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Sampling (reconstructing) Hilbert-Schmidt operators: Why and how to do it?

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Outline

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- A practical motivation
- The Weyl and Kohn-Nirenberg transforms for Hilbert-Schmidt operators
- Λ-shift-invariant subspaces: An expression for the samples
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A mathematical motivation

Average sampling in shift-invariant subspaces in $L^2(\mathbb{R}^d)$

- In a **classical shift-invariant subspaces** in $L^2(\mathbb{R}^d)$ with a stable generator set $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$

$$V_\Phi^2 = \Big\{ \sum_{n=1}^{N} \sum_{\alpha\in\mathbb{Z}^d} c_n(\alpha)\, \varphi_n(t-\alpha) \;:\; \{c_n(\alpha)\}_{\alpha\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)\,,\, n=1,2,\dots,N \Big\}$$

For $f \in V_\Phi^2$ we consider **average samples** $\{\langle f, \psi_m(\cdot-\alpha)\rangle\}_{\alpha\in\mathbb{Z}^d}$
ψ_m, $m = 1, 2, \dots, M$, are the *average functions* (not necessarily in V_Φ^2)

- Under appropriate hypotheses there exist $M (\geq N)$ sampling functions $S_m \in V_\Phi^2$ such that, for each $f \in V_\Phi^2$:

$$f(t) = \sum_{m=1}^{M} \sum_{\alpha\in\mathbb{Z}^d} \langle f, \psi_m(\cdot-\alpha)\rangle\, S_m(t-\alpha)\,, \quad t \in \mathbb{R}^d$$

and the sequence $\{S_m(t-\alpha)\}_{\alpha\in\mathbb{Z}^d;\, m=1,2,\dots,M}$ is a frame for V_Φ^2

Average sampling in shift-invariant-like subspaces of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

► *The **translation of an operator*** $S : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $z = (x, \omega)$ in the *phase space* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d (\simeq \mathbb{R}^{2d})$ is defined by

$$\alpha_z(S) := \pi(z) S \pi(z)^*, \quad z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$$

where $\pi(z)$ denotes the ***time-frequency shift*** which acts on $f \in L^2(\mathbb{R}^d)$ as

$$\pi(z) f(t) = \mathrm{e}^{2\pi i \omega \cdot t} f(t - x), \quad t \in \mathbb{R}^d$$

► The set of translations $\{\alpha_z\}_{z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d}$ is a *unitary representation* of the additive group $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on the Hilbert space $\big(\mathcal{HS}(\mathbb{R}^d), \langle \cdot, \cdot \rangle_{\mathcal{HS}}\big)$ of Hilbert-Schmidt operators on $L^2(\mathbb{R}^d)$

▶ Let Λ be a ***full rank lattice*** in $\mathbb{R}^{2d}$, i.e., $\Lambda = A\mathbb{Z}^{2d}$ where A is a $2d \times 2d$ real invertible matrix.
For $S_n \in \mathcal{HS}(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$, we could consider the subspace

$$V_{\mathbf{S}}^2 = \Big\{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \ : \ \{c_n(\lambda)\}_{\lambda \in \Lambda} \in \ell^2(\Lambda)\,,\, n = 1, 2, \ldots, N \Big\}$$

▶ We define for any $T \in V_{\mathbf{S}}^2$, its ***average samples*** at Λ by

$$\big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}}\,, \quad \lambda \in \Lambda\,,\ \ m = 1, 2, \ldots, M$$

from M fixed operators $Q_1, Q_2, \ldots, Q_M$ in $\mathcal{HS}(\mathbb{R}^d)$, the *average operators* (not necessarily in $V_{\mathbf{S}}^2$)

Under which hypotheses there exist $M(\geq N)$ sampling operators $H_m \in V^2_{\mathbf{S}}$ such that for each $T \in V^2_{\mathbf{S}}$

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \big\langle T, \alpha_\lambda(Q_m) \big\rangle_{\mathcal{HS}} \, \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

where the sequence $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a *frame* for the Hilbert space $V^2_{\mathbf{S}}$?

