Ibero-American Seminar on Mathematical Analysis and Applied Mathematics

From classical sampling to average sampling in shift-invariant-like subspaces of Hilbert-Schmidt operators

Antonio García García

Universidad Carlos III de Madrid
Departamento de Matemáticas

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Classical sampling theory at a glance

The WKS sampling theorem

Any function f in the *Paley-Wiener space*:

$$PW_\pi := \left\{ f \in L^2(\mathbb{R}) \cap C(\mathbb{R}), \quad \operatorname{supp} \widehat{f} \subseteq [-\pi, \pi] \right\}$$

i.e., *bandlimited* to the interval $[-\pi, \pi]$, can be expressed as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (t-n)}{\pi (t-n)}, \quad t \in \mathbb{R}$$

The series converges in $L^2(\mathbb{R})$-sense and also absolutely y uniformly on $\mathbb{R}$.

WKS means Whittaker–Kotel'nikov–Shannon

- Any $f \in PW_\pi$ can be expressed as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{f}(w)\, \mathrm{e}^{itw} dw = \langle \widehat{f}, \frac{e^{-itw}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]}, \quad t \in \mathbb{R}$$

In particular,

$$f(n) = \langle \widehat{f}, \frac{e^{-inw}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]}, \quad n \in \mathbb{Z}$$

- The sampling period is $T_s = \frac{2\pi}{2\pi} = 1$
- Observe that it is a Lagrange-type interpolation formula

$$f(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{G(t)}{G'(n)(t-n)}, \quad t \in \mathbb{R}, \quad \text{with} \quad G(t) = \frac{\sin \pi t}{\pi}$$

Shannon's original statement

THEOREM 1: *If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart.*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-2\pi W}^{2\pi W} \widehat{f}(w)\, \mathrm{e}^{itw} dw\,, \quad t \in \mathbb{R}$$

Here the sampling period is $T_s = \frac{2\pi}{4\pi W} = \frac{1}{2W}$, and the sampling formula reads:

$$f(t) = \sum_{n=-\infty}^{\infty} f\Big(\frac{n}{2W}\Big) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}\,, \quad t \in \mathbb{R}$$

A hilbertian easy proof:

1. We expand $\widehat{f}$ following the orthonormal basis $\{e^{-inw}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ for $L^2[-\pi,\pi]$:

$$\underbrace{\widehat{f}} = \sum_{n=-\infty}^{\infty} \langle \widehat{f}, \frac{e^{-inw}}{\sqrt{2\pi}} \rangle_{L^2[-\pi,\pi]} \frac{e^{-inw}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} \underbrace{f(n)} \frac{e^{-inw}}{\sqrt{2\pi}}$$

2. We apply the inverse Fourier transform $\mathcal{F}^{-1}$:

$$f = \sum_{n=-\infty}^{\infty} f(n) \underbrace{\mathcal{F}^{-1}} \Big(\frac{e^{-inw}}{\sqrt{2\pi}} \chi_{[-\pi,\pi]}(w) \Big)(t) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$$

Convergence in the L^2-norm sense

Observe that $\mathcal{F}^{-1}\Big(\frac{1}{\sqrt{2\pi}}\chi_{[-\pi,\pi]}(w)\Big)(t) = \frac{\sin \pi t}{\pi t}(t)$

Some consequences and comments:

- Convergence in L^2-norm implies pointwise convergence, which is also uniform on $\mathbb{R}$: the Paley-Wiener space PW_π is a RKHS (reproducing kernel Hilbert space)

$$|f(t)| = \left|\left\langle \widehat{f}, \frac{e^{-itw}}{\sqrt{2\pi}} \right\rangle_{L^2[-\pi,\pi]}\right| \leq \|f\|_{L^2(\mathbb{R})}, \quad \forall t \in \mathbb{R}$$

- The uncondicional character of an orthonormal basis implies absolute convergence
- The sequence $\left\{ \dfrac{\sin \pi(t-n)}{\pi(t-n)} \right\}_{n\in\mathbb{Z}}$ is an orthonormal basis for PW_π
- The Parseval identity gives the *conservation of the signal energy*

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |f(n)|^2, \quad \forall f \in PW_\pi$$

- **Average sampling:** For a fixed function $\psi \in L^2(\mathbb{R})$ the samples are taken from the LTI system $\mathcal{A}_\psi$, i.e.,

$$\mathcal{A}_\psi f(n) = \langle f, \psi(\cdot - n)\rangle = \langle f, T_n\psi\rangle = (f * \widetilde{\psi})(n)\,, \quad n \in \mathbb{Z}\,,$$

  where $\widetilde{\psi}(t) = \overline{\psi(-t)}$ is the average function
- **It includes also pointwise sampling:** For any $f \in PW_\pi$,

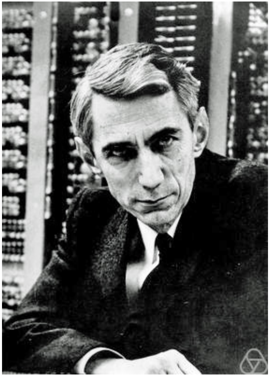
$$f(n) = (f * \mathrm{sinc})(n)\,, \quad n \in \mathbb{Z}$$

- Due to the drawbacks of the sine cardinal function, one considers shift-invariant subspaces as

$$V_\varphi^2 = \Big\{ f(t) = \sum_{n\in\mathbb{Z}} a_n\, \varphi(t-n) \;:\; \{a_n\} \in \ell^2(\mathbb{Z}) \Big\} \subset L^2(\mathbb{R})$$

  In particular, PW_π is a shift-invariant subspace with generator the sine cardinal function, i.e., $\varphi(t) = \mathrm{sinc}(t) := \dfrac{\sin \pi t}{\pi t}\,,\ t \in \mathbb{R}$

Claude E. Shannon (1916–2001)



