\} | n \in \mathbb{Z} \in \ell^2(\mathbb{Z})\}$.

Examples: Translation and Modulation operator on $L^2(\mathbb{R})$

$(T_a f)(t) = f(t-a)$

$(M_a f)(t) = f(t) e^{i a t}$
Sampling theory in $U$-invariant spaces

Let $U$ be an unitary operator in a separable Hilbert space $\mathcal{H}$; for a fixed $a \in \mathcal{H}$, consider the closed subspace given by

$$\mathcal{A}_a := \overline{\text{span}}\{U^n a, \ n \in \mathbb{Z}\}.$$
Sampling theory in $U$-invariant spaces

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Sampling theory in $U$-invariant spaces

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**Examples:** Translation and Modulation operator on $L^2(\mathbb{R})$

$$(T_a f)(t) = f(t - a)$$

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The sequence $\{U^na\}_{n \in \mathbb{Z}}$
The sequence \( \{ U^n a \}_{n \in \mathbb{Z}} \)

- The *auto-covariance* function admits the integral representation

\[
R_a(k) := \langle U^k a, a \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu_a(\theta), \quad k \in \mathbb{Z}.
\]
The sequence \( \{U^n a\}_{n \in \mathbb{Z}} \)

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\]

- The positive Borel spectral measure \( \mu_a \) can be decomposed as \( d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu^s_a(\theta) \).
The sequence $\{U^na\}_{n \in \mathbb{Z}}$

- The auto-covariance function admits the integral representation

$$R_a(k) := \langle U^ka, a \rangle_H = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu_a(\theta), \quad k \in \mathbb{Z}.$$ 

- The positive Borel spectral measure $\mu_a$ can be decomposed as $d\mu_a(\theta) = \phi_a(\theta)d\theta + d\mu^s_a(\theta)$.

**Theorem**

The sequence $\{U^na\}_{n \in \mathbb{Z}}$ is a Riesz basis for $A_a$ if and only if the singular part $\mu^s_a \equiv 0$ and

$$0 < \mathrm{ess \, inf}_{\theta \in (-\pi, \pi)} \phi_a(\theta) \leq \mathrm{ess \, sup}_{\theta \in (-\pi, \pi)} \phi_a(\theta) < \infty.$$ 

If $\{b_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ are the Fourier coefficients of the function $1/\phi_a(\theta) \in L^2(-\pi, \pi)$ then the vector $b = \sum_{k \in \mathbb{Z}} b_k U^k a \in A_a$ generates the dual Riesz basis $\{U^nb\}_{n \in \mathbb{Z}}$ with spectral measure $\phi_b(\theta) = 1/\phi_a(\theta)$. 
We define the isomorphism $\mathcal{T}_{U,a}$ which maps the orthonormal basis $\{e^{2\pi inw}\}_{n \in \mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n \in \mathbb{Z}}$ for $A_a$, that is,

$$\mathcal{T}_{U,a} : \quad L^2(0,1) \rightarrow A_a \quad \text{such that} \quad F = \sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi inw} \mapsto x = \sum_{n \in \mathbb{Z}} \alpha_n U^n a.$$
We define the isomorphism $T_{U,a}$ which maps the orthonormal basis $\{e^{2\pi inw}\}_{n \in \mathbb{Z}}$ for $L^2(0,1)$ onto the Riesz basis $\{U^n a\}_{n \in \mathbb{Z}}$ for $A_a$, that is,

$$T_{U,a} : \quad L^2(0,1) \quad \longrightarrow \quad A_a$$

$$F = \sum_{n \in \mathbb{Z}} \alpha_n e^{2\pi inw} \quad \longmapsto \quad x = \sum_{n \in \mathbb{Z}} \alpha_n U^n a.$$

The following $U$-shift property holds: For any $F \in L^2(0,1)$ and $N \in \mathbb{Z}$, we have

$$T_{U,a} \left( F e^{2\pi iNw} \right) = U^N \left( T_{U,a} F \right).$$
An expression for the generalized samples

For \( x \in \mathcal{A}_a \) let \( F \in L^2(0, 1) \) such that \( \mathcal{T}_{U,a} F = x \);

\[
\mathcal{L}_j x(rm) = \left\langle F, g_j(w) e^{2\pi i rmw} \right\rangle_{L^2(0,1)} \quad \text{for } m \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, s,
\]

where the function

\[
g_j(w) := \sum_{k \in \mathbb{Z}} \mathcal{L}_j a(k) e^{2\pi i kw}
\]

belongs to \( L^2(0, 1) \) for each \( j = 1, 2, \ldots, s \).
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belongs to \( L^2(0, 1) \) for each \( j = 1, 2, \ldots, s \).