- The *adjoint operator* is $\pi(z)^* = e^{-2\pi i x \cdot \omega}\, \pi(-z)$ for $z = (x, \omega) \in \mathbb{R}^{2d}$
- The *short-time Fourier transform* (Gabor transform) $V_\psi \varphi$ of φ with window ψ, both in $L^2(\mathbb{R}^d)$, is defined by

$$V_\psi \varphi(z) = \big\langle \varphi, \pi(z)\psi \big\rangle_{L^2(\mathbb{R}^d)}, \quad z \in \mathbb{R}^{2d}$$

A practical motivation

LTV versus LTI systems

- Linear time-invariant system

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h(s)\,x(t-s)\,ds = \int_{-\infty}^{\infty} \widehat{h}(w)\,\widehat{x}(w)\mathrm{e}^{2\pi iwt}dw$$

- Linear time-varying system

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h(t,s)\,x(t-s)\,ds = \int_{-\infty}^{\infty} \sigma(t,w)\,\widehat{x}(w)\mathrm{e}^{2\pi iwt}dw$$

where

$$\sigma = \mathcal{F}_2\,h\,, \text{ i.e.,} \quad \sigma(t,w) = \int_{-\infty}^{\infty} h(t,s)\,\mathrm{e}^{-2\pi iws}ds$$

Thus, H is a *pseudo-differential operator with symbol* σ

In particular, **Hilbert-Schmidt operators model LTV systems**:

$$Hf(t) = \int_{-\infty}^{\infty} \kappa(t,s)f(s)\,ds = \int_{-\infty}^{\infty} \kappa(t,t-s)f(t-s)\,ds$$

In *Orthogonal frequency-division multiplexing* (OFDM) the digital information, i.e., a sequence of numbers $\{c_\lambda\}$, λ in the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ $(a, b > 0)$, is used as the coefficients of the input signal $x(t) = \sum_{\mu\in\Lambda} c_\mu\, \pi(\mu)g(t)$ of a time-varying system H producing the output $y(t) = Hx(t)$. Then, it is considered the sequence of numbers

$$d_\lambda = \big\langle y, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)} = \sum_{\mu\in\Lambda} c_\mu \,\big\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda,$$

The task is to recover the original data $\{c_\lambda\}$ from the received data $\{d_\lambda\}$

The matrix $A = [a_{\lambda,\mu}]$, where $a_{\lambda,\mu} = \big\langle H\pi(\mu)g, \pi(\lambda)\widetilde{g}\,\big\rangle_{L^2(\mathbb{R}^d)}$ is the so-called *channel matrix* associated with H and the functions (windows) $g, \widetilde{g}$ in $L^2(\mathbb{R}^d)$

The diagonal channel samples of H with respect to $g, \widetilde{g}$ are

$$\left\langle H\pi(\lambda)g, \pi(\lambda)\widetilde{g} \right\rangle_{L^2(\mathbb{R}^d)}, \quad \lambda \in \Lambda$$

They are also known as:

- The *lower symbol of the operator* H with respect $g, \widetilde{g} \in L^2(\mathbb{R}^d)$ and lattice Λ used in *time-frequency analysis*
- The samples of the *Berezin transform* of H

$$\mathcal{B}^{g,\widetilde{g}} H(z) := \left\langle H\pi(z)g, \pi(z)\widetilde{g} \right\rangle_{L^2(\mathbb{R}^d)}, \quad z \in \mathbb{R}^{2d}$$

  at the lattice Λ used in *quantum physics*
- **Diagonal channel samples are a particular case of average samples**

The Weyl and Kohn-Nirenberg transforms

A brief on Hilbert-Schmidt operators

- For a compact operator S on $L^2(\mathbb{R}^d)$ there exist two orthonormal sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$ and a bounded sequence of positive numbers $\{s_n(S)\}_{n\in\mathbb{N}}$ (*singular values of S*) such that

$$S = \sum_{n\in\mathbb{N}} s_n(S)\, x_n \otimes y_n$$

with convergence of the series in the operator norm (SVD)

Here, $x_n \otimes y_n$ denotes the rank-one operator

$$\big(x_n \otimes y_n\big)(f) = \big\langle f, y_n \big\rangle_{L^2} x_n \text{ for } f \in L^2(\mathbb{R}^d)$$

- The class of *Hilbert-Schmidt operators* is $\mathcal{HS}(\mathbb{R}^d) := \mathcal{T}^2$

