Sampling associated with a unitary representation of a finite group

Miguel A. Hernández Medina[1]
(based in a joint work with Antonio G. García and Alberto Ibort)

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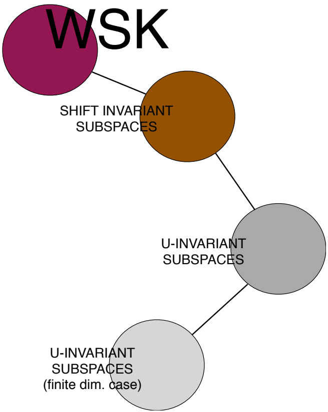
WSK
SHIFT INVARIANT
SUBSPACES
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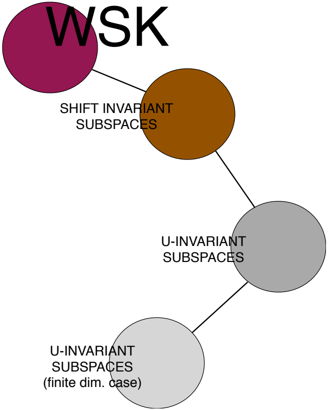
(finite dim. case)

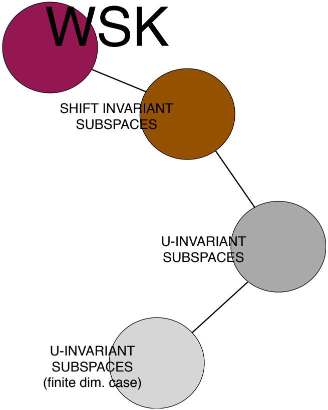
U-Invariant subspaces

Unitary representaton of $\mathbb{Z}$



$$n \in \mathbb{Z} \mapsto U^n \in \mathcal{U}(\mathcal{H})$$

where $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator.



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U-Invariant subspaces (finite case)

Unitary representaton of $\mathbb{Z}_N$

$$g \in \mathbb{Z}_N \mapsto \pi(g) \in \mathcal{U}(\mathbb{C}^N)$$

where $\pi(g)$ is a unitary operator.

Mathematical Physics

Towards a quantum sampling theory: the case of finite groups

Antonio G. García, Miguel A. Hernández-Medina, A. Ibort

(Submitted on 27 Oct 2015)

Nyquist–Shannon sampling theorem, instrumental in classical telecommunication technologies, is extended to quantum systems supporting a unitary representation of a finite group G. Two main ideas from the classical theory having natural counterparts in the quantum setting: frames and invariant subspaces, provide the mathematical background for the theory. The main ingredients of classical sampling theorems are discussed and their quantum counterparts are thoroughly analyzed in this simple situation. A few examples illustrating the obtained results are discussed.

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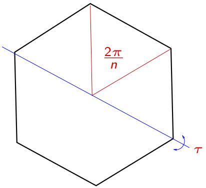
Cite as: arXiv:1510.08134 **[math-ph]**
(or arXiv:1510.08134v1 **[math-ph]** for this version)

The Dihedral Group D_n



$$D_n = \{e, g, g^2, \ldots, g^{n-1}, \tau, \tau g, \ldots, \tau g^{n-1}\}$$

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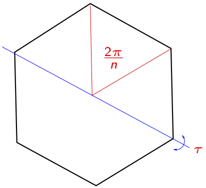

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Unitary representation $\mathcal{H}$ Hilbert space

$$h \in D_n \mapsto U(h) \in \mathcal{U}(\mathcal{H})$$

Given $a \in \mathcal{H}$ such that $\{U(h)a \mid h \in D_n\}$ l.i., let

$$\mathcal{A}_a = span\{U(h)a \mid h \in D_n\}$$

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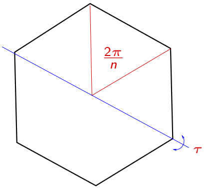