- B. S. Engineering Mathematics, University of Michigan, 1936
  Ph. D. Mathematics, MIT, 1940
- Research Mathematician, Bell Labs, 1941–1972; MIT Faculty Member, 1956–1978; Donner Professor of Science 1958
- Major publication: **A Mathematical Theory of Communication**, Bell System Technical Report, 1948
- Honorary Degree, Univ. of Michigan, 1961; The National Medal of Science, 1966; The Audio Engineering Society Gold Medal, 1985; The Kyoto Prize, 1985

The american mathematician, computer scientist, communication engineer, and the founder ot the field of **Information Theory***, whose work has laid the foundation for the telecommunication networks that lace the globe*

Frames in a separable Hilbert space

- The sequence $\{x_n\}_{n=1}^{\infty}$ is a frame for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ (frame bounds) such that:

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2 \leq B\|x\|^2\,, \quad \forall x \in \mathcal{H}$$

- The preframe operator T is defined by:

$$\ell^2(\mathbb{Z}) \ni \{c_n\}_{n=1}^{\infty} \overset{T}{\longmapsto} \sum_{n=1}^{\infty} c_n\, x_n \in \mathcal{H}$$

It characterizes a frame:

$\{x_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ $\iff$ T is bounded and surjective

- Two important particular cases:
  - T is also unitary $\iff$ $\{x_n\}_{n=1}^{\infty}$ is an orthonormal basis
  - T is also injective (isomorphism) $\iff$ $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis
- Given a frame $\{x_n\}_{n=1}^{\infty}$ there exist frames $\{y_n\}_{n=1}^{\infty}$ (dual frames) such that:

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle y_n = \sum_{n=1}^{\infty} \langle x, y_n \rangle x_n \quad \forall x \in \mathcal{H}$$

- In case of bases, the dual frame is unique:
  - For orthonormal bases $\{y_n\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$
  - For Riesz bases $\{y_n\}_{n=1}^{\infty}$ is its dual Riesz basis of $\{x_n\}_{n=1}^{\infty}$
- For overcomplete frames, there exist infinite dual frames $\{y_n\}_{n=1}^{\infty}$ satisfying the above equalities

Summarizing the key points in the WKS theorem proof:

(a) $\mathcal{H}$ is a Hilbert space where we are sampling. The samples of any $f \in \mathcal{H}$, can be expressed as frame coefficients $f(n) = \langle x, x_n \rangle_{\mathcal{K}}$ in a frame expansion

$$x = \sum_n \langle x, x_n \rangle \, y_n = \sum_n f(n) \, y_n \quad \text{in } \mathcal{K},$$

where $\{x_n\}$ and $\{y_n\}$ dual frames for an auxiliary Hilbert space $\mathcal{K}$

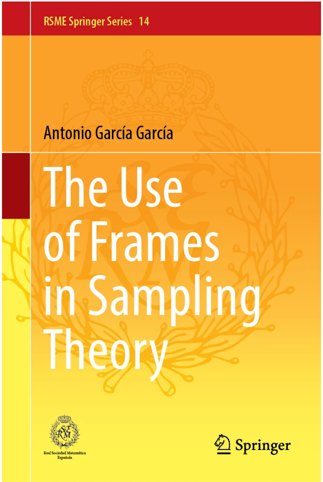
(b) There exists an isomorphism $\mathcal{T} : \mathcal{K} \longrightarrow \mathcal{H}$ relating both spaces, i.e., $\mathcal{T}x = f$

(c) By means of this isomorphism $\mathcal{T}$, the frame expansion in $\mathcal{K}$ in the first item yields a sampling expansion in $\mathcal{H}$:

$$f = (\mathcal{T}x) = \sum_n f(n) \, \mathcal{T}(y_n) \quad \text{in } \mathcal{H}$$

The isomorphism $\mathcal{T}$ should respect the structure in $\mathcal{H}$

The above argument, borrowed from the hilbertian proof of the WSK sampling theorem, has been profusely used in my research and, as a consequence...



Average sampling in shift-invariant subspace in $L^2(\mathbb{R}^d)$

- Consider a classical shift-invariant subspaces in $L^2(\mathbb{R}^d)$ with a stable set of generators $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$

$$V_\Phi^2 = \Big\{ \sum_{n=1}^{N} \sum_{\alpha\in\mathbb{Z}^d} c_n(\alpha)\, \varphi_n(t-\alpha) \ : \ \{c_n(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\,,\, n=1,\dots,N \Big\}$$

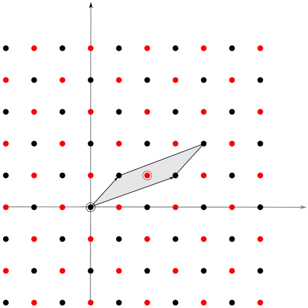
In other words, the sequence $\{\varphi_n(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, n=1,2,\dots,N}$ is a Riesz basis for V_Φ^2

- For any $f \in V_\Phi^2$ we consider average samples

$$\{\langle f, \psi_m(\cdot - \alpha)\rangle\}_{\alpha\in P\mathbb{Z}^d}\,, \quad m = 1, 2, \dots, M\,,$$

ψ_m are the *average functions* (not necessarily in V_Φ^2), $m = 1, 2, \dots, M$
P is a $d \times d$ matrix with integer entries and $\det P > 0$
The set $\Lambda := P\,\mathbb{Z}^d$ is a full rank lattice in $\mathbb{R}^d$

For instance, in $\mathbb{Z}^2$, the lattice $\Lambda = P\,\mathbb{Z}^2$ where $P = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ with $\det P = 2$, is depicted as the black points:



- $P[0,1)^2$ is the fundamental parallelepided of Λ
- $\mathbb{Z}^2 \cap P[0,1)^2 = \{(0,0)^\top, (2,1)^\top\}$
- $\mathbb{Z}^2/\Lambda = \{\Lambda, (2,1)^\top + \Lambda\}$