$\mathcal{T}^2$ is the Schatten-2 class, i.e., singular values in $\ell^2(\mathbb{N})$

The space $\mathcal{HS}(\mathbb{R}^d)$ is a Hilbert space with the inner product

$$\big\langle S, T \big\rangle_{\mathcal{HS}} = \mathrm{tr}(ST^*)\,, \quad S, T \in \mathcal{HS}(\mathbb{R}^d)$$

Remind that the trace $\operatorname{tr}(S) = \sum_{n\in\mathbb{N}} \langle Se_n, e_n\rangle_{L^2}$ is a well-defined bounded linear functional on $\mathcal{T}^1$, and independent of the used orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ in $L^2(\mathbb{R}^d)$

- For the norm of $S \in \mathcal{HS}(\mathbb{R}^d)$ we have

$$\|S\|^2_{\mathcal{HS}} = \operatorname{tr}(SS^*) = \sum_{n\in\mathbb{N}} \|S^*(e_n)\|^2_{L^2} = \sum_{n\in\mathbb{N}} \|S(e_n)\|^2_{L^2} = \sum_{n\in\mathbb{N}} s_n^2(S)$$

- A Hilbert-Schmidt operator $S \in \mathcal{HS}(\mathbb{R}^d)$ can be seen also as a compact operator on $L^2(\mathbb{R}^d)$ defined for each $f \in L^2(\mathbb{R}^d)$ by

$$Sf(t) = \int_{\mathbb{R}^d} \kappa_S(t,x) f(x)\, dx \quad \text{a.e. } t \in \mathbb{R}^d$$

with kernel $\kappa_S \in L^2(\mathbb{R}^{2d})$. Besides, $\langle S, T\rangle_{\mathcal{HS}} = \langle \kappa_S, \kappa_T\rangle_{L^2(\mathbb{R}^{2d})}$ for $S, T \in \mathcal{HS}(\mathbb{R}^d)$

The Weyl transform

The Weyl transform $L^2(\mathbb{R}^{2d}) \ni f \longmapsto L_f \in \mathcal{HS}(\mathbb{R}^d)$ is a **unitary operator** where $L_f : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\langle L_f \phi, \psi \rangle_{L^2(\mathbb{R}^d)} = \langle f, W(\psi, \phi) \rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d)$$

here

$$W(\psi, \phi)(x, \omega) = \int_{\mathbb{R}^d} \psi\Big(x + \frac{t}{2}\Big) \overline{\phi\Big(x - \frac{t}{2}\Big)} \, \mathrm{e}^{-2\pi i \omega \cdot t} dt, \quad (x, \omega) \in \mathbb{R}^{2d},$$

is the ***cross-Wigner distribution*** of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$
For each $S, T \in \mathcal{HS}(\mathbb{R}^d)$ with Weyl symbols a_S, a_T in $L^2(\mathbb{R}^{2d})$ we have

$$\langle S, T \rangle_{\mathcal{HS}} = \langle a_S, a_T \rangle_{L^2(\mathbb{R}^{2d})}$$

The Kohn-Nirenberg transform

The ***Kohn-Nirenberg transform*** $L^2(\mathbb{R}^{2d}) \ni \sigma \longmapsto K_\sigma \in \mathcal{HS}(\mathbb{R}^d)$ is a **unitary operator** where $K_\sigma : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the Hilbert-Schmidt operator defined in weak sense by

$$\left\langle K_\sigma \phi, \psi \right\rangle_{L^2(\mathbb{R}^d)} = \left\langle \sigma, R(\psi, \phi) \right\rangle_{L^2(\mathbb{R}^{2d})}, \quad \phi, \psi \in L^2(\mathbb{R}^d)$$

here

$$R(\psi, \phi)(x, \omega) = \psi(x)\, \overline{\widehat{\phi}(\omega)}\, \mathrm{e}^{-2\pi i\, x\cdot\omega}, \quad (x, \omega) \in \mathbb{R}^{2d},$$

is the ***Rihaczek distribution*** of the functions $\psi, \phi \in L^2(\mathbb{R}^d)$
For each $S, T \in \mathcal{HS}(\mathbb{R}^d)$ with Kohn-Nirenberg symbols σ_S, σ_T in $L^2(\mathbb{R}^{2d})$ we have

$$\langle S, T \rangle_{\mathcal{HS}} = \langle \sigma_S, \sigma_T \rangle_{L^2(\mathbb{R}^{2d})}$$