As a consequence, the stable recovery of any \( x \in \mathcal{A}_a \) depends on whether the sequence

\[
\left\{ \overline{g_j(w)} e^{2\pi irmw} \right\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}
\]

forms a frame for \( L^2(0, 1) \).
### The matrix $G(w)$

The matrix $G(w)$ is defined as:

$$
G(w) := \begin{bmatrix}
    g_1(w) & g_1\left(w + \frac{1}{r}\right) & \cdots & g_1\left(w + \frac{r-1}{r}\right) \\
    g_2(w) & g_2\left(w + \frac{1}{r}\right) & \cdots & g_2\left(w + \frac{r-1}{r}\right) \\
    \vdots & \vdots & \ddots & \vdots \\
    g_s(w) & g_s\left(w + \frac{1}{r}\right) & \cdots & g_s\left(w + \frac{r-1}{r}\right)
\end{bmatrix}
$$
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\vdots & \vdots & \ddots & \vdots 
g_s(w) & g_s(w + \frac{1}{r}) & \cdots & g_s(w + \frac{r-1}{r})
\end{bmatrix}
\]

\[
\alpha_{G} := \operatorname{ess \ inf}_{w \in (0, 1/r)} \lambda_{\min}[G^*(w)G(w)],
\]

\[
\beta_{G} := \operatorname{ess \ sup}_{w \in (0, 1/r)} \lambda_{\max}[G^*(w)G(w)],
\]
Theorem

Assume that the function $g_j, j = 1, 2, \ldots, s$ belongs to $L^\infty(0, 1)$. The following statements are equivalent:

(a) $\alpha_G > 0$

(b) There exists a vector $[h_1(w), h_2(w), \ldots, h_s(w)]$ with entries in $L^\infty(0, 1)$ satisfying

$$[h_1(w), h_2(w), \ldots, h_s(w)] G(w) = [1, 0, \ldots, 0] \text{ a.e. in } (0, 1).$$

(c) There exist $c_j \in A_a, j = 1, 2, \ldots, s$, such that the sequence

$\{ U^{rk} c_j \}_{k \in \mathbb{Z}; j = 1,2,\ldots s}$

is a frame for $A_a$, and for any $x \in A_a$ we have the expansion

$$x = \sum_{j=1}^s \sum_{k \in \mathbb{Z}} L_j x(rk) U^{rk} c_j \text{ in } H,$$

(d) There exists a frame $\{ C_{j,k} \}_{k \in \mathbb{Z}; j = 1,2,\ldots s}$ for $A_a$ such that, for each $x \in A_a$ we have the expansion

$$x = \sum_{j=1}^s \sum_{k \in \mathbb{Z}} L_j x(rk) C_{j,k} \text{ in } H,$$
Another approach

\[ \langle U^k a, U^{rn} b_j \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-rn)\theta} \phi_{a,b_j}(e^{i\theta}) d\theta. \]

- The **left-shift operator** $S$ defined as
  \[
  S : \quad L^2(\mathbb{T}) \quad \longrightarrow \quad L^2(\mathbb{T}) \quad \sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \quad \longmapsto \quad \sum_{k \in \mathbb{Z}} a_{k+1} e^{ik\theta},
  \]
  or equivalently, by $(Sf)(e^{i\theta}) = f(e^{i\theta})e^{-i\theta}$.