• Then, under appropriate hypotheses (similar to those in the second part of this talk) there exist $M(\geq N \det P)$ sampling functions $S_m \in V_\Phi^2$ such that, for each $f \in V_\Phi^2$:

$$f(t) = \sum_{m=1}^{M} \sum_{\alpha \in P\mathbb{Z}^d} \langle f, \psi_m(\cdot - \alpha) \rangle \, S_m(t - \alpha) \,, \quad t \in \mathbb{R}^d$$

and the sequence $\{S_m(t-\alpha)\}_{\alpha \in P\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for V_Φ^2

Sampling in subspaces of Hilbert-Schmidt operators

A mathematical motivation: Average sampling in shift-invariant-like subspaces of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

► A Hilbert-Schmidt operator $H : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is an integral operator

$$Hf(t) = \int_{\mathbb{R}^d} \kappa_H(t,s) f(s)\, ds$$

where the kernel $\kappa_H \in L^2(\mathbb{R}^{2d})$ and $\|H\|_{\mathcal{HS}} = \|\kappa_H\|_{L^2(\mathbb{R}^{2d})}$

► *The translation of an operator* $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $z = (x, \omega)$ in the *phase space* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d (\simeq \mathbb{R}^{2d})$ is defined by

$$\alpha_z(S) := \pi(z) S \pi(z)^*, \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$$

where $\pi(z)$ denotes the time-frequency shift which acts on $f \in L^2(\mathbb{R}^d)$ as

$$\pi(z) f(t) = \mathrm{e}^{2\pi i \omega \cdot t} f(t - x), \quad t \in \mathbb{R}^d$$

- The set of translations $\{\alpha_z\}_{z\in\mathbb{R}^{2d}}$ is a *unitary representation* of the additive group $\mathbb{R}^{2d}$ on the Hilbert space $\left(\mathcal{HS}(\mathbb{R}^d), \langle\cdot,\cdot\rangle_{\mathcal{HS}}\right)$ of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

- Let Λ be a full rank lattice in $\mathbb{R}^{2d}$, i.e., $\Lambda = A\mathbb{Z}^{2d}$ where A is a $2d \times 2d$ real invertible matrix.

For $S_n \in \mathcal{HS}(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$, we could consider the (closed) subspace of $\mathcal{HS}(\mathbb{R}^d)$ given by

$$V_{\mathbf{S}}^2 = \Big\{ \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \ : \ \{c_n(\lambda)\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)\,,\ n = 1, 2, \ldots, N \Big\}$$

in the case that $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for $V_{\mathbf{S}}^2$

- We could define, for any $T \in V^2_{\mathbf{S}}$ its average samples at Λ by

$$\big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}}, \quad \lambda \in \Lambda, \; m = 1, 2, \ldots, M$$

from M fixed operators $Q_1, Q_2, \ldots, Q_M$ in $\mathcal{HS}(\mathbb{R}^d)$, the *average operators* (not necessarily in $V^2_{\mathbf{S}}$)

A mathematical question

Under which hypotheses there exist $M(\geq N)$ operators $H_m \in V^2_{\mathbf{S}}$ such that for each $T \in V^2_{\mathbf{S}}$

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}} \, \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

where the sequence $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a *frame* for the Hilbert space $V^2_{\mathbf{S}}$?

- The *adjoint operator* is $\pi(z)^* = e^{-2\pi i x\cdot\omega}\,\pi(-z)$ for $z = (x,\omega) \in \mathbb{R}^{2d}$
- The *short-time Fourier transform* (Gabor transform) $V_\psi\varphi$ of φ with window ψ, both in $L^2(\mathbb{R}^d)$, is defined by

$$V_\psi\varphi(z) = \langle \varphi, \pi(z)\psi \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \varphi(t)\, e^{-2\pi i t\cdot\omega}\, \overline{\psi(t-x)}\, dt\,, \quad z \in \mathbb{R}^{2d}$$

$V_\psi\varphi(z)$ gives time-frequency information on φ at $z = (x,\omega)$

A practical question

Why to sample Hilbert-Schmidt operators with precisely the average samples?

A practical motivation: LTV versus LTI systems

- Linear time-invariant (LTI) system

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h(s)\, x(t-s)\, ds = \int_{-\infty}^{\infty} \widehat{h}(w)\, \widehat{x}(w) \mathrm{e}^{2\pi i w t} dw$$

- Linear time-varying (LTV) system

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h(t,s)\, x(t-s)\, ds = \int_{-\infty}^{\infty} \sigma(t,w)\, \widehat{x}(w) \mathrm{e}^{2\pi i w t} dw$$

where

$$\sigma = \mathcal{F}_2\, h\,, \text{ i.e.,} \quad \sigma(t,w) = \int_{-\infty}^{\infty} h(t,s)\, \mathrm{e}^{-2\pi i w s} ds$$

Thus, operator H is a pseudo-differential operator with symbol σ

In particular, Hilbert-Schmidt operators model LTV systems:

$$Hf(t) = \int_{-\infty}^{\infty} \kappa_H(t,s) f(s)\, ds$$

where the integral kernel $\kappa_H \in L^2(\mathbb{R}^2)$

Besides

$$Hf(t) = \int_{-\infty}^{\infty} \kappa_H(t,s) f(s)\, ds = \int_{-\infty}^{\infty} \kappa_H(t,t-s) f(t-s)\, ds$$

is a LTV system where $h(t,s) = \kappa_H(t,t-s)$

Concerning the average samples:

In *Orthogonal frequency-division multiplexing* (OFDM) the digital information, i.e., a sequence of numbers $\{c_\lambda\}$, λ in the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ $(a, b > 0)$, is used as the coefficients of the input signal $x(t) = \sum_{\mu\in\Lambda} c_\mu\, \pi(\mu)g(t)$ of a time-varying system H producing the output $y(t) = Hx(t)$. Then, it is considered the sequence of numbers

$$d_\lambda = \big\langle y, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)} = \sum_{\mu\in\Lambda} c_\mu \left\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\right\rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda\,,$$