A crucial property for both transforms

- There is a transition between Weyl and Kohn-Nirenberg calculus:

$$\sigma_s = Ua_s\,, \text{ where } \widehat{Ua_s}(\xi,u) = \mathrm{e}^{\pi i u\cdot\xi}\,\widehat{a_s}(\xi,u)\,,\ (\xi,u)\in\mathbb{R}^{2d}$$

- The Weyl and Kohn-Nirenberg transforms in $\mathcal{HS}(\mathbb{R}^d)$ respect both the translations in the sense:

For $f \in L^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ and $z \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we have:

$$\mathcal{L}(T_z f) = \alpha_z(\mathcal{L} f)$$

where $\mathcal{L}$ denotes the Weyl or the Kohn-Nirenberg transform

- In addition to the unitary character we obtain that

Properties of V_S^2 in $\mathcal{HS}(\mathbb{R}^d) \longleftrightarrow$ Properties of $V_{\sigma_S}^2$ (or $V_{a_S}^2$) in $L^2(\mathbb{R}^{2d})$

Λ-shift-invariant subspaces

Let $\mathbf{S} = \{S_1, S_2, \ldots, S_N\}$ be a fixed subset of $\mathcal{HS}(\mathbb{R}^d)$ and let Λ be a lattice in $\mathbb{R}^{2d}$. We are searching for a necessary and sufficient condition such that $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N}$ is a Riesz sequence for $\mathcal{HS}(\mathbb{R}^d)$, i.e., a Riesz basis for the closed subspace

$$V_{\mathbf{S}}^2 := \overline{\text{span}}_{\mathcal{HS}}\big\{\alpha_\lambda(S_n)\big\}_{\lambda\in\Lambda;\, n=1,2,\ldots,N} \subset \mathcal{HS}(\mathbb{R}^d)$$

In this case, $\mathbf{S} = \{S_1, S_2, \ldots, S_N\}$ **is a *set of generators* for the** Λ***-shift-invariant subspace*** $V_{\mathbf{S}}^2$ which can be described by

$$V_{\mathbf{S}}^2 = \Big\{ \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \;:\; \{c_n(\lambda)\}_{\lambda\in\Lambda} \in \ell^2(\Lambda)\,,\, n = 1, 2, \ldots, N \Big\}$$

Theorem

Let Λ *be a lattice and* $S_n \in \mathcal{B}$, $n = 1, 2, \dots, N$. *Then,* $\{\alpha_\lambda(S_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ *is a Riesz sequence for* $\mathcal{HS}(\mathbb{R}^d)$ *if and only if there exist two constants* $0 < m \leq M$ *such that*

$$m\,\mathbb{I}_N \leq G_{\mathbf{S}}^W(z) \leq M\,\mathbb{I}_N \quad \text{for any } z \in \mathbb{R}^{2d}\,,$$

where $G_{\mathbf{S}}^W(z)$ *denotes the* $N \times N$ *matrix-valued function*

$$G_{\mathbf{S}}^W(z) := \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)\,\overline{\mathcal{F}_W(\mathbf{S})(z+\lambda^\circ)}^\top, \quad z \in \mathbb{R}^{2d}$$

and $\mathcal{F}_W(\mathbf{S}) = \big(\mathcal{F}_W(S_1), \mathcal{F}_W(S_2), \dots, \mathcal{F}_W(S_N)\big)^\top$

where:

- $S \in \mathcal{B}$ is the Banach space of continuous operators with Weyl symbol a_s in the *Feichtinger's algebra* $\mathcal{S}_0(\mathbb{R}^{2d})$. In essence, $\mathcal{B}$ consists of trace class operators on $L^2(\mathbb{R}^d)$ with a norm-continuous inclusion $\iota : \mathcal{B} \hookrightarrow \mathcal{T}^1$
  Recall that $\psi \in \mathcal{S}_0(\mathbb{R}^{2d})$ iff $V_{\varphi_0}\psi \in L^1(\mathbb{R}^{2d})$
- Λ° is the adjoint lattice of the lattice Λ. Its associated matrix is $A^{-\top}\Omega_d$ in case $\Lambda = A\mathbb{Z}^{2d}$, where

$$\Omega_d = \begin{pmatrix} O & I_d \\ -I_d & O \end{pmatrix}$$

- $\mathcal{F}_W(S)$ denotes the Fourier-Wigner transform of an operator S defined as the function

$$\mathcal{F}_W(S)(z) := \mathrm{e}^{-\pi i x\cdot\omega}\, \mathrm{tr}[\pi(-z)S]\,, \quad z = (x,\omega) \in \mathbb{R}^{2d}$$

