Sampling formulas involving differences in shift-invariant subspaces: a unified approach

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Celebrating the 60th birthday of the authors

Statement of the problem

Given a generador $\varphi \in L^2(\mathbb{R})$ with good properties there exists a Shannon-type sampling formula in the shift-invariant subspace:

$$V_\varphi^2 = \Big\{ \sum_{n\in\mathbb{Z}} a_n \, \varphi(t-n) \ : \ \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R}) \,,$$

which reads as

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n) \, S_a(t-n) \,, \quad t \in \mathbb{R} \,,$$

where $S_a \in V_\varphi^2$ and the sequence $\{S_a(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for the shift-invariant subspace V_φ^2.

- $\exists \ 0 < A \leq B$ such that $A\|f\|^2 \leq \sum_{n\in\mathbb{Z}} |f(a+n)|^2 \leq B\|f\|^2, \ f \in V_\varphi^2$
- $S_a(a+n) = \delta_n, \ n \in \mathbb{Z}$

Goal: To investigate the existence of stable sampling formulas in V_φ^2 involving the data

$$\{f(a+pn)\}_{n\in\mathbb{Z}}; \quad \{\Delta_+^k f(a+pn)\}_{n\in\mathbb{Z}}; \quad \text{and/or} \quad \{\Delta_-^k f(a+pn)\}_{n\in\mathbb{Z}},$$

where

$$\Delta_+^k f(a+pn) := \Delta_+^{k-1} f(a+pn+1) - \Delta_+^{k-1} f(a+pn)$$

$$\Delta_-^k f(a+pn) := \Delta_-^{k-1} f(a+pn) - \Delta_-^{k-1} f(a+pn-1), \quad \Delta_\pm^0 = I$$

denote forward/backward differences.

(We could also use central differences $\Delta_0 f(a+pn)$ or averages $\mu_+ f(a+pn)$, $\mu_- f(a+pn)$ or $\mu_0 f(a+pn)$)

An easy example

Adding and subtracting $f(a+2n)\,S_a(t-2n-1)$ in each summand of

$$f(t) = \sum_{n\in\mathbb{Z}} \left\{ f(a+2n)\,S_a(t-2n) + f(a+2n+1)\,S_a(t-2n-1) \right\}$$

yields

$$f(t) = \sum_{n\in\mathbb{Z}} \left\{ f(a+2n)\,\left[S_a(t-2n) + S_a(t-2n-1) \right] + \Delta_+ f(a+2n)\,S_a(t-2n-1) \right\}$$

Stable recovery of any $f \in V^2_\varphi$ from the new samples turns out to be

$$\left\{ S_a(t-2n) + S_a(t-2n-1) \right\}_{n\in\mathbb{Z}} \cup \left\{ S_a(t-2n-1) \right\}_{n\in\mathbb{Z}}$$

a Riesz basis for V^2_φ, or equivalently, there exist constants $0 < A \le B$ such that for any $f \in V^2_\varphi$

$$A\|f\|^2 \le \sum_{n\in\mathbb{Z}} \left\{ |f(a+2n)|^2 + |\Delta_+ f(a+2n)|^2 \right\} \le B\|f\|^2$$

The mathematical setting

The shift-invariant subspace V_φ^2

$$V_\varphi^2 := \overline{\text{span}}\{\varphi(t-n)\}_{n\in\mathbb{Z}} = \Big\{ \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})\,,$$

We are assuming that:

- The sequence $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_φ^2
- φ is continuous on $\mathbb{R}$
- The series $\sum_{n\in\mathbb{Z}} |\varphi(t-n)|^2$ is bounded on $\mathbb{R}$

The isomorphism $\mathcal{T}_\varphi$

The space V_φ^2 is the image $L^2(0,1)$ by means of the isomorphism:

$$\mathcal{T}_\varphi : L^2(0,1) \longrightarrow V_\varphi^2$$
$$\{e^{-2\pi inx}\}_{n\in\mathbb{Z}} \longmapsto \{\varphi(t-n)\}_{n\in\mathbb{Z}}$$

- For each $f \in V_\varphi^2$ we have

$$f(t) = \langle F, K_t\rangle_{L^2(0,1)}\,, \quad t \in \mathbb{R}$$

where $F = \mathcal{T}_\varphi^{-1} f$ and

$$K_t(x) = \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)} e^{-2\pi inx} = \overline{Z\varphi(t,x)}$$

($Z\varphi$ is the **Zak transform** of φ)

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- $K_{t+m}(x) = e^{-2\pi imx}K_t(x)\,,\ t\in\mathbb{R}\,,\ \ m\in\mathbb{Z}$
- $\mathcal{T}_\varphi\big[e^{-2\pi imx}F(x)\big](t) = f(t-m)\,,\ t\in\mathbb{R} \quad (f=\mathcal{T}_\varphi(F))$
- V_φ^2 is a RKHS of continuous functions