The task: to recover the original data $\{c_\lambda\}$ from the received data $\{d_\lambda\}$

The matrix $A = [a_{\lambda,\mu}]$, where $a_{\lambda,\mu} = \left\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\right\rangle_{L^2(\mathbb{R}^d)}$ is the so-called *channel matrix* associated with H and the functions (windows) $g, \widetilde{g}$ in $L^2(\mathbb{R}^d)$

Diagonal channel samples

The diagonal channel samples of H with respect to $g, \widetilde{g}$ are

$$\left\langle H\pi(\lambda)g, \pi(\lambda)\widetilde{g} \right\rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda$$

They are also known as:

- The *lower symbol of the operator H* with respect $g, \widetilde{g} \in L^2(\mathbb{R}^d)$ and lattice Λ used in *time-frequency analysis*
- The samples of the *Berezin transform* of H

$$\mathcal{B}^{g,\widetilde{g}} H(z) := \left\langle H\pi(z)g, \pi(z)\widetilde{g} \right\rangle_{L^2(\mathbb{R}^d)}, \quad z \in \mathbb{R}^{2d}$$

  at the lattice Λ used in *quantum physics*
- Diagonal channel samples are a particular case of average samples

Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

- For a compact operator S on $L^2(\mathbb{R}^d)$ there exist two orthonormal sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a bounded sequence of positive numbers $\{s_n(S)\}_{n\in\mathbb{N}}$ (*singular values* of S) such that

$$S = \sum_{n\in\mathbb{N}} s_n(S)\, x_n \otimes y_n \qquad \text{(SVD)}$$

with convergence of the series in the operator norm. The sequence $\{s_n(S)\}$ consists of the eigenvalues of the operator $|T| = (S^*S)^{1/2}$
Here, $x_n \otimes y_n$ denotes the rank-one operator

$$\big(x_n \otimes y_n\big)(f) = \big\langle f, y_n \big\rangle_{L^2} x_n \quad \text{for } f \in L^2(\mathbb{R}^d)$$

- The class of *Hilbert-Schmidt operators* is $\mathcal{HS}(\mathbb{R}^d) := \mathcal{T}^2$
$\mathcal{T}^2$ is the Schatten 2-class, i.e., singular values in $\ell^2(\mathbb{N})$
The space $\mathcal{HS}(\mathbb{R}^d)$ is a Hilbert space with the inner product

$$\big\langle S, T \big\rangle_{\mathcal{HS}} = \mathrm{tr}(ST^*)\,, \quad S, T \in \mathcal{HS}(\mathbb{R}^d)$$

Remind that the trace $\mathrm{tr}(S) = \sum_{n\in\mathbb{N}} \langle Se_n, e_n \rangle_{L^2}$ is a well-defined bounded linear functional on $\mathcal{T}^1$, and independent of the used orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$

- Concerning the norm of S in $\mathcal{HS}(\mathbb{R}^d)$ we have

$$\|S\|_{\mathcal{HS}}^2 = \mathrm{tr}(SS^*) = \sum_{n\in\mathbb{N}} \|S^*(e_n)\|_{L^2}^2 = \sum_{n\in\mathbb{N}} \|S(e_n)\|_{L^2}^2 = \sum_{n\in\mathbb{N}} s_n^2(S)$$

- A Hilbert-Schmidt operator $S \in \mathcal{HS}(\mathbb{R}^d)$ can be seen also as an integral operator on $L^2(\mathbb{R}^d)$ defined for each $f \in L^2(\mathbb{R}^d)$ by

$$Sf(t) = \int_{\mathbb{R}^d} \kappa_S(t,x) f(x)\, dx \quad \text{a.e. } t \in \mathbb{R}^d$$

with kernel $\kappa_S \in L^2(\mathbb{R}^{2d})$. Besides, $\left\langle S, T \right\rangle_{\mathcal{HS}} = \left\langle \kappa_S, \kappa_T \right\rangle_{L^2(\mathbb{R}^{2d})}$ for $S, T \in \mathcal{HS}(\mathbb{R}^d)$

The Weyl transform

The Weyl transform is a unitary operator

$$L^2(\mathbb{R}^{2d}) \ni f \longmapsto L_f \in \mathcal{HS}(\mathbb{R}^d)$$

where $L_f : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\big\langle L_f \phi, \psi \big\rangle_{L^2(\mathbb{R}^d)} = \big\langle f, W(\psi, \phi) \big\rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d)$$

here

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi\big(x + \frac{t}{2}\big) \overline{\phi\big(x - \frac{t}{2}\big)} \, \mathrm{e}^{-2\pi i \omega \cdot t} dt \,, \quad (x, \omega) \in \mathbb{R}^{2d} \,,$$

is the *cross-Wigner distribution* of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$
For each $S, T \in \mathcal{HS}(\mathbb{R}^d)$ with Weyl symbols a_S, a_T in $L^2(\mathbb{R}^{2d})$ we have

$$\langle S, T \rangle_{\mathcal{HS}} = \langle a_S, a_T \rangle_{L^2(\mathbb{R}^{2d})}$$

The Kohn-Nirenberg transform

The Kohn-Nirenberg transform is a unitary operator

$$L^2(\mathbb{R}^{2d}) \ni \sigma \longmapsto K_\sigma \in \mathcal{HS}(\mathbb{R}^d)$$

where $K_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\left\langle K_\sigma \phi, \psi \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle \sigma, R(\psi, \phi) \right\rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d)$$

here

$$R(\psi, \phi)(x, \omega) = \psi(x)\, \overline{\widehat{\phi}(\omega)}\, \mathrm{e}^{-2\pi i x \cdot \omega}, \quad (x, \omega) \in \mathbb{R}^{2d},$$

is the *Rihaczek distribution* of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$