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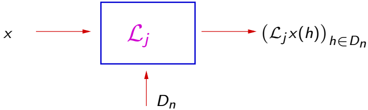
Sampling: $\mathbf{b} = (b_1, b_2, \dots, b_s) \in \mathcal{H}^s$, $s \geq n$, $\mathcal{L}_j x(\cdot) = \langle U(\cdot)b_j, x \rangle$



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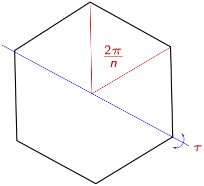
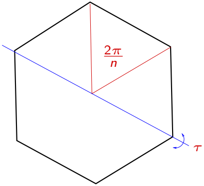


Sampling result: If $\mathbf{b} = (b_1, b_2, \dots, b_s)$ verifies (C) then there exists s vectors $c_1, c_2, \dots c_s \in \mathcal{A}_a$ such that for each $x \in \mathcal{A}_a$

$$x = \sum_{j=1}^{s} [\mathcal{L}_j x(e) U(e) c_j + \mathcal{L}_j x(\tau) U(\tau) c_j]$$

The Dihedral Group D_n



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$$x = \sum_{j=1}^{s} [\mathcal{L}_j x(e) U(e) c_j + \mathcal{L}_j x(\tau) U(\tau) c_j]$$

Let $R_{b_j,a}(h) := \langle U(h) b_j, a \rangle$, $j = 1, 2, \dots, s$ and

$$\mathbf{R}_{\mathbf{b},a} = \begin{pmatrix} R_{b_1,a}(e) & \cdots & R_{b_1,a}(g^{n-1}) & R_{b_1,a}(\tau) & \cdots & R_{b_1,a}(\tau g^{n-1}) \\ R_{b_1,a}(\tau) & \cdots & R_{b_1,a}(\tau g^{n-1}) & R_{b_1,a}(e) & \cdots & R_{b_1,a}(g^{n-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ R_{b_s,a}(e) & \cdots & R_{b_s,a}(g^{n-1}) & R_{b_s,a}(\tau) & \cdots & R_{b_s,a}(\tau g^{n-1}) \\ R_{b_s,a}(\tau) & \cdots & R_{b_s,a}(\tau g^{n-1}) & R_{b_s,a}(e) & \cdots & R_{b_s,a}(g^{n-1}) \end{pmatrix} \in \mathbb{C}^{2s \times 2n}$$

(C) $\operatorname{rank} \mathbf{R}_{\mathbf{b},a} = 2n$

Mathematical setting

Let G be a **finite** group (not neccesarily conmmutative)
Unitary representation $\mathcal{H}$ Hilbert space

$$h \in G \mapsto U(h) \in \mathcal{U}(\mathcal{H})$$

Let $a \in \mathcal{H}$ with $\{U(h)a \mid h \in G\}$ l.i., let

$$\mathcal{A}_a = span\{U(h)a \mid h \in G\}$$

Main goal
Find K a subgroup of G, $\mathbf{b} = (b_1, b_2, \ldots, b_N) \in \mathcal{H}^N$ and $\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathcal{A}^N$ such that for each $x \in \mathcal{A}$

$$x = \sum_{j=1}^{N} \sum_{t \in K} \mathcal{L}_j x(t) U(t) c_j$$

The isomorphism $\mathcal{T}_a^G$

$\mathbb{C}[G] \sim \boldsymbol{\alpha} : G \to \mathbb{C}$, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}(g))_{g \in G}$

$$\mathbb{C}[G] \sim L^2(G) \sim \mathbb{C}^{|G|}$$

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Left regular representation $G \ni t \mapsto L_t \in \mathcal{U}(L^2(G))$

$$L_t \boldsymbol{\alpha}(g) = \boldsymbol{\alpha}(t^{-1} g)$$

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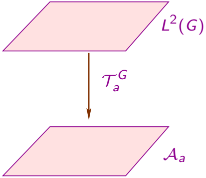
The Isomorphism



$$L^2(G) \xrightarrow{\mathcal{T}_a^G} \mathcal{A}_a$$

$$\boldsymbol{\alpha} \longrightarrow x = \sum_{t \in G} \boldsymbol{\alpha}(t) U(t) a$$