The sampling formula in V_φ^2

For a fixed $a \in [0,1)$ and $n \in \mathbb{Z}$ we have

$$f(a+n) = \langle F, K_{a+n} \rangle_{L^2(0,1)} = \langle F, \mathrm{e}^{-2\pi i n x} K_a \rangle_{L^2(0,1)}$$

$\{x_n := \mathrm{e}^{-2\pi i n x} K_a(x)\}_{n\in\mathbb{Z}}$ **Riesz basis** $\iff 0 < \|K_a\|_0 \leq \|K_a\|_\infty < \infty$, where $\|K_a\|_0 := \operatorname*{ess\,inf}_{x\in(0,1)} |K_a(x)|$ and $\|K_a\|_\infty := \operatorname*{ess\,sup}_{x\in(0,1)} |K_a(x)|$.

Its **dual Riesz basis** is $\{y_n := \mathrm{e}^{-2\pi i n x}/\overline{K_a(x)}\}_{n\in\mathbb{Z}}$

For each $F \in L^2(0,1)$

$$F = \sum_{n\in\mathbb{Z}} \langle F, \mathrm{e}^{-2\pi i n x} K_a \rangle \frac{\mathrm{e}^{-2\pi i n x}}{\overline{K_a(x)}} = \sum_{n\in\mathbb{Z}} f(a+n) \frac{\mathrm{e}^{-2\pi i n x}}{\overline{K_a(x)}} \quad \text{in } L^2(0,1)\,.$$

Applying the isomorphism $\mathcal{T}_\varphi$ and its shifting property we obtain

$$f(t) = \sum_{n\in\mathbb{Z}} f(a+n)\, S_a(t-n)\,, \quad t \in \mathbb{R}\,,$$

where $S_a := \mathcal{T}_\varphi\big(1/\overline{K_a}\big) \in V_\varphi^2$.

Some well-known examples

- The **Paley-Wiener space** PW_π coincides with V^2_{sinc} whose generator is the *sine cardinal function*. For any $a \in [0,1)$, the following sampling formula holds:

$$f(t) = \sum_{n\in\mathbb{Z}} f(n+a)\frac{\sin \pi(t-n-a)}{\pi(t-n-a)}\,, \quad t \in \mathbb{R}\,,$$

- Consider $\varphi := N_m$ where N_m is the B**-spline** of order $m-1$, i.e., $N_m := N_1 * N_1 * \cdots * N_1$ (m times) where $N_1 := \chi_{[0,1]}$.
  It is known that the sequence $\{N_m(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $V^2_{N_m}$.
  For quadratic and cubic splines the following sampling formulas hold:

- **Quadratic Spline** N_3. Here, taking $a = 1/2$, for any $f \in V^2_{N_3}$ we obtain the sampling formula

$$f(t) = \sum_{n \in \mathbb{Z}} f(n + \frac{1}{2})\, S_{1/2}(t-n)\,, \quad t \in \mathbb{R}\,,$$

where $S_{1/2}(t) = \sqrt{2} \sum_{n \in \mathbb{Z}} (2\sqrt{2}-3)^{|n+1|}\, N_3(t-n)$.
For $a = 0$ the sampling result fails since $\|K_0\|_0 = 0$.

- **Cubic Spline** N_4. Here, taking $a = 0$, for any $f \in V^2_{N_4}$ we obtain the sampling formula

$$f(t) = \sum_{n \in \mathbb{Z}} f(n)\, S_0(t-n)\,, \quad t \in \mathbb{R}\,,$$

where $S_0(t) = \sqrt{3} \sum_{n \in \mathbb{Z}} (-1)^n (2-\sqrt{3})^{|n|}\, N_4(t-n+2)$.

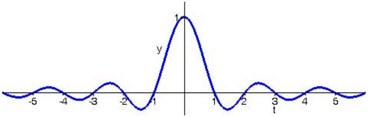

Figure: Sine cardinal funtion

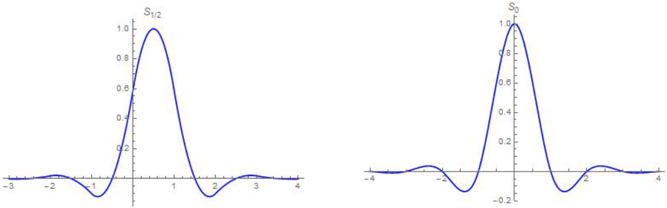

Figure: Quadratic $S_{1/2}$ and cubic S_0 sampling funtions

Generalized finite differences

Forward, backward or central difference operators:

$$\Delta_+ f(t) := f(t+1) - f(t)\,,\ \Delta_- f(t) := f(t) - f(t-1)$$
$$\Delta_0 f(t) := f(t+1) - f(t-1)$$