- In our case, $\mathcal{F}_W(S_n) = \mathcal{F}_s(a_{S_n})$ for $n = 1, 2, \ldots, N$, where $\mathcal{F}_s$ denotes the *symplectic Fourier transform* of a_{S_n} defined by

$$\mathcal{F}_s(a_{S_n})(z) := \int_{\mathbb{R}^{2d}} a_{S_n}(z')\, \mathrm{e}^{-2\pi i\, \sigma(z,z')} dz' \,, \quad z \in \mathbb{R}^{2d}$$

$\sigma(z, z') = \omega \cdot x' - \omega' \cdot x$ is the *standard symplectic form* in $\mathbb{R}^{2d}$

The isomorphism $\mathcal{T}_{\mathbf{S}}$

$$
\begin{array}{ccccc}
\mathcal{T}_{\mathbf{S}} : \ell^2_N(\Lambda) & \longrightarrow & V^2_{\sigma_{\mathbf{S}}} \subset L^2(\mathbb{R}^{2d}) & \longrightarrow & V^2_{\mathbf{S}} \subset \mathcal{HS}(\mathbb{R}^d) \\
(c_1, c_2, \dots, c_N)^\top & \longmapsto & \displaystyle\sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, T_\lambda \sigma_{S_n} & \longmapsto & \displaystyle\sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, \alpha_\lambda(S_n)
\end{array}
$$

The isomorphism $\mathcal{T}_{\mathbf{S}}$ is the composition of the isomorphism $\mathcal{T}_{\sigma_{\mathbf{S}}} : \ell^2_N(\Lambda) \to V^2_{\sigma_{\mathbf{S}}}$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_\lambda\}_{\lambda \in \Lambda}$ for $\ell^2_N(\Lambda)$ onto the Riesz basis $\{T_\lambda \sigma_{S_n}\}_{\lambda \in \Lambda;\, n=1,2,\dots,N}$ for $V^2_{\sigma_{\mathbf{S}}}$, and the Kohn-Nirenberg (Weyl) transform transform between $V^2_{\sigma_{\mathbf{S}}}$ ($V^2_{a_{\mathbf{S}}}$) and $V^2_{\mathbf{S}}$

An expression for the average samples

The **average samples** of any $T = \sum_{n=1}^{N} \sum_{\mu \in \Lambda} c_n(\mu)\, \alpha_\mu(S_n)$ in $V_{\mathbf{S}}^2$ can be expressed as the output of a discrete convolution system in $\ell_N^2(\Lambda)$:

$$\langle T, \alpha_\lambda(Q_m) \rangle_{\mathcal{HS}} = \sum_{n=1}^{N} \big(a_{m,n} *_\Lambda c_n\big)(\lambda) = \langle \mathbf{c}, T_\lambda\, \mathbf{a}_m^* \rangle_{\ell_N^2(\Lambda)}, \quad \lambda \in \Lambda$$

where $\mathbf{a}_m^* = (a_{m,1}^*, a_{m,2}^*, \dots, a_{m,N}^*)^\top$, $a_{m,n}^*(\lambda) = \overline{a_{m,n}(-\lambda)}$, and being

$$a_{m,n}(\mu) = \langle \sigma_{S_n}, T_\mu \sigma_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})} = \langle a_{S_n}, T_\mu a_{Q_m} \rangle_{L^2(\mathbb{R}^{2d})}$$
$$= \langle S_n, \alpha_\mu(Q_m) \rangle_{\mathcal{HS}}, \quad \mu \in \Lambda$$

The sampling property will depend on the $M \times N$ matrix-valued function $A(\lambda) = [a_{m,n}(\lambda)]\,,\ \lambda \in \Lambda$, whose entries are in $\ell^2(\Lambda)$