The isomorphism $\mathcal{T}_a^G$

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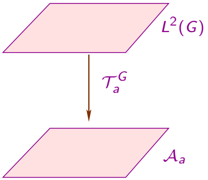

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\mathcal{T}_a^G} & \mathcal{A}_a \\ \boldsymbol{\alpha} & \longrightarrow & x = \sum_{t \in G} \boldsymbol{\alpha}(t) U(t) a \end{array}$$

Shifting property: for any $s \in G$ and $\boldsymbol{\alpha} \in L^2(G)$ we have

$$\mathcal{T}_a^G \big(L_s \boldsymbol{\alpha}\big) = U(s)\, \mathcal{T}_a^G (\boldsymbol{\alpha})$$

The group G

We assume that

- G has two subgroups K and H such that
  - K is **Abelian**
  - $G = KH$ y $H \cap K = \{e\}$ (H is a complement of K).

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Theorem (Schur-Zassenhaus)

If K is an Abelian normal subgroup of G such that $|K|$ and $|G/K|$ are coprime then there exists a complement H of K in G. In such case the group G is the semidirect product *of K and H, and denoted as $G = H \rtimes K$.*

The group G

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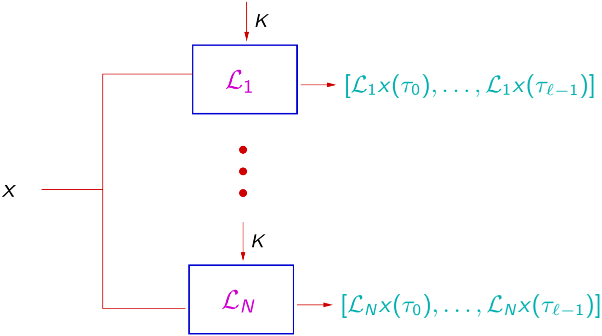
Notation

- $\ell := |K| = |G|/|H|$, $K = \left\{\tau_0 = e, \tau_1, \cdots, \tau_{\ell-1}\right\}$
- $G/H = \left\{[e = \tau_0], [\tau_1], \cdots, [\tau_{\ell-1}]\right\}$
- For any fixed way of writing the elements of H

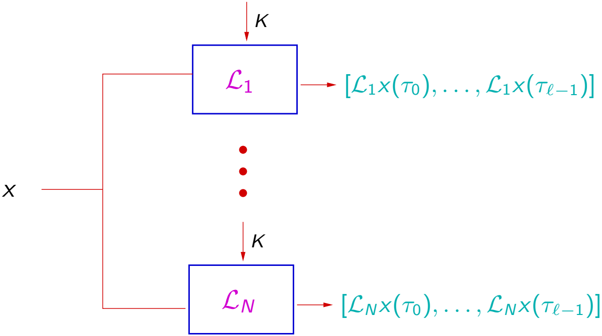
$$G = \left\{\tau_0^{-1}H, \tau_1^{-1}H, \cdots, \tau_{\ell-1}^{-1}H\right\}.$$

Generalized samples



- $b_1, b_2, \ldots, b_N \in \mathcal{H}$
- $\mathcal{L}_j x(\cdot) = \langle U(\cdot) b_j, x \rangle$

Generalized samples



- $b_1, b_2, \ldots, b_N \in \mathcal{H}$
- $\mathcal{L}_j x(\cdot) = \langle U(\cdot) b_j, x \rangle$
- $\ell = |K|$, $\ell |H| = |G|$.