- The **decimation operator** $D_r$, $r$ a positive integer, defined as
  \[
  D_r : \quad L^2(\mathbb{T}) \quad \longrightarrow \quad L^2(\mathbb{T}) \quad \sum_{k \in \mathbb{Z}} a_k e^{ik\theta} \quad \longmapsto \quad \sum_{k \in \mathbb{Z}} a_{rk} e^{ik\theta},
  \]
  which can equivalently be written as
  \[
  (D_r f)(e^{i\theta}) = \frac{1}{r} \sum_{k=0}^{r-1} f(e^{i\theta + \frac{2k\pi}{r}}).
  \]
The $U$-systems

For any fixed $b \in \mathcal{H}$ we define the $U$-system $\mathcal{L}_b$ as the linear operator between $\mathcal{H}$ and the set $C(\mathbb{R})$ of the continuous functions on $\mathbb{R}$ given by

$$\mathcal{L}_b : \mathcal{H} \to C(\mathbb{R})$$

$$x \mapsto \mathcal{L}_b x,$$

where $\mathcal{L}_b x(t) := \langle x, U^t b \rangle_{\mathcal{H}}, \quad t \in \mathbb{R}.$
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In this case $U$ should coincide with $U^1$ on a continuous group of unitary operators $\{ U^t \}_{t \in \mathbb{R}}$. 
A brief walk on continuous groups of unitary operators

Definition \( \{ U_t \} \) \( t \in \mathbb{R} \) is a family of unitary operators in \( H \) satisfying:

1. \( U_t U_{t'} = U_{t+t'} \),
2. \( U_0 = I_H \),
3. \( \langle U_t x, y \rangle_H \) is a continuous function of \( t \) for any \( x, y \in H \).

Classical Stone's theorem assures us the existence of a self-adjoint operator \( T \) (possibly unbounded) such that \( U_t \equiv e^{i t T} \). This self-adjoint operator \( T \) is defined on the dense domain \( D_T \) of \( H \).

Notice that, whenever the self-adjoint operator \( T \) is bounded, \( D_T = H \) and \( e^{i t T} \) can be defined as the usual exponential series; in any case, \( U_t \equiv e^{i t T} \) means that \( \langle U_t x, y \rangle_H = \int_{-\infty}^{\infty} e^{i w t} \langle E_w x, y \rangle_H \), \( t \in \mathbb{R} \), where \( x \in D_T \) and \( y \in H \).
A brief walk on continuous groups of unitary operators

**Definition**

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\[
\langle U^t x, y \rangle = \int_{-\infty}^{\infty} e^{iwt} d\langle E_w x, y \rangle, \quad t \in \mathbb{R},
\]

where \( x \in D_T \) and \( y \in \mathcal{H} \).
Asymmetric sampling
Asymmetric sampling

We have at our disposal the asymmetric samples

\[ \{ \mathcal{L}_j x(\sigma_j + r_j m) \}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \]

where \( \sigma_j \in \mathbb{R} \) and \( r_j \in \mathbb{N} \).
Asymmetric sampling

We have at our disposal the asymmetric samples

\[ \{ L_j x(\sigma_j + r_j m) \}_{m \in \mathbb{Z}; j=1,2,\ldots,s} \]

where \( \sigma_j \in \mathbb{R} \) and \( r_j \in \mathbb{N}. \)

The recovery formula

\[ x = \sum_{j=1}^{s} \sum_{l_j=1}^{r_j} \sum_{k \in \mathbb{Z}} L_j x(\sigma_j + r k + r_j(l_j - 1)) U^{rk} c_{j,l_j} \]

where \( r := \text{lcm}\{r_j\}_{j=1,\ldots,s} \)
Time-jitter error: irregular sampling in $A_a$
Time-jitter error: irregular sampling in $A_a$

The perturbed samples

$$\{(L_j x)(rm + \epsilon_{mj})\}$$
The perturbed samples

\[
\{(L_j x)(rm + \epsilon_{mj})\} = \left\{ \langle F, g_{m,j}(w) e^{2\pi irmw} \rangle_{L^2(0,1)} \right\} m \in \mathbb{Z}; j = 1, 2, \ldots, s
\]

where

\[
g_{m,j}(w) := \sum_{k \in \mathbb{Z}} L_j a(k + \epsilon_{mj}) e^{2\pi ikw},
\]
Time-jitter error: irregular sampling in $A_a$