Average operators:

$$\mu_+ f(t) := \frac{1}{2}[f(t+1) + f(t)]\,,\ \mu_- f(t) := \frac{1}{2}[f(t) + f(t-1)]$$
$$\mu_0 f(t) := \frac{1}{2}[f(t+1) + f(t-1)]$$

and their iterates are a particular case of the generalized finite difference $\Delta^{\mathbf{a}}_{M,N} f(t) := \displaystyle\sum_{k=-M}^{N} a_k f(t+k)$, where $\mathbf{a} = (a_k) \subset \mathbb{C}$ and $M, N \in \mathbb{Z}$

In particular, for forward/backward differences in V_φ^2:

$$\Delta_+ f(t) = \big\langle F, K_{t+1} - K_t \big\rangle_{L^2(0,1)} = \big\langle F, (e^{-2\pi i x} - 1)\, K_t \big\rangle_{L^2(0,1)}, \quad t \in \mathbb{R}$$

$$\Delta_- f(t) = \big\langle F, K_t - K_{t-1} \big\rangle_{L^2(0,1)} = \big\langle F, (1 - e^{2\pi i x})\, K_t \big\rangle_{L^2(0,1)}, \quad t \in \mathbb{R}$$

and iterating

$$\Delta_+^k f(t) = \big\langle F, (e^{-2\pi i x} - 1)^k\, K_t \big\rangle_{L^2(0,1)}, \quad t \in \mathbb{R}$$

$$\Delta_-^k f(t) = \big\langle F, (1 - e^{2\pi i x})^k\, K_t \big\rangle_{L^2(0,1)}, \quad t \in \mathbb{R}$$

Similar expressions can be obtained for other differences or averages

Going back to the initial example, for the samples $f(a+2n)$ and $\Delta_+ f(a+2n)$, $n \in \mathbb{Z}$:

$$f(a+2n) = \left\langle F, e^{-2\pi i(2n)x} K_a \right\rangle_{L^2(0,1)},$$
$$\Delta_+ f(a+2n) = \left\langle F, e^{-2\pi i(2n+1)x} K_a - e^{-2\pi i(2n)x} K_a \right\rangle_{L^2(0,1)}.$$

Denoting $x_n := e^{-2\pi inx} K_a(x)$, $n \in \mathbb{Z}$, the questions are:

- whether the sequence $\{x_{2n}\}_{n\in\mathbb{Z}} \cup \{-x_{2n} + x_{2n+1}\}_{n\in\mathbb{Z}}$ is a Riesz basis for $L^2(0,1)$, and
- how to obtain, in the affirmative case, its dual basis in terms of $y_n := e^{-2\pi inx}/\overline{K_a(x)}$, $n \in \mathbb{Z}$, the dual Riesz basis of $\{x_n\}_{n\in\mathbb{Z}}$ Here, it is $\{y_{2n} + y_{2n+1}\}_{n\in\mathbb{Z}} \cup \{y_{2n+1}\}_{n\in\mathbb{Z}}$

Let $\{x_n\}_{n\in\mathbb{Z}}$ and $\{y_n\}_{n\in\mathbb{Z}}$ be a pair of dual Riesz bases for $\mathcal{H}$. For a fixed $p\in\mathbb{N}$, consider a partition $\{x_{1n}\}_{n\in\mathbb{Z}}\cup\{x_{2n}\}_{n\in\mathbb{Z}}\cup\cdots\cup\{x_{pn}\}_{n\in\mathbb{Z}}$ of $\{x_n\}_{n\in\mathbb{Z}}$ and let $M=\big(a_{jk}\big)\in\mathbb{C}^{p\times p}$ be a matrix of scalars. Then, the new sequence defined by using the rows of M as

$$\Big\{\sum_{k=1}^{p}a_{1k}x_{kn}\Big\}_{n\in\mathbb{Z}}\cup\Big\{\sum_{k=1}^{p}a_{2k}x_{kn}\Big\}_{n\in\mathbb{Z}}\cup\cdots\cup\Big\{\sum_{k=1}^{p}a_{pk}x_{kn}\Big\}_{n\in\mathbb{Z}}$$

is a Riesz basis for $\mathcal{H}$ if and only if $\det M\neq 0$. In this case, its dual Riesz basis is given by

$$\Big\{\sum_{k=1}^{p}\overline{b}_{k1}y_{kn}\Big\}_{n\in\mathbb{Z}}\cup\Big\{\sum_{k=1}^{p}\overline{b}_{k2}y_{kn}\Big\}_{n\in\mathbb{Z}}\cup\cdots\cup\Big\{\sum_{k=1}^{p}\overline{b}_{kp}y_{kn}\Big\}_{n\in\mathbb{Z}}$$

where $\big(b_{1k}\ b_{2k}\ \ldots\ b_{pk}\big)^\top$ denotes the k-th column of the inverse matrix M^{-1}, $k=1,2,\ldots,p$.