Generalized samples

$$x = \sum_{s \in G} \alpha_s \, U(s)a \in \mathcal{A}_a, \quad \boldsymbol{\alpha} = (\alpha_s)_{s \in G} \in L^2(G)$$

$$\mathcal{L}_j x(\tau_n) = \langle x, U(\tau_n) b_j \rangle_{\mathcal{H}} = \Big\langle \sum_{s \in G} \alpha_s \, U(s)a, U(\tau_n) b_j \Big\rangle_{\mathcal{H}}$$
$$= \langle \boldsymbol{\alpha}, G_{j,\tau_n} \rangle_{L^2(G)}$$

- $G_{j,\tau_n} = (\overline{\langle U(s)a, U(\tau_n) b_j \rangle})_{s \in G} \in L^2(G).$

Generalized samples

$$x = \sum_{s \in G} \alpha_s \, U(s) a \in \mathcal{A}_a, \;\; \mathcal{L}_j x(\tau_n) = \left\langle \boldsymbol{\alpha}, G_{j,\tau_n} \right\rangle_{L^2(G)}$$

Proposition

Any $x \in \mathcal{A}_a$ can be recovered from its samples

$$\left\{ \mathcal{L}_j x(\tau_n) \right\}_{\substack{j=1,2,\ldots,N \\ n=0,1,\ldots,\ell-1}}$$

if and only if the set of vectors

$$\left\{ G_{j,\tau_n} \right\}_{\substack{j=1,2,\ldots,N \\ n=0,1,\ldots,\ell-1}} \in L^2(G)$$

form a frame (a spanning set) for $L^2(G)$.

Generalized samples

$\{G_{j,\tau_n}\}$ is frame in $L^2(G)$ if and only if the $(|G| \times N\ell)$-matrix

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ G_{1,\tau_0} & \cdots & G_{1,\tau_{\ell-1}} & G_{2,\tau_0} & \cdots & G_{2,\tau_{\ell-1}} & \cdots & G_{N,\tau_0} & \cdots & G_{N,\tau_{\ell-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

has rank $|G| = \dim L^2(G)$.

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has rank $|G| = \dim L^2(G)$.

Corollary:

$$N \geq |H|$$

An expression for generalized samples

$$G_{j,\tau_n} = \big(\langle U(\tau_n)b_j, U(s)a\rangle\big)_{s\in G} = \big(\langle U(s^{-1}\tau_n)b_j, a\rangle\big)_{s\in G}$$
$$= \big(\overline{R_{b_j,a}(\tau_n^{-1}s)}\big)_{s\in G}.$$

where $R_{a,b}(g) := \langle U(g)b, a\rangle_{\mathcal{H}}$, $g \in G$.

An expression for generalized samples

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where $R_{a,b}(g) := \langle U(g)b, a\rangle_{\mathcal{H}}$, $g \in G$. The vectors G_{j,τ_n} can be written as column matrices as

$$G_{j,\tau_n} = \Big(\overline{R_{b_j,a}(\tau_n^{-1}\tau_0^{-1}H)}, \overline{R_{b_j,a}(\tau_n^{-1}\tau_1^{-1}H)}, \dots, \overline{R_{b_j,a}(\tau_n^{-1}\tau_{\ell-1}^{-1}H)}\Big)^\top$$

An expression for generalized samples

For each $j = 1, 2, \ldots, N$ let $\mathbf{R}_{b_j,a}$ be the $\ell \times |G|$ matrix

$$\mathbf{R}_{b_j,a} = \begin{pmatrix} R_{b_j,a}(\tau_0^{-1}\tau_0^{-1}H) & R_{b_j,a}(\tau_0^{-1}\tau_1^{-1}H) & \ldots & R_{b_j,a}(\tau_0^{-1}\tau_{\ell-1}^{-1}H) \\ R_{b_j,a}(\tau_1^{-1}\tau_0^{-1}H) & R_{b_j,a}(\tau_1^{-1}\tau_1^{-1}H) & \ldots & R_{b_j,a}(\tau_1^{-1}\tau_{\ell-1}^{-1}H) \\ \vdots & \vdots & \ldots & \vdots \\ R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_0^{-1}H) & R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_1^{-1}H) & \ldots & R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_{\ell-1}^{-1}H) \end{pmatrix}$$