For each $S, T \in \mathcal{HS}(\mathbb{R}^d)$ with Kohn-Nirenberg symbols σ_S, σ_T in $L^2(\mathbb{R}^{2d})$ we have

$$\langle S, T \rangle_{\mathcal{HS}} = \langle \sigma_S, \sigma_T \rangle_{L^2(\mathbb{R}^{2d})}$$

- There is a transition between Weyl and Kohn-Nirenberg calculus:

$$\sigma_s = Ua_s \,, \text{ where } \widehat{Ua_s}(\xi, u) = e^{\pi i u \cdot \xi}\, \widehat{a_s}(\xi, u)\,,\ (\xi, u) \in \mathbb{R}^{2d}$$

- The Weyl and Kohn-Nirenberg transforms in $\mathcal{HS}(\mathbb{R}^d)$ respect both the translations in the sense:

A key property for both transforms

For $f \in L^2(\mathbb{R}^{2d})$ and $z \in \mathbb{R}^{2d}$ we have:

$$\underline{\mathcal{L}(T_z f) = \alpha_z(\mathcal{L} f)}$$

where $\mathcal{L}$ denotes the Weyl or the Kohn-Nirenberg transform

- As a consequence:

Properties of V_S^2 in $\mathcal{HS}(\mathbb{R}^d) \longleftrightarrow$ Properties of $V_{\sigma_S}^2$ (or $V_{a_S}^2$) in $L^2(\mathbb{R}^{2d})$

Λ-shift-invariant subspaces in $\mathcal{HS}(\mathbb{R}^d)$

Let $\mathbf{S} = \{S_1, S_2, \ldots, S_N\}$ be a fixed subset of $\mathcal{HS}(\mathbb{R}^d)$ and let Λ be a lattice in $\mathbb{R}^{2d}$. We are searching for a necessary and sufficient condition such that $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for the closed subspace

$$V_{\mathbf{S}}^2 := \overline{\operatorname{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S_n)\big\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N} \subset \mathcal{HS}(\mathbb{R}^d)$$

In this case, $\mathbf{S} = \{S_1, S_2, \ldots, S_N\}$ is a *set of stable generators* for the Λ*-shift-invariant subspace* $V_{\mathbf{S}}^2$ which can be described by

$$V_{\mathbf{S}}^2 = \Big\{ \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \ : \ \{c_n(\lambda)\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)\,,\, n = 1, 2, \ldots, N \Big\}$$

Theorem

Let Λ be a lattice and $S_n \in \mathcal{B}$, $n = 1, 2, \ldots, N$. Then, $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$ if and only if there exist two constants $0 < m \le M$ such that

$$m\,\mathbb{I}_N \le G_{\mathbf{S}}^W(z) \le M\,\mathbb{I}_N \quad \text{for any } z \in \mathbb{R}^{2d},$$

where $G_{\mathbf{S}}^W(z)$ denotes the $N \times N$ matrix-valued function

$$G_{\mathbf{S}}^W(z) := \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)\,\overline{\mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)}^\top, \quad z \in \mathbb{R}^{2d}$$

and $\mathcal{F}_W(\mathbf{S}) = \big(\mathcal{F}_W(S_1), \mathcal{F}_W(S_2), \ldots, \mathcal{F}_W(S_N)\big)^\top$

where:

- $S \in \mathcal{B}$ is the Banach space of continuous operators on $L^2(\mathbb{R}^d)$ with Weyl symbol a_s in the *Feichtinger's algebra* $\mathcal{S}_0(\mathbb{R}^{2d})$. In essence, $\mathcal{B}$ consists of trace class operators on $L^2(\mathbb{R}^d)$ with a norm-continuous inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{T}^1$
  Recall that $\psi \in \mathcal{S}_0(\mathbb{R}^{2d})$ iff $V_{\varphi_0}\psi \in L^1(\mathbb{R}^{2d})$ where $\varphi_0(x) = 2^{d/4}\mathrm{e}^{-\pi x\cdot x}$ is the normalized d-dimensional gaussian function
- Λ° is the adjoint lattice of the lattice Λ. Its associated matrix is $A^{-\top}\Omega_d$ in case $\Lambda = A\mathbb{Z}^{2d}$, where

$$\Omega_d = \begin{pmatrix} O & I_d \\ -I_d & O \end{pmatrix}$$

- $\mathcal{F}_W(S)$ denotes the Fourier-Wigner transform of an operator S defined as the function

$$\mathcal{F}_W(S)(z) := \mathrm{e}^{-\pi i x\cdot\omega}\,\mathrm{tr}[\pi(-z)S]\,, \quad z = (x,\omega) \in \mathbb{R}^{2d}$$

- In our case, $\mathcal{F}_W(S_n) = \mathcal{F}_s(a_{S_n})$ for $n = 1, 2, \ldots, N$, where $\mathcal{F}_s$ denotes the *symplectic Fourier transform* of a_{S_n} defined by

$$\mathcal{F}_s(a_{S_n})(z) := \int_{\mathbb{R}^{2d}} a_{S_n}(z')\, \mathrm{e}^{-2\pi i\, \sigma(z,z')} dz' \,, \quad z \in \mathbb{R}^{2d}$$

  $\sigma(z, z') = \omega \cdot x' - \omega' \cdot x$ is the *standard symplectic form* in $\mathbb{R}^{2d}$

  Observe that $\mathcal{F}_s f(x, w) = \mathcal{F} f(w, -x)$, $(x, w) \in \mathbb{R}^{2d}$, where $\mathcal{F}$ denotes the classical $2d$-dimensional Fourier transform