Some sampling results

Using first and second forward differences

$$f(a+3n) = \left\langle F, e^{-2\pi i(3n)x} K_a \right\rangle_{L^2(0,1)}$$
$$\Delta_+ f(a+3n) = \left\langle F, e^{-2\pi i(3n+1)x} K_a - e^{-2\pi i(3n)x} K_a \right\rangle_{L^2(0,1)}$$
$$\Delta_+^2 f(a+3n) = \left\langle F, e^{-2\pi i(3n+2)x} K_a - 2e^{-2\pi i(3n+1)x} K_a + e^{-2\pi i(3n)x} K_a \right\rangle_{L^2(0,1)}$$

The sequence

$$\{x_{3n}\}_{n\in\mathbb{Z}} \cup \{x_{3n+1} - x_{3n}\}_{n\in\mathbb{Z}} \cup \{x_{3n+2} - 2x_{3n+1} + x_{3n}\}_{n\in\mathbb{Z}}$$

is a Riesz basis for $L^2(0,1)$ and its dual Riesz basis is

$$\{y_{3n} + y_{3n+1} + y_{3n+2}\}_{n\in\mathbb{Z}} \cup \{y_{3n+1} + 2y_{3n+2}\}_{n\in\mathbb{Z}} \cup \{y_{3n+2}\}_{n\in\mathbb{Z}}$$

since

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

Expanding F with respect to the above dual basis we obtain

$$F = \sum_{n\in\mathbb{Z}} \Big\{ f(a+3n)\big[y_{3n} + y_{3n+1} + y_{3n+2}\big] + \Delta_+ f(a+3n)\big[y_{3n+1} + 2y_{3n+2}\big] + \Delta_+^2 f(a+3n)\, y_{3n+2} \Big\} \qquad \text{in } L^2(0,1)$$

Using $\mathcal{T}_\varphi$ and its shifting property $\mathcal{T}_\varphi(y_{pn+k}) = S_a\big(t-(pn+k)\big)$ we get in V_φ^2 the sampling formula:

$$f(t) = \sum_{n\in\mathbb{Z}} \Big\{ f(a+3n)\big[S_a(t-3n) + S_a(t-3n-1) + S_a(t-3n-2)\big] + \Delta_+ f(a+3n)\big[S_a(t-3n-1) + 2S_a(t-3n-2)\big] + \Delta_+^2 f(a+3n)\, S_a(t-3n-2) \Big\}, \quad t\in\mathbb{R}\,.$$

Using backward and forward differences

$$\Delta_- f(a+4n) = \left\langle F, e^{-2\pi i(4n)x}K_a - e^{-2\pi i(4n-1)x}K_a \right\rangle_{L^2(0,1)}$$
$$f(a+4n) = \left\langle F, e^{-2\pi i(4n)x}K_a \right\rangle_{L^2(0,1)},$$
$$\Delta_+ f(a+4n) = \left\langle F, e^{-2\pi i(4n+1)x}K_a - e^{-2\pi i(4n)x}K_a \right\rangle_{L^2(0,1)}$$
$$\Delta_+^2 f(a+4n) = \left\langle F, e^{-2\pi i(4n+2)x}K_a - 2e^{-2\pi i(4n+1)x}K_a + e^{-2\pi i(4n)x}K_a \right\rangle_{L^2(0,1)}$$

The sequence

$$\{-x_{4n-1}+x_{4n}\}_{n\in\mathbb{Z}} \cup \{x_{4n}\}_{n\in\mathbb{Z}} \cup \{-x_{4n}+x_{4n+1}\}_{n\in\mathbb{Z}} \cup \{x_{4n}-2x_{4n+1}+x_{4n+2}\}_{n\in\mathbb{Z}}$$

is a Riesz basis for $L^2(0,1)$ and its dual Riesz basis is

$$\{-y_{4n-1}\}_{n\in\mathbb{Z}} \cup \{y_{4n-1}+y_{4n}+y_{4n+1}+y_{4n+2}\}_{n\in\mathbb{Z}} \cup \{y_{4n+1}+2y_{4n+2}\} \cup \{y_{4n+2}\}_{n\in\mathbb{Z}}$$

since

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

The resulting sampling formula in V_φ^2 reads:

$$\begin{aligned} f(t) = \sum_{n\in\mathbb{Z}} \Big\{ &-\Delta_- f(a+4n) S_a(t-4n+1) \\ &+ f(a+4n)\big[S_a(t-4n+1) + S_a(t-4n) \\ &+ S_a(t-4n-1) + S_a(t-4n-2)\big] \\ &+ \Delta_+ f(a+4n)\big[S_a(t-4n-1) + 2S_a(t-4n-2)\big] \\ &+ \Delta_+^2 f(a+4n)\, S_a(t-4n-2)\Big\}, \quad t \in \mathbb{R}\,. \end{aligned}$$