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Since K is an **Abelian subgroup** of G the cosets $\tau_n^{-1}\tau_p^{-1}H$ and $\tau_p^{-1}\tau_n^{-1}H$ coincide. As a consequence,

$$\mathbf{R}_{b_j,a} = \begin{pmatrix} R_{b_j,a}(\tau_0^{-1}\tau_0^{-1}H) & R_{b_j,a}(\tau_1^{-1}\tau_0^{-1}H) & \ldots & R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_0^{-1}H) \\ R_{b_j,a}(\tau_0^{-1}\tau_1^{-1}H) & R_{b_j,a}(\tau_1^{-1}\tau_1^{-1}H) & \ldots & R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_1^{-1}H) \\ \vdots & \vdots & \ldots & \vdots \\ R_{b_j,a}(\tau_0^{-1}\tau_{\ell-1}^{-1}H) & R_{b_j,a}(\tau_1^{-1}\tau_{\ell-1}^{-1}H) & \ldots & R_{b_j,a}(\tau_{\ell-1}^{-1}\tau_{\ell-1}^{-1}H) \end{pmatrix}$$

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Se define la $N\ell \times |G|$-matriz

$$\mathbf{R}_{\mathbf{b},a} := \begin{pmatrix} \mathbf{R}_{b_1,a} \\ \mathbf{R}_{b_2,a} \\ \vdots \\ \mathbf{R}_{b_N,a} \end{pmatrix} .$$

An expression for generalized samples

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$\{G_{j,\tau_n}\}_{\substack{j=1,2,\dots,N \\ n=0,1,\dots,\ell-1}}$ **frame** for $L^2(G) \Leftrightarrow \operatorname{rank} \mathbf{R}_{\mathbf{b},a} = |G|$.

An expression for generalized samples

Proposition

For any $x = \sum_{s \in G} \alpha_s \, U(s)a \in \mathcal{A}_a$ *consider its samples vector*

$$\mathcal{L}_{\text{samp}} x = \left(\mathcal{L}_1 x(\tau_0) \ldots \mathcal{L}_1 x(\tau_{\ell-1}) \cdots \mathcal{L}_N x(\tau_0) \ldots \mathcal{L}_N x(\tau_{\ell-1}) \right)^{\top} .$$

Then, the matrix relationship

$$\mathcal{L}_{\text{samp}} x = \mathbf{R}_{\mathbf{b},a} \, \boldsymbol{\alpha}$$

holds.

Left inverses

The Moore-Penrose pseudoinverse of $\mathbf{R}_{\mathbf{b},a}$ is the $|G| \times N\ell$ matrix

$$\mathbf{R}^+_{\mathbf{b},a} = [\mathbf{R}^*_{\mathbf{b},a}\mathbf{R}_{\mathbf{b},a}]^{-1}\mathbf{R}^*_{\mathbf{b},a}$$

$$\mathbf{R}^+_{\mathbf{b},a} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{R}^+_{1,\tau_0} & \cdots & \mathbf{R}^+_{1,\tau_{\ell-1}} & \cdots & \mathbf{R}^+_{N,\tau_0} & \cdots & \mathbf{R}^+_{N,\tau_{\ell-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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$$\mathcal{L}_{\mathrm{samp}} x = \mathbf{R}_{\mathbf{b},a}\, \boldsymbol{\alpha}$$

$$\updownarrow$$

$$\boldsymbol{\alpha} = \mathbf{R}^+_{\mathbf{b},a}\, \mathcal{L}_{\mathrm{samp}} x = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n) \mathbf{R}^+_{j,\tau_n} = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \langle \boldsymbol{\alpha}, G_{j,\tau_n} \rangle_{L^2(G)} \mathbf{R}^+_{j,\tau_n}$$