The perturbed samples

$$\{(L_j x)(rm + \epsilon_{mj})\} = \left\{ \langle F, \overline{g_{m,j}(w)} e^{2\pi irmw} \rangle_{L^2(0,1)} \right\} m \in \mathbb{Z}; j = 1, 2, ..., s$$

where

$$g_{m,j}(w) := \sum_{k \in \mathbb{Z}} L_j a(k + \epsilon_{mj}) e^{2\pi ikw},$$

We can study

$$\left\{ \overline{g_{m,j}(w)} e^{2\pi irmw} \right\} m \in \mathbb{Z}; j = 1, 2, ..., s$$

as a perturbation of the frame

$$\left\{ \overline{g_j(w)} e^{2\pi irmw} \right\} m \in \mathbb{Z}; j = 1, 2, ..., s$$
Theorem

For sufficiently small errors $\epsilon := \{\epsilon_m\}_{m \in \mathbb{Z}}; j=1,\ldots,s$ there exists a frame $\{C_{j,m}^\epsilon\}_{m \in \mathbb{Z}; j=1,2,\ldots,s}$ for $\mathcal{A}_a$ such that, for any $x \in \mathcal{A}_a$, the sampling expansion

$$x = \sum_{j=1}^{s} \sum_{m \in \mathbb{Z}} \mathcal{L}_j x (r m + \epsilon_m) C_{j,m}^\epsilon \quad \text{in } \mathcal{H},$$

holds.
How small should be the errors?

\[ \tilde{M}_{a,b,j}(\gamma) := \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma,\gamma]} |L_{j,a}(n+t) - L_{j,a}(n)|, \]

\[ \tilde{N}_{a,b,j}(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma,\gamma]} |L_{j,a}(rn+k+t) - L_{j,a}(rn+k)|. \]

Theorem

Given an error sequence \( \epsilon := \{\epsilon_{mj}\}_{m \in \mathbb{Z}}; j = 1,\ldots,s \), define the constant \( \gamma_j := \sup_{m \in \mathbb{Z}} |\epsilon_{mj}| \) for each \( j = 1,2,\ldots,s \). The condition

\[ s \sum_{j=1}^{s} \tilde{M}_{a,b,j}(\gamma_j) \tilde{N}_{a,b,j}(\gamma_j) < \alpha \]

ensures that reconstruction is possible.

\( b_j \in D_T \Rightarrow L_{j,a}(t) \in C^1(\mathbb{R}) \) and condition \( (L_{j,a})'(t) = O(|t|^{-1+\eta_j}) \) implies that \( \tilde{N}_{a,b,j}(\gamma_j) \) and \( \tilde{M}_{a,b,j}(\gamma_j) \) are continuous near to 0.
How small should be the errors?

\[
\tilde{M}_{a,b_j}(\gamma) := \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma,\gamma]} |\mathcal{L}_j a(n + t) - \mathcal{L}_j a(n)|,
\]

\[
\tilde{N}_{a,b_j}(\gamma) := \max_{k=0,1,\ldots,r-1} \sum_{n \in \mathbb{Z}} \max_{t \in [-\gamma,\gamma]} |\mathcal{L}_j a(rn + k + t) - \mathcal{L}_j a(rn + k)|.
\]
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**Theorem**

Given an error sequence \( \epsilon := \{\epsilon_{mj}\}_{m \in \mathbb{Z}, j=1,\ldots,s} \), define the constant \( \gamma_j := \sup_{m \in \mathbb{Z}} |\epsilon_{mj}| \) for each \( j = 1, 2, \ldots, s \). The condition

\[ \sum_{j=1}^{s} \tilde{M}_{a,b_j}(\gamma_j) \tilde{N}_{a,b_j}(\gamma_j) < \frac{\alpha G}{r} \]

ensures that **reconstruction is possible**.

\[ b_j \in D_{T} \Rightarrow \mathcal{L}_j a(t) \in C^1(\mathbb{R}) \] and condition \((\mathcal{L}_j a)'(t) = O(|t|^{-(1+\eta_j)})\)

implies that \( \tilde{N}_{a,b_j}(\gamma) \) and \( \tilde{M}_{a,b_j}(\gamma) \) are continuous near to 0.
The perturbed sequence \( \{ U^{r_k+\epsilon_{kj}} b_j \}_{k \in \mathbb{Z}; j=1,2,...,r} \)