Using central averages and differences

$$\mu_0 f(a+3n) = \left\langle F, \frac{1}{2}(e^{-2\pi i(3n+1)x}K_a + e^{-2\pi i(3n-1)x}K_a \right\rangle_{L^2(0,1)}$$

$$f(a+3n) = \left\langle F, e^{-2\pi i(3n)x}K_a \right\rangle_{L^2(0,1)}$$

$$\Delta_0 f(a+3n) = \left\langle F, e^{-2\pi i(3n+1)x}K_a - e^{-2\pi i(3n-1)x}K_a \right\rangle_{L^2(0,1)}$$

The sequence

$$\{x_{3n}\}_{n\in\mathbb{Z}} \cup \{\frac{1}{2}x_{3n-1} + \frac{1}{2}x_{3n+1}\}_{n\in\mathbb{Z}} \cup \{-x_{3n-1} + x_{3n+1}\}_{n\in\mathbb{Z}}$$

is a Riesz basis for $L^2(0,1)$ and its dual Riesz basis is

$$\{y_{3n}\}_{n\in\mathbb{Z}} \cup \{y_{3n-1} + y_{3n+1}\}_{n\in\mathbb{Z}} \cup \{-\frac{1}{2}y_{3n-1} + \frac{1}{2}y_{3n+1}\}_{n\in\mathbb{Z}}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ -1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & -1/2 \\ 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{pmatrix}$$

The resulting sampling formula in V_φ^2 reads:

$$\begin{aligned} f(t) = \sum_{n\in\mathbb{Z}} \Big[& f(a+3n)\, S_a(t-3n) \\ & + \mu_0 f(a+3n)\big[S_a(t-3n+1) + S_a(t-3n-1)\big] \\ & + \Delta_0 f(a+3n)\frac{1}{2}\big[-S_a(t-3n+1) + S_a(t-3n-1)\big]\Big], \quad t\in\mathbb{R} \end{aligned}$$

Many other formulas

- Using $p-1$ forward differences with sampling period p
- Using $p-1$ backward differences with sampling period p
- Using p forward and $q-1$ backward differences with sampling period $p+q$
- etc.

The two-dimensional case

The sampling formula: general case

$$V_\Phi^2 = \Big\{ \sum_{n,m\in\mathbb{Z}} a_{nm}\, \Phi(t-n, s-m) \ : \ \{a_{nm}\} \in \ell^2(\mathbb{Z}^2) \Big\}$$

The **isomorphism** $\mathcal{T}_\Phi : L^2(0,1)^2 \longrightarrow V_\Phi^2$ mapping the orthonormal basis $\{e^{-2\pi i n x}\, e^{-2\pi i m y}\}_{n,m\in\mathbb{Z}}$ onto the Riesz basis $\{\Phi(t-n, s-m)\}_{n,m\in\mathbb{Z}}$ satisfies the **shifting property**

$$\mathcal{T}_\Phi\big[e^{-2\pi i n x} e^{-2\pi i m y} F(x,y)\big](t,s) = \mathcal{T}_\Phi[F](t-n, s-m)\,, \quad t,s \in \mathbb{R}\,,\ n,m\in\mathbb{Z}\,.$$

Any $f = \mathcal{T}_\Phi F \in V_\varphi^2$ can be expressed as

$$f(t,s) = \Big\langle F(x,y), K_{t,s}(x,y) \Big\rangle_{L^2(0,1)^2}\,, \quad t,s \in \mathbb{R}\,,$$

where $K_{t,s}(x,y) = \sum_{n,m\in\mathbb{Z}} \overline{\Phi(t-n, s-m)}\, e^{-2\pi i n x} e^{-2\pi i m y} \in L^2(0,1)^2$

For fixed $a, b \in [0, 1)$ and $n, m \in \mathbb{Z}$ we get

$$f(a+n, b+m) = \langle F, K_{a+n,b+m} \rangle_{L^2(0,1)^2} = \langle F, \mathrm{e}^{-2\pi inx} \mathrm{e}^{-2\pi imy} K_{a,b}(x,y) \rangle$$

The sequence $\{x_{n,m} := \mathrm{e}^{-2\pi inx}\, \mathrm{e}^{-2\pi imy} K_{a,b}(x,y)\}_{n,m\in\mathbb{Z}}$ is a **Riesz basis** for $L^2(0,1)^2 \iff 0 < \|K_{a,b}\|_0 \leq \|K_{a,b}\|_\infty < \infty$

Its **dual Riesz basis** is $\{y_{n,m} := \mathrm{e}^{-2\pi inx}\, \mathrm{e}^{-2\pi imy} / \overline{K_{a,b}}(x,y)\}_{n,m\in\mathbb{Z}}$