Left inverses

The Moore-Penrose pseudoinverse of $\mathbf{R}_{\mathbf{b},a}$ is the $|G| \times N\ell$ matrix

$$\mathbf{R}^+_{\mathbf{b},a} = \begin{pmatrix} \vdots & & \vdots & & \vdots & & \vdots \\ \mathbf{R}^+_{1,\tau_0} & \cdots & \mathbf{R}^+_{1,\tau_{\ell-1}} & \cdots & \mathbf{R}^+_{N,\tau_0} & \cdots & \mathbf{R}^+_{N,\tau_{\ell-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

$$\alpha = \mathbf{R}^+_{\mathbf{b},a} \mathcal{L}_{\text{samp}} x = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n) \mathbf{R}^+_{j,\tau_n} = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \langle \alpha, G_{j,\tau_n} \rangle_{L^2(G)} \mathbf{R}^+_{j,\tau_n}$$

$\{R^+_{j,\tau_n}\}_{\substack{j=1,2,\ldots,N \\ n=0,1,\ldots,\ell-1}}$ is the canonical dual frame of $\{G_{j,\tau_n}\}_{\substack{j=1,2,\ldots,N \\ n=0,1,\ldots,\ell-1}}$

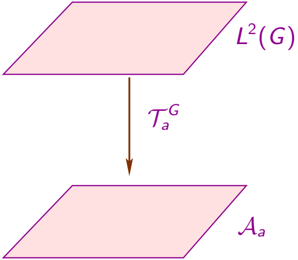
The sampling result (I)

$x = \sum_{s\in G} \alpha_s\, U(s)a \in \mathcal{A}_a$

$$\boldsymbol{\alpha} = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n) \mathbf{R}^{+}_{j,\tau_n}$$

$$\Big\downarrow \mathcal{T}_a^G$$

$$x = \mathcal{T}_a^G(\boldsymbol{\alpha}) = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n)\, \mathcal{T}_a^G(\mathbf{R}^{+}_{j,\tau_n})$$



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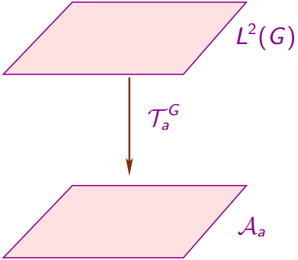

The sampling function $\mathcal{T}_a^G(\mathbf{R}^{+}_{j,\tau_n})$ do not have, in principle, any special structure since $\mathbf{R}^{+}_{\mathbf{b},a}$ does not have it.

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$$G_{j,\tau_n} = \left(\overline{R_{b_j,a}(\tau_n^{-1}s)}\right)_{s\in G} = L_{\tau_n} G_{j,\tau_0}$$

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There exists $\mathbf{M}$ such that

- $\mathbf{M}\mathbf{R}_{\mathbf{b},a} = I$ ($\Leftrightarrow$ the set of columns of $\mathbf{M}$ is a dual frame of $\{G_{j,\tau_n}\}$)
- $\mathbf{M}_{n,\tau_j} = L_{\tau_j}\mathbf{M}_{n,\tau_0}$?