- Condition $m\, \mathbb{I}_N \leq G^W_{\mathbf{S}}(z) \leq M\, \mathbb{I}_N$ means:

$$m\|x\|^2 \leq \left\langle G^W_{\mathbf{S}}(z)x, x \right\rangle_{\mathbb{C}^N} \leq M\|x\|^2 \,, \quad \forall x \in \mathbb{C}^N$$

The isomorphism $\mathcal{T}_{\mathbf{S}}$ (via the Kohn-Nirenberg transform)

The isomorphism $\mathcal{T}_{\mathbf{S}}$ between $\ell^2_N(\Lambda)$ and $V^2_{\mathbf{S}}$

$$\begin{array}{ccccc} \mathcal{T}_{\mathbf{S}} : \ell^2_N(\Lambda) & \longrightarrow & V^2_{\sigma_{\mathbf{S}}} \subset L^2(\mathbb{R}^{2d}) & \longrightarrow & V^2_{\mathbf{S}} \subset \mathcal{HS}(\mathbb{R}^d) \\ (c_1, c_2, \dots, c_N)^\top & \longmapsto & \displaystyle\sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, T_\lambda \sigma_{S_n} & \longmapsto & \displaystyle\sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \end{array}$$

The isomorphism $\mathcal{T}_{\mathbf{S}}$ is the composition of the isomorphism $\mathcal{T}_{\sigma_{\mathbf{S}}} : \ell^2_N(\Lambda) \to V^2_{\sigma_{\mathbf{S}}}$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_\lambda\}_{\lambda\in\Lambda}$ for $\ell^2_N(\Lambda)$ onto the Riesz basis $\{T_\lambda \sigma_{S_n}\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ for $V^2_{\sigma_{\mathbf{S}}}$, and the Kohn-Nirenberg (Weyl) transform transform between $V^2_{\sigma_{\mathbf{S}}}$ ($V^2_{a_{\mathbf{S}}}$) and $V^2_{\mathbf{S}}$

An expression for the average samples

The average samples of any $T=\sum_{n=1}^{N}\sum_{\mu\in\Lambda} c_n(\mu)\,\alpha_\mu(S_n)$ in $V^2_{\mathbf{S}}$ can be expressed as the output of a discrete convolution system in $\ell^2_N(\Lambda)$:

$$\big\langle T,\alpha_\lambda(Q_m)\big\rangle_{\mathcal{HS}}=\sum_{n=1}^{N}\big(a_{m,n}*_\Lambda c_n\big)(\lambda)=\big\langle \mathbf{c},T_\lambda\,\mathbf{a}^*_m\big\rangle_{\ell^2_N(\Lambda)}\,,\quad \lambda\in\Lambda$$

where $\mathbf{a}^*_m=(a^*_{m,1},a^*_{m,2},\dots,a^*_{m,N})^\top$, $a^*_{m,n}(\lambda)=\overline{a_{m,n}(-\lambda)}$, and being

$$a_{m,n}(\mu)=\big\langle \sigma_{S_n},T_\mu\sigma_{Q_m}\big\rangle_{L^2(\mathbb{R}^{2d})}=\big\langle S_n,\alpha_\mu(Q_m)\big\rangle_{\mathcal{HS}}\,,\quad \mu\in\Lambda$$

The sampling condition will depend on the $M\times N$ matrix-valued function $A(\lambda)=[a_{m,n}(\lambda)]\,,\ \lambda\in\Lambda$, whose entries are in $\ell^2(\Lambda)$

The diagonal channel samples revisited

For the diagonal channel samples of the operator T

$$\left\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m \right\rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda, \quad m = 1, 2, \ldots, M$$

we have

Diagonal channel samples as average samples

$$\left\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle T, \alpha_\lambda(\widetilde{g}_m \otimes g_m) \right\rangle_{\mathcal{HS}}, \quad \lambda \in \Lambda$$

In particular, for $\varphi, \psi \in L^2(\mathbb{R}^d)$,

$$\alpha_z(\varphi \otimes \psi) = [\pi(z)\varphi] \otimes [\pi(z)\psi], \quad z \in \mathbb{R}^{2d}$$

The symplectic Fourier transform in $\ell^2(\Lambda)$

- The dual group $\widehat{\Lambda}$ is identified with $\mathbb{R}^{2d}/\Lambda^\circ$, where Λ° is the ***annihilator group*** (adjoint lattice of Λ)

$$\Lambda^\circ = \left\{\lambda^\circ \in \mathbb{R}^{2d} \ : \ \mathrm{e}^{2\pi i\,\sigma(\lambda^\circ,\lambda)} = 1 \ \text{ for all } \lambda \in \Lambda\right\}$$

$\sigma(z,z') = \omega \cdot x' - \omega' \cdot x$ for $z = (x,\omega)$ and $z' = (x',\omega')$ in $\mathbb{R}^{2d}$ is the *standard symplectic form*

- The symplectic Fourier transform of $c \in \ell^1(\Lambda)$ is defined as

$$\mathcal{F}_s^\Lambda(c)(\dot{z}) := \sum_{\lambda\in\Lambda} c(\lambda)\,\mathrm{e}^{2\pi i\,\sigma(\lambda,z)}\,, \quad \dot{z} \in \mathbb{R}^{2d}/\Lambda^\circ\,,$$

where $\dot{z}$ denotes the image of z under the natural quotient map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda^\circ$

- Since $\mathcal{F}_s^\Lambda$ is a Fourier transform it extends to a unitary mapping

Symplectic Fourier transform in $\ell^2(\Lambda)$

$$\begin{array}{rcl} \mathcal{F}_s^\Lambda : \ell^2(\Lambda) & \longrightarrow & L^2(\widehat{\Lambda}) \\ c & \longmapsto & \mathcal{F}_s^\Lambda(c) \end{array}$$

It satisfies:

- $\mathcal{F}_s^\Lambda(c *_\Lambda d) = \mathcal{F}_s^\Lambda(c)\, \mathcal{F}_s^\Lambda(d)$, for $c \in \ell^1(\Lambda)$ and $d \in \ell^2(\Lambda)$
- If $c, d \in \ell^2(\Lambda)$ and $\mathcal{F}_s^\Lambda(c) \in L^\infty(\widehat{\Lambda}) \Rightarrow \mathcal{F}_s^\Lambda(c *_\Lambda d) = \mathcal{F}_s^\Lambda(c)\, \mathcal{F}_s^\Lambda(d)$

As usual, the convolution $*_\Lambda$ of two sequences c, d is defined by

$$\big(c *_\Lambda d\big)(\lambda) = \sum_{\mu \in \Lambda} c(\mu)\, d(\lambda - \mu), \quad \lambda \in \Lambda$$

Generalized stable sampling procedure in $V_{\mathbf{S}}^2$

- It is a map $\mathcal{S}_{\mathsf{samp}} : V_{\mathbf{S}}^2 \longrightarrow \ell_M^2(\Lambda)$ defined as

$$T = \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \in V_{\mathbf{S}}^2 \longmapsto \mathbf{s}_T := A *_\Lambda \mathbf{c} \in \ell_M^2(\Lambda)$$

where $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(\Lambda)\big)$

That is: $\mathbf{s}_T(\lambda) = \big(s_{T,1}(\lambda), s_{T,2}(\lambda), \dots, s_{T,M}(\lambda)\big)^\top, \ \lambda \in \Lambda$, where

$$s_{T,m}(\lambda) = \sum_{n=1}^{N} \big(a_{m,n} *_\Lambda c_n\big)(\lambda)\,, \quad \lambda \in \Lambda\,, \quad m = 1, 2, \cdots, M$$

- Besides, the involved matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(\Lambda)\big)$ satisfies:

$$0 < \alpha_A := \operatorname*{ess\,inf}_{\xi \in \widehat{\Lambda}} \lambda_{\mathsf{min}}[\widehat{A}(\xi)^* \widehat{A}(\xi)] \le \beta_A := \operatorname*{ess\,sup}_{\xi \in \widehat{\Lambda}} \lambda_{\mathsf{max}}[\widehat{A}(\xi)^* \widehat{A}(\xi)] < \infty$$

- The matrix-valued function $\widehat{A}(\xi) := \left[\mathcal{F}_s^\Lambda(a_{m,n})(\xi)\right]$, a.e. $\xi \in \widehat{\Lambda}$ is the transfer matrix of A where $\mathcal{F}_s^\Lambda$ denotes the symplectic Fourier transform in $\ell^2(\Lambda)$
- A crucial fact concerning the samples:

$$\sum_{n=1}^{N} \big(a_{m,n} *_\Lambda c_n\big)(\lambda) = \big\langle \mathbf{c}, T_\lambda\, \mathbf{a}_m^* \big\rangle_{\ell^2_N(\Lambda)}\,, \quad \lambda \in \Lambda$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)^\top$ and $\mathbf{a}_m^* = (a_{m,1}^*, a_{m,2}^*, \ldots, a_{m,N}^*)^\top$

$\big\{T_\lambda\, \mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\ldots,M}$ is a frame for $\ell^2_N(\Lambda) \iff 0 < \alpha_A \le \beta_A < \infty$

- For the average sampling the corresponding matrix A has entries

$$a_{m,n}(\lambda) = \big\langle \sigma_{S_n}, T_\lambda \sigma_{Q_m} \big\rangle_{L^2(\mathbb{R}^{2d})} = \langle S_n, \alpha_\lambda(Q_m)\rangle_{\mathcal{HS}}\,, \quad \lambda \in \Lambda\,,$$

i.e., the columns of A are the sequences of average samples of the generators S_n of $V^2_{\mathbf{S}}$

► Concerning its dual frames $\{T_\lambda \mathbf{b}_m\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ with the same structure, they are obtained from the left-inverses $\widehat{B} \in \mathcal{M}_{N\times M}(L^\infty(\widehat{\Lambda}))$ of the matrix $\widehat{A}$, i.e., $\widehat{B}(\xi)\,\widehat{A}(\xi) = \mathbb{I}_N$, a.e. $\xi \in \widehat{\Lambda}$

The $\mathbf{b}_m$ are the columns of the matrix $B \in \mathcal{M}_{N\times M}\big(\ell^2(\Lambda)\big)$

► For instance, we can choose as $\widehat{B}$ its Moore-Penrose pseudo-inverse $\widehat{A}(\xi)^\dagger = \big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$

► All the suitable left-inverses $\widehat{B}$ of the matrix $\widehat{A}$ can be written from the Moore-Penrose pseudo-inverse as the $N \times M$ matrices

$$\widehat{B}(\xi) = \widehat{A}(\xi)^\dagger + C(\xi)\big[\mathbb{I}_M - \widehat{A}(\xi)\widehat{A}(\xi)^\dagger\big], \quad \text{a.e. } \xi \in \widehat{\Lambda},$$

where C denotes any $N \times M$ matrix with entries in $L^\infty(\widehat{\Lambda})$

Theorem (Sampling theorem in $V_{\mathbf{S}}^2$)

(1) *Given a sampling procedure $\mathcal{S}_{samp}$ in $V_{\mathbf{S}}^2$, there exist $M \geq N$ elements $H_m \in V_{\mathbf{S}}^2$, $m = 1, 2, \ldots, M$, such that the sampling formula*

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda)\, \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

holds for each $T \in V_{\mathbf{S}}^2$ where $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a frame for $V_{\mathbf{S}}^2$

(2) *The convergence of the series is unconditional in $\mathcal{HS}$-norm, and besides, $\|\mathbf{s}_T\|_{\ell^2_M} \asymp \|T\|_{\mathcal{HS}}$ in $V_{\mathbf{S}}^2$*