For any $f \in V_\Phi^2$ we obtain the **sampling formula**

$$f(t,s) = \sum_{n,m\in\mathbb{Z}} f(a+n, b+m)\, S_{a,b}(t-n, s-m)\,, \quad t, s \in \mathbb{R}$$

where $S_{a,b} = \mathcal{T}_\Phi\big(1/\overline{K_{a,b}}\big) \in V_\Phi^2$

The sampling formula: separable variables

Consider a generator $\Phi(x,y) := \varphi(x)\psi(y)$ in $L^2(\mathbb{R}^2)$ with $\varphi(x)$ and $\psi(y)$ in $L^2(\mathbb{R})$

Its associated shift-invariant subspace $V^2_{\varphi\psi} := V^2_{\Phi} = V^2_{\varphi} \otimes V^2_{\psi}$

Straightforward, $\mathcal{T}_{\varphi\psi} := \mathcal{T}_{\Phi} = \mathcal{T}_{\varphi} \otimes \mathcal{T}_{\psi}$ holds and, consequently, for $H, G \in L^2(0,1)$ and $n, m \in \mathbb{Z}$, the **shifting property**

$$\mathcal{T}_{\varphi\psi}\left[e^{-2\pi inx}e^{-2\pi imy}H(x)G(y)\right](t,s) = \mathcal{T}_{\varphi}[H](t-n)\mathcal{T}_{\psi}[G](s-m),\ t,s \in \mathbb{R}\,.$$

Any $f = \mathcal{T}_{\varphi\psi}F \in V^2_{\varphi\psi}$ can be expressed as

$$f(t,s) = \left\langle F(x,y), K_t(x)\widetilde{K}_s(y)\right\rangle_{L^2(0,1)^2}, \quad t,s \in \mathbb{R}\,,$$

where $K_t(x) = \sum_{n\in\mathbb{Z}} \overline{\varphi(t-n)}\, e^{-2\pi inx}$ and $\widetilde{K}_s(y) = \sum_{m\in\mathbb{Z}} \overline{\psi(s-m)}\, e^{-2\pi imy}$

For fixed $a, b \in [0,1)$ and $n, m \in \mathbb{Z}$ we get

$$f(a+n, b+m) = \big\langle F, K_{a+n}\,\widetilde{K}_{b+m}\big\rangle_{L^2(0,1)^2} = \big\langle F, \mathrm{e}^{-2\pi i n x} K_a(x)\, \mathrm{e}^{-2\pi i m y} \widetilde{K}_b(y)\big\rangle$$

If $0 < \|K_a\|_0 \leq \|K_a\|_\infty < \infty$ and $0 < \|\widetilde{K}_b\|_0 \leq \|\widetilde{K}_b\|_\infty < \infty$, the sequence $\big\{\mathrm{e}^{-2\pi i n x}\mathrm{e}^{-2\pi i m y} K_a(x)\widetilde{K}_b(y)\big\}_{n,m\in\mathbb{Z}}$ is a **Riesz basis** for $L^2(0,1)^2$ with **dual Riesz basis** $\big\{\mathrm{e}^{-2\pi i n x}\mathrm{e}^{-2\pi i m y}/(\overline{K_a(x)\widetilde{K}_b(y)})\big\}_{n,m\in\mathbb{Z}}$
In tensor notation, $\big\{x_n \otimes \widetilde{x}_m\big\}_{n,m\in\mathbb{Z}}$ and $\big\{y_n \otimes \widetilde{y}_m\big\}_{n,m\in\mathbb{Z}}$ form a pair of dual Riesz bases in $L^2(0,1)^2$
For any $f \in V^2_{\varphi\psi}$ the **sampling formula**

$$f(t,s) = \sum_{n,m\in\mathbb{Z}} f(a+n, b+m)\, S_a(t-n)\, \widetilde{S}_b(s-m)\,, \quad t, s \in \mathbb{R}\,,$$

where $S_a := \mathcal{T}_\varphi\big(1/\overline{K_a}\big) \in V^2_\varphi$ and $\widetilde{S}_b := \mathcal{T}_\psi\big(1/\overline{\widetilde{K}_b}\big) \in V^2_\psi$

Now we will use **forward differences** defined as:

$$\Delta_+^{k,k'} f = \Delta_+^{1,0}\big(\Delta_+^{k-1,k'} f\big) = \Delta_+^{0,1}\big(\Delta_+^{k,k'-1} f\big)$$

where

$$\Delta_+^{1,0} f(t,s) = f(t+1,s) - f(t,s) \quad \text{and} \quad \Delta_+^{0,1} f(t,s) = f(t,s+1) - f(t,s)$$