$$\mathbf{M} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{M}_{1,\tau_0} & \cdots & \mathbf{M}_{1,\tau_{\ell-1}} & \cdots & \mathbf{M}_{N,\tau_0} & \cdots & \mathbf{M}_{N,\tau_{\ell-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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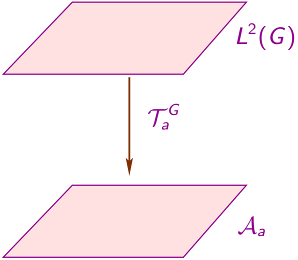

$$\mathcal{T}_a^G(\mathbf{M}_{j,\tau_n}) = \mathcal{T}_a^G(L_{\tau_n}\mathbf{M}_{j,\tau_0}) = U(\tau_n)\mathbf{M}_{j,\tau_0},$$

$$j = 1, \ldots, N, \;\; n = 0, \ldots, \ell - 1.$$

Sampling. G-compatible left-inverses

Let $\mathbf{S}$ be the first $|H|$ rows of **any left-inverse** of the matrix $\mathbf{R}_{\mathbf{b},a}$

$$\mathbf{S}\,\mathbf{R}_{\mathbf{b},a} = \left(\mathbf{I}_{|H|} \;\; \mathbf{O}_{|H|\times(|G|-|H|)}\right). \tag{1}$$

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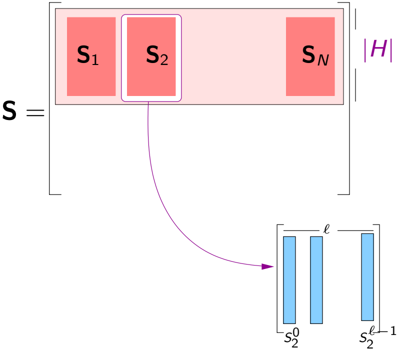
We write the $|H| \times N\ell$ matrix $\mathbf{S}$ as

$$\mathbf{S} = \begin{pmatrix}\mathbf{S}_1\,\mathbf{S}_2\cdots\mathbf{S}_N\end{pmatrix}$$

where each block $\mathbf{S}_j$ is a $|H| \times \ell$ matrix

$$\mathbf{S}_j = \begin{pmatrix}S_j^0\,S_j^1\,\ldots\,S_j^{\ell-1}\end{pmatrix}$$



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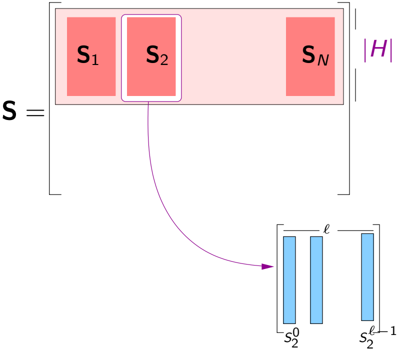
$$\sum_{j=1}^{N}\sum_{n=0}^{\ell-1} S_j^n\, R_{b_j,a}(\tau_0^{-1}\tau_n^{-1}H) = \mathbf{I},$$

$$\sum_{j=1}^{N}\sum_{n=0}^{\ell-1} S_j^n\, R_{b_j,a}(\tau_k^{-1}\tau_n^{-1}H) = \mathbf{O},$$

$$k = 1, 2, \dots, \ell-1\,,$$



G-compatible left-inverses

From $\mathbf{S}$ we form the $|G| \times N\ell$ matrix

$$\widetilde{\mathbf{S}} = \left(\widetilde{\mathbf{S}}_1\, \widetilde{\mathbf{S}}_2 \ldots \widetilde{\mathbf{S}}_N\right)$$

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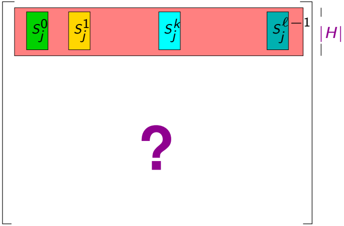

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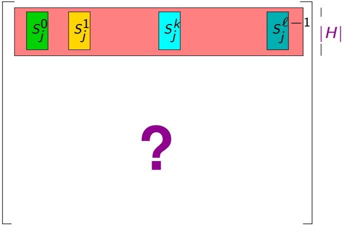