(3) *Reciprocally, if a sampling formula like above holds in $V_{\mathbf{S}}^2$ where*

$$\mathbf{s}_T(\lambda) = \big(s_{T,1}(\lambda), s_{T,2}(\lambda), \ldots, s_{T,M}(\lambda)\big)^\top := \big(A *_\Lambda \mathbf{c}\big)(\lambda)\,, \quad \lambda \in \Lambda\,,$$

where $\beta_A < +\infty$, and $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a frame for $V_{\mathbf{S}}^2$, then $\alpha_A > 0$

Scketch of the proof

For any $T=\sum_{n=1}^{N}\sum_{\mu\in\Lambda} c_n(\mu)\,\alpha_\mu(S_n)$ in $V_{\mathbf{S}}^2$ we have:

- For its samples $s_{T,m}(\lambda)=\big\langle \mathbf{c}, T_\lambda\,\mathbf{a}_m^*\big\rangle_{\ell^2_N(\Lambda)}$, $\lambda\in\Lambda$, where the sequence $\big\{T_\lambda\,\mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ is a frame for $\ell^2_N(\Lambda)$
- By using a dual frame $\big\{T_\lambda\,\mathbf{b}_m\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ of the above frame $\big\{T_\lambda\,\mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\, m=1,2,\dots,M}$ we obtain

$$\mathbf{c}=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda}\big\langle \mathbf{c}, T_\lambda\mathbf{a}_m^*\big\rangle_{\ell^2_N(\Lambda)}\,T_\lambda\mathbf{b}_m \quad \text{for each } \mathbf{c}\in\ell^2_N(\Lambda)$$

- Finally, applying the isomorphism $\mathcal{T}_{\mathbf{S}}$

$$T=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\,\mathcal{T}_{\mathbf{S}}[T_\lambda\mathbf{b}_m]=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\,K_{T_\lambda(\mathcal{T}_{\sigma_S}\mathbf{b}_m)}$$

- That is

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda)\, \alpha_\lambda[K_{h_m}] = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda)\, \alpha_\lambda(H_m)$$

where $H_m = K_{h_m}$ and $h_m = \mathcal{T}_{\sigma_S}(\mathbf{b}_m)$
We have used that

$$\mathcal{T}_{\sigma_S}(T_\lambda \mathbf{b}_m) = T_\lambda(\mathcal{T}_{\sigma_S}\mathbf{b}_m) = T_\lambda(h_m)$$

Observe that $\mathbf{b}_m = \big(b_{1,m}(\lambda), b_{2,m}(\lambda), \dots, b_{N,m}(\lambda)\big)^\top$ is the m-th column of B (remind that $\widehat{B}\,\widehat{A} = \mathbb{I}_N$). As a consequence,

$$H_m = \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} b_{n,m}(\lambda)\, \alpha_\lambda(S_n)\,, \quad m = 1, 2, \dots, M$$

- Under the above hypotheses, the average sampling formula reads

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}} \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

- Under the same hypotheses, the case $M = N$ implies that $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\dots,N}$ is a Riesz basis for $V_{\mathbf{S}}^2$
- Since convergence in $\mathcal{HS}$-norm implies convergence in operator norm, for each $f \in L^2(\mathbb{R}^d)$ we get the pointwise expansion:

$$T(f) = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}} [\alpha_\lambda(H_m)](f) \quad \text{in } L^2\text{-norm}$$

An illustrative example

- Assume $V_{\mathbf{S}}^2$ with N stable generators of the form $S_n = \varphi_n \otimes \widetilde{\varphi}_n$ with $\varphi_n, \widetilde{\varphi}_n \in \mathcal{S}_0(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$. In this regard,

$$\mathcal{F}_W(\varphi_n \otimes \widetilde{\varphi}_n)(z) = \mathrm{e}^{\pi i x\cdot\omega}\, V_{\widetilde{\varphi}_n}\varphi_n(z)\,, \quad z = (x, \omega) \in \mathbb{R}^{2d}$$

- For each $T \in V_{\mathbf{S}}^2$ we consider the diagonal channel samples

$$\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda$$

with $g_m, \widetilde{g}_m \in \mathcal{S}_0(\mathbb{R}^d)$, $m = 1, 2, \ldots, M$. In this case,

$$\begin{aligned} a_{m,n}(\lambda) &= \big\langle \big(\varphi_n \otimes \widetilde{\varphi}_n\big)\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)} \\ &= \overline{V_{g_m}\widetilde{\varphi}_n(\lambda)}\, V_{\widetilde{g}_m}\varphi_n(\lambda)\,, \quad \lambda \in \Lambda \end{aligned}$$

- It is known that the sequences $\big\{a_{m,n}(\lambda)\big\}_{\lambda\in\Lambda}$ belong to $\ell^1(\Lambda)$ and, as a consequence, the entries of $\widehat{A}$ are continuous functions on the compact $\widehat{\Lambda}$

► Thus, the sampling conditions in the definition of generalized stable sampling procedure reduce to

$$\det[\widehat{A}(\xi)^*\widehat{A}(\xi)] \neq 0 \quad \text{for all } \xi \in \widehat{\Lambda}$$

► Under the above circumstances:

Any $T = \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda) \alpha_\lambda(\varphi_n \otimes \widetilde{\varphi}_n) \in V_{\mathbf{S}}^2$ can be recovered, in a stable way, from its diagonal channel samples $\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m \big\rangle_{L^2(\mathbb{R}^d)}$

Moreover, since

$$T(\eta) = \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda) V_{\widetilde{\varphi}_n}\eta(\lambda)\pi(\lambda)\varphi_n = \sum_{n=1}^{N} \mathcal{G}_{\mathbf{c}_n}^{\widetilde{\varphi}_n,\varphi_n}(\eta)\,, \quad \eta \in L^2(\mathbb{R}^d)$$

the operator T is nothing but a sum of Gabor multipliers $\mathcal{G}_{\mathbf{c}_n}^{\widetilde{\varphi}_n,\varphi_n}$

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