We adopt the convention $\Delta_+^{0,0} = I$

For any $f \in V_{\varphi\psi}^2$ (such that $\mathcal{T}_{\varphi\psi} F = f$) or $f \in V_\Phi^2$ (such that $\mathcal{T}_\Phi F = f$), we respectively get

$$\Delta_+^{k,k'} f(t,s) = \big\langle F, (e^{-2\pi i x} - 1)^k \, (e^{-2\pi i y} - 1)^{k'} \, K_t(x) \, \widetilde{K}_s(y) \big\rangle_{L^2(0,1)^2}$$

or

$$\Delta_+^{k,k'} f(t,s) = \big\langle F, (e^{-2\pi i x} - 1)^k \, (e^{-2\pi i y} - 1)^{k'} \, K_{t,s}(x,y) \big\rangle_{L^2(0,1)^2}$$

Using $\Delta_+^{k,k'} f(a+2n, b+3m)$, $(k = 0, 1\,;\, k' = 0, 1, 2)$ in $V^2_{\varphi\psi}$

$$\Delta_+^{k,k'} f(a+2n, b+3m) = \left\langle F, (e^{-2\pi i x} - 1)^k (e^{-2\pi i y} - 1)^{k'} e^{-2\pi i (2n)x} K_a(x) e^{-2\pi i (3m)y} \widetilde{K}_b(y) \right\rangle$$

where $n, m \in \mathbb{Z}$ and $k = 0, 1\,;\, k' = 0, 1, 2$.

Consider the Riesz basis $\{x_n \otimes \widetilde{x}_m\}_{n,m\in\mathbb{Z}}$ for $L^2(0,1)^2$, where $x_n := e^{-2\pi i n x} K_a(x)$ and $\widetilde{x}_m := e^{-2\pi i m y} \widetilde{K}_b(y)$, $n, m \in \mathbb{Z}$.

For the partition

$$\{x_{2n} \otimes \widetilde{x}_{3m}\} \cup \{x_{2n} \otimes \widetilde{x}_{3m+1}\} \cup \{x_{2n} \otimes \widetilde{x}_{3m+2}\} \cup \{x_{2n+1} \otimes \widetilde{x}_{3m}\} \cup \{x_{2n+1} \otimes \widetilde{x}_{3m+1}\} \cup \{x_{2n+1} \otimes \widetilde{x}_{3m+2}\},$$

the matrix of the Riesz basis associated with the above samples will be the **Kronecker product** of the corresponding matrices derived for the one-dimensional case. That is:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 2 & -1 & 1 & -2 & 1 \end{pmatrix}$$

Its inverse is

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 \end{pmatrix}$$

Thus we obtain the sampling formula:

$$f(t,s) = \sum_{n,m\in\mathbb{Z}} \Big\{ \sum_{k=0}^{1} \sum_{k'=0}^{2} \Delta_+^{k,k'} f(a+2n, b+3m)\, \widetilde{T}_{a,b}^{k,k'}(t-2n, s-3m) \Big\},\ t,s\in\mathbb{R}$$

where

$$\begin{aligned}
\widetilde{T}_{a,b}^{0,0}(t,s) &= S_a(t)\,\widetilde{S}_b(s) + S_a(t)\,\widetilde{S}_b(s-1) + S_a(t)\,\widetilde{S}_b(s-2) + S_a(t-1)\,\widetilde{S}_b(s) \\
&\quad + S_a(t-1)\,\widetilde{S}_b(s-1) + S_a(t-1)\,\widetilde{S}_b(s-2) \\
\widetilde{T}_{a,b}^{0,1}(t,s) &= S_a(t)\,\widetilde{S}_b(s-1) + 2S_a(t)\,\widetilde{S}_b(s-2) + S_a(t-1)\,\widetilde{S}_b(s-1) \\
&\quad + 2S_a(t-1)\,\widetilde{S}_b(s-2) \\
\widetilde{T}_{a,b}^{0,2}(t,s) &= S_a(t)\,\widetilde{S}_b(s-2) + S_a(t-1)\,\widetilde{S}_b(s-2)
\end{aligned}$$

$$\widetilde{T}^{1,0}_{a,b}(t,s) = S_a(t-1)\,\widetilde{S}_b(s) + S_a(t-1)\,\widetilde{S}_b(s-1) + S_a(t-1)\,\widetilde{S}_b(s-2)$$
$$\widetilde{T}^{1,1}_{a,b}(t,s) = S_a(t-1)\,\widetilde{S}_b(s-1) + 2S_a(t-1)\,\widetilde{S}_b(s-2)$$
$$\widetilde{T}^{1,2}_{a,b}(t,s) = S_a(t-1)\,\widetilde{S}_b(s-2)\,, \quad t,s \in \mathbb{R}$$

Observe that

$$\mathcal{T}_{\varphi\psi}\big(y_{2n+k} \otimes \widetilde{y}_{3m+k'}\big) = S_a(t-(2n+k))\widetilde{S}_b(s-(3m+k'))$$

where $y_n := e^{-2\pi i n x}/\overline{K_a(x)}$ and $\widetilde{y}_m := e^{-2\pi i m y}/\overline{\widetilde{K}_b(y)}$, $\quad n,m \in \mathbb{Z}$

Using $\Delta_+^{k,k'} f(a+3n, b+3m)$, $(k, k' = 0, 1, 2)$ in V_Φ^2

$$\Delta_+^{k,k'} f(a+3n, b+3m) = \left\langle F, (e^{-2\pi ix}-1)^k (e^{-2\pi iy}-1)^{k'} e^{-2\pi i(3n)x} e^{-2\pi i(3m)y} K_{a,b}(x,y) \right\rangle$$

where $n, m \in \mathbb{Z}$ and $k, k' = 0, 1, 2$.