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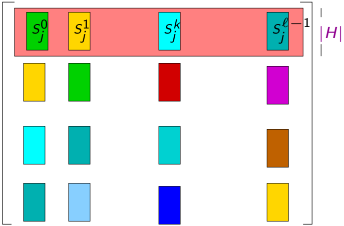

$$S_j^{n,i} := S_j^k \quad \text{whenever} \quad \tau_i\, \tau_n = \tau_k\,.$$

The sampling result

Lemma

- $\widetilde{\mathbf{S}} \in \mathbb{C}^{|G| \times N\ell}$ *is a left-inverse of* $\mathbf{R}_{\mathbf{b},a}$, *i.e,* $\widetilde{\mathbf{S}}\,\mathbf{R}_{\mathbf{b},a} = \mathbf{I}_{|G|}$.
- $\widetilde{\mathbf{S}}_{j,n} = L_{\tau_n}\widetilde{\mathbf{S}}_{j,0}$, $j = 1, 2, \ldots, N$, $n = 0, 1, \ldots, \ell - 1$.

The sampling result

Theorem. The following statements are equivalent:

1. $\operatorname{rank} \mathbf{R}_{\mathbf{b},a} = |G|$
2. $\exists \mathbf{S} \in \mathbb{C}^{|H| \times N\ell}$ such that $\mathbf{S}\,\mathbf{R}_{\mathbf{b},a} = \big(\mathbf{I}_{|H|}\ \mathbf{O}_{|H|\times(|G|-|H|)}\big)$
3. $\exists \{c_j\}_{j=1}^{N} \in \mathcal{A}_a$, such that $\{U(\tau_n)c_j\}_{\substack{j=1,2,\dots,N\\ n=0,1,\dots,\ell-1}}$ is a frame for $\mathcal{A}_a$, and for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n)\, U(\tau_n) c_j$$

holds.

4. $\exists \{C_{j,n}\}_{\substack{j=1,2,\dots,N\\ n=0,1,\dots,\ell-1}}$ frame in $\mathcal{A}_a$ such that, for each $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{N} \sum_{n=0}^{\ell-1} \mathcal{L}_j x(\tau_n)\, C_{j,n}$$

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Next steps ...

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- Unitary representation of G

$$\begin{aligned} G &\xrightarrow{\ \pi\ } \mathcal{U}(\mathcal{H}) \\ g &\longrightarrow \pi(g) \end{aligned}$$

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2. A frame sequence in $\mathcal{H}$

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We denote by $(e_g)_{g\in G}$ the canonical orthornomal basis in $\ell^2(G)$. Let T_a^G be the operator

$$\ell^2(G) \xrightarrow{T_a^G} \mathcal{A}_a$$

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T_a^G is a bounded invertible operator

Next steps ...

Let $K < G$ be a subgroup, and $b_1, \dots, b_N \in \mathcal{H}$. We define

$$\mathcal{L}_j x(k) := \langle x, \pi(g) b_j \rangle_{\mathcal{H}}, \quad j = 1, 2, \dots, N.$$

Lemma Let $x \in \mathcal{A}_a := \overline{\text{span}}\{\pi(g)a \mid g \in G\}$ and $T_a^G F = x$, then

$$\mathcal{L}_j x(k) = \langle F, G_{j,k} \rangle_{\ell^2(G)}, \quad j = 1, 2, \dots, N, \ k \in K$$
$$G_{j,k} = L_k G_j$$

$G \ni g \mapsto L_g \in \mathcal{U}(\ell^2(G))$ is the **left regular representation** of G.

Next steps ...

Find G, $K < G$ and $b_1, \ldots, b_n \in \mathcal{H}$ such that

1. $\{L_k G_j\}_{\substack{j=1,2,\ldots,N \\ k \in K}}$ is a frame in $\ell^2(G)$
2. If $\{L_k G_j\}_{\substack{j=1,2,\ldots,N \\ k \in K}}$ is a frame in $\ell^2(G)$ then to find a dual frame of the form $\{L_k H_j\}_{\substack{j=1,2,\ldots,N \\ k \in K}}$