**Theorem**

Assume that for certain \( b_j \in D_T, j = 1, 2, \ldots, r \), the sequence \( \{ U^{r_k} b_j \}_{k \in \mathbb{Z}; j=1,2,...,r} \) is a **Riesz basis** for \( A_a \) with Riesz bounds \( 0 < A_\psi \leq B_\psi < \infty \). For a sequence \( \epsilon := \{ \epsilon_{kj} \}_{k \in \mathbb{Z}; j=1,2,...,r} \) of errors, let \( R \) be the constant given by

\[
R := \| \epsilon \|^2 \max_{j=1,2,...,r} \left\{ \int_{-\infty}^{\infty} w^2 d \| E_w b_j \|^2 \right\},
\]

where \( \| \epsilon \| \) denotes the \( \ell^2 \)-norm of the sequence \( \epsilon \).

If \( R < A_\psi \), then the perturbed sequence

\[
\{ U^{r_k+\epsilon_{kj}} b_j \}_{k \in \mathbb{Z}; j=1,2,...,r}
\]

is a **Riesz sequence** in \( \mathcal{H} \) with Riesz bounds

\[
A_\psi (1 - \sqrt{R/A_\psi})^2 \quad \text{and} \quad B_\psi (1 + \sqrt{R/B_\psi})^2.
\]
The case of multiple generators

Sampling in multiple generated $U$-invariant subspaces can be analogously derived, $\mathcal{A}_a := \overline{\text{span}}\{U^na_l, \ n \in \mathbb{Z}; \ l = 1, 2, \ldots, L\}$.

The sequence $\{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ can be thought as an $L$-dimensional stationary sequence. Its covariance matrix $R_a(k)$ is the $L \times L$ matrix

$$R_a(k) = \left[\langle U^k a_m, a_n \rangle_h\right]_{m,n=1,2,\ldots,L} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ik\theta} d\mu_a(\theta), \quad k \in \mathbb{Z}.$$

**Theorem**

Let $\{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ be a sequence obtained from an unitary operator with spectral measure $d\mu_a(\theta) = \Phi_a(\theta)d\theta + d\mu^s_a(\theta)$, and let $\mathcal{A}_a$ be the closed subspace spanned by $\{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$. Then the sequence $\{U^n a_l\}_{n \in \mathbb{Z}; l=1,2,\ldots,L}$ is a Riesz basis for $\mathcal{A}_a$ if and only if the singular part $\mu^s_a \equiv 0$ and

$$0 < \text{ess inf}_{\theta \in (-\pi,\pi)} \lambda_{\min} [\Phi_a(\theta)] \leq \text{ess sup}_{\theta \in (-\pi,\pi)} \lambda_{\max} [\Phi_a(\theta)] < \infty.$$
Summarizing:

<table>
<thead>
<tr>
<th></th>
<th>$V_\Phi^2$</th>
<th>$V_{L^2}(\Phi)$</th>
<th>$A_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Reconstruction formula</strong></td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td><strong>Time-jitter error</strong></td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td><strong>prescribed properties</strong></td>
<td>✓</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td><strong>Other results</strong></td>
<td>$L^2$-approximation properties</td>
<td>Dirac sampling case</td>
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</tr>
</tbody>
</table>
Future Work

- To carry out a deeper study of the weighted sampling framework
- Uniform average sampling in frame generated weighted shift-invariant spaces
- Sampling in finite $U$-invariant subspaces with multiple generators. We have assumed that the stationary sequence $\{U_n\}_{n \in \mathbb{Z}}$ in $H$ has infinite different elements. It could happen that for some $a \in H$ there exists $N \in \mathbb{N}$ such that $U_N a = a$.
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\[
\{ \mathcal{L}_j x(t_n) := \langle x, U^{t_n} b_j \rangle \}_{n \in \mathbb{Z}; j=1,2,...s},
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Héctor R. Fernández
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Introduction to
sampling theory

Generalized sampling
in $L^2(\mathbb{R})$ shift-invariant subspaces with
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Uniform average
sampling in frame
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Sampling theory in
$U$-invariant spaces

Publications


  *Generalized sampling in $U$-invariant subspaces.* Proceedings of the 10th International Conference on Sampling Theory and Applications, Eurasip Open Library, 208-211, 2013. (Ch. 4)


THANKS