Consider the Riesz basis $\{x_{n,m}\}_{n,m\in\mathbb{Z}}$ for $L^2(0,1)^2$, where $x_{n,m} := \mathrm{e}^{-2\pi inx} \mathrm{e}^{-2\pi imy} K_{a,b}(x,y)$, $n, m \in \mathbb{Z}$.

For the partition

$$\begin{aligned} &\{x_{3n,3m}\} \cup \{x_{3n,3m+1}\} \cup \{x_{3n,3m+2}\} \\ &\quad \cup \{x_{3n+1,3m}\} \cup \{x_{3n+1,3m+1}\} \cup \{x_{3n+1,3m+2}\} \\ &\quad \cup \{x_{3n+2,3m}\} \cup \{x_{3n+2,3m+1}\} \cup \{x_{3n+2,3m+2}\} \end{aligned}$$

the matrix of the associate Riesz basis with the given samples is the **Kronecker product**:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & -2 & 0 & -1 & 1 & 0 \\ 1 & -2 & 1 & -2 & 4 & -2 & 1 & -2 & 1 \end{pmatrix}$$

Its inverse is:

$$\begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 2 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0\\ 1 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0\\ 1 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 1\end{pmatrix}$$

Thus we obtain the sampling formula:

$$f(t,s) = \sum_{n,m\in\mathbb{Z}} \Big\{ \sum_{k,k'=0}^{2} \Delta_{+}^{k,k'} f(a+3n, b+3m)\, T_{a,b}^{k,k'}(t-3n, s-3m) \Big\},\ t,s \in \mathbb{R}$$

where

$$\begin{aligned}
T_{a,b}^{0,0}(t,s) &= S_{a,b}(t,s) + S_{a,b}(t,s-1) + S_{a,b}(t,s-2) \\
&+ S_{a,b}(t-1,s) + S_{a,b}(t-1,s-1) + S_{a,b}(t-1,s-2) \\
&+ S_{a,b}(t-2,s) + S_{a,b}(t-2,s-1) + S_{a,b}(t-2,s-2) \\
T_{a,b}^{0,1}(t,s) &= S_{a,b}(t,s-1) + 2S_{a,b}(t,s-2) + S_{a,b}(t-1,s-1) \\
&+ 2S_{a,b}(t-1,s-2) + S_{a,b}(t-2,s-1) + 2S_{a,b}(t-2,s-2) \\
T_{a,b}^{0,2}(t,s) &= S_{a,b}(t,s-2) + S_{a,b}(t-1,s-2) + S_{a,b}(t-2,s-2)
\end{aligned}$$

$$\begin{aligned}
T_{a,b}^{1,0}(t,s) &= S_{a,b}(t-1,s) + S_{a,b}(t-1,s-1) + S_{a,b}(t-1,s-2) \\
&\quad + 2S_{a,b}(t-2,s-2) + 2S_{a,b}(t-2,s-1) + 2S_{a,b}(t-2,s-2) \\
T_{a,b}^{1,1}(t,s) &= S_{a,b}(t-1,s-1) + 2S_{a,b}(t-1,s-2) \\
&\quad + 2S_{a,b}(t-2,s-1) + 4S_{a,b}(t-2,s-2) \\
T_{a,b}^{1,2}(t,s) &= S_{a,b}(t-1,s-2) + 2S_{a,b}(t-2,s-2) \\
T_{a,b}^{2,0}(t,s) &= S_{a,b}(t-2,s) + S_{a,b}(t-2,s-1) + S_{a,b}(t-2,s-2) \\
T_{a,b}^{2,1}(t,s) &= S_{a,b}(t-2,s-1) + 2S_{a,b}(t-2,s-2) \\
T_{a,b}^{2,2}(t,s) &= S_{a,b}(t-2,s-2)\,, \quad t,s \in \mathbb{R}
\end{aligned}$$

Observe that

$$\mathcal{T}_\Phi\left(y_{3n+k,3m+k'}\right) = S_{a,b}(t-(3n+k), s-(3m+k'))\,,$$

where $y_{n,m} := \mathrm{e}^{-2\pi inx}\,\mathrm{e}^{-2\pi imy}/\overline{K_{a,b}(x,y)}$, $n,m \in \mathbb{Z}$

That's all!