The diagonal channel samples revisited

For the diagonal channel samples of the operator T

$$\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda\,, \quad m = 1, 2, \dots, M$$

we have

$$\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)} = \big\langle T, \alpha_\lambda(\widetilde{g}_m \otimes g_m)\big\rangle_{\mathcal{HS}}\,, \quad \lambda \in \Lambda$$

We have also

$$\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)} = \big\langle \alpha_{-\lambda}(T)g_m, \widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda$$

For $\varphi, \psi \in L^2(\mathbb{R}^d)$, $\alpha_z(\varphi \otimes \psi) = [\pi(z)\varphi] \otimes [\pi(z)\psi]$, $z \in \mathbb{R}^{2d}$

The sampling result

Definition (Generalized stable sampling procedure in $V_{\mathbf{S}}^2$)

This is a map $\mathcal{S}_{samp} : V_{\mathbf{S}}^2 \to \ell_M^2(\Lambda)$ *defined as*

$$T = \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} c_n(\lambda)\, \alpha_\lambda(S_n) \in V_{\mathbf{S}}^2 \longmapsto \mathbf{s}_T := A *_\Lambda \mathbf{c} \in \ell_M^2(\Lambda)$$

where the matrix $A = [a_{m,n}] \in \mathcal{M}_{M \times N}\big(\ell^2(\Lambda)\big)$ *satisfies the conditions:*

$$0 < \alpha_A := \operatorname*{ess\,inf}_{\xi \in \widehat{\Lambda}} \lambda_{min}[\widehat{A}(\xi)^* \widehat{A}(\xi)] \le \beta_A := \operatorname*{ess\,sup}_{\xi \in \widehat{\Lambda}} \lambda_{max}[\widehat{A}(\xi)^* \widehat{A}(\xi)] < \infty$$

For average sampling the corresponding matrix A has entries

$$a_{m,n}(\lambda) := \big\langle \sigma_{S_n}, T_\lambda \sigma_{Q_m} \big\rangle_{L^2(\mathbb{R}^{2d})} = \langle S_n, \alpha_\lambda(Q_m) \rangle_{\mathcal{HS}}\,, \quad \lambda \in \Lambda\,,$$

i.e., the columns of A are the sequences of samples of the generators of $V_{\mathbf{S}}^2$

▶ The matrix-valued function $\widehat{A}(\xi) := \left[\mathcal{F}_s^{\Lambda}(a_{m,n})(\xi)\right]$, a.e. $\xi \in \widehat{\Lambda}$ is the **transfer matrix of** A where $\mathcal{F}_s^{\Lambda}$ denotes the symplectic Fourier transform in $\ell^2(\Lambda)$

▶ The dual group $\widehat{\Lambda}$ is identified with $\mathbb{R}^{2d}/\Lambda^\circ$, where Λ° is the ***annihilator group*** (adjoint lattice of Λ)

$$\Lambda^\circ = \left\{\lambda^\circ \in \mathbb{R}^{2d} \;:\; \mathrm{e}^{2\pi i\,\sigma(\lambda^\circ,\lambda)} = 1 \;\text{ for all } \lambda \in \Lambda\right\}$$

$\sigma(z,z') = \omega \cdot x' - \omega' \cdot x$ for $z = (x,\omega)$ and $z' = (x',\omega')$ in $\mathbb{R}^{2d}$ is the *standard symplectic form*

▶ The Fourier transform of $c \in \ell^1(\Lambda)$ is the *symplectic Fourier series*

$$\mathcal{F}_s^{\Lambda}(c)(\dot{z}) := \sum_{\lambda \in \Lambda} c(\lambda)\, \mathrm{e}^{2\pi i\,\sigma(\lambda,z)}\,, \quad \dot{z} \in \mathbb{R}^{2d}/\Lambda^\circ\,,$$

where $\dot{z}$ denotes the image of z under the natural quotient map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}/\Lambda^\circ$

- Since $\mathcal{F}_s^\Lambda$ is a Fourier transform it extends to a unitary mapping

$$\mathcal{F}_s^\Lambda : \ell^2(\Lambda) \to L^2(\widehat{\Lambda})$$

It satisfies:

  - $\mathcal{F}_s^\Lambda(c *_\Lambda d) = \mathcal{F}_s^\Lambda(c)\,\mathcal{F}_s^\Lambda(d)$, for $c \in \ell^1(\Lambda)$ and $d \in \ell^2(\Lambda)$
  - If $c, d \in \ell^2(\Lambda)$ and $\mathcal{F}_s^\Lambda(c) \in L^\infty(\widehat{\Lambda}) \Rightarrow \mathcal{F}_s^\Lambda(c *_\Lambda d) = \mathcal{F}_s^\Lambda(c)\,\mathcal{F}_s^\Lambda(d)$

As usual, the convolution $*_\Lambda$ of two sequences c, d is defined by

$$\big(c *_\Lambda d\big)(\lambda) = \sum_{\mu \in \Lambda} c(\mu)\, d(\lambda - \mu), \quad \lambda \in \Lambda$$

- Finally,

$\{T_\lambda \mathbf{a}_m^*\}_{\lambda \in \Lambda;\, m=1,2,\dots,M}$ is a frame for $\ell^2_N(\Lambda) \iff 0 < \alpha_A \le \beta_A < \infty$

Theorem

Given a sampling procedure $\mathcal{S}_{samp}$ in $V^2_{\mathbf{S}}$, there exist $M \geq N$ elements $H_m \in V^2_{\mathbf{S}}$, $m = 1, 2, \ldots, M$, such that the sampling formula

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} s_{T,m}(\lambda)\, \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

holds for each $T \in V^2_{\mathbf{S}}$ where $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a frame for $V^2_{\mathbf{S}}$. The convergence of the series is unconditional in $\mathcal{HS}$-norm, and the ℓ^2-norm of the samples $\|\mathbf{s}_T\|_{\ell^2_M}$ is an equivalent norm to $\|T\|_{\mathcal{HS}}$ in $V^2_{\mathbf{S}}$. Reciprocally, if a sampling formula like above holds in $V^2_{\mathbf{S}}$ where

$$\mathbf{s}_T(\lambda) = \big(s_{T,1}(\lambda), s_{T,2}(\lambda), \ldots, s_{T,M}(\lambda)\big)^\top := \big(A *_\Lambda \mathbf{c}\big)(\lambda)\,, \quad \lambda \in \Lambda\,,$$

where $\beta_A < +\infty$, and $\{\alpha_\lambda(H_m)\}_{\lambda \in \Lambda;\, m=1,2,\ldots,M}$ is a frame for $V^2_{\mathbf{S}}$, then $\alpha_A > 0$

Scketch of the proof

For any $T=\sum_{n=1}^{N}\sum_{\mu\in\Lambda} c_n(\mu)\,\alpha_\mu(S_n)$ in $V_{\mathbf{S}}^2$ we have:

- For its samples $s_{T,m}(\lambda)=\big\langle \mathbf{c}, T_\lambda\,\mathbf{a}_m^*\big\rangle_{\ell_N^2(\Lambda)}$, $\lambda\in\Lambda$, and the sequence $\big\{T_\lambda\,\mathbf{a}_m^*\big\}_{\lambda\in\Lambda;\,m=1,2,\dots,M}$ is a frame for $\ell_N^2(\Lambda)$
- Its dual frames $\big\{T_\lambda\,\mathbf{b}_m\big\}_{\lambda\in\Lambda;\,m=1,2,\dots,M}$ are obtained from the left-inverses $\widehat{B}\in\mathcal{M}_{N\times M}(L^\infty(\widehat{\Lambda}))$ of the matrix $\widehat{A}$ (for instance, $\widehat{A}(\xi)^\dagger=\big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$) obtaining

$$\mathbf{c}=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda}\big\langle \mathbf{c}, T_\lambda\mathbf{a}_m^*\big\rangle_{\ell_N^2(\Lambda)}\,T_\lambda\mathbf{b}_m \quad \text{for each } \mathbf{c}\in\ell_N^2(\Lambda)$$

- Finally, applying the isomorphism $\mathcal{T}_{\mathbf{S}}$

$$T=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\,\mathcal{T}_{\mathbf{S}}[T_\lambda\mathbf{b}_m]=\sum_{m=1}^{M}\sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\,K_{T_\lambda(\mathcal{T}_{\sigma_S}\mathbf{b}_m)}$$

- That is

$$T = \sum_{m=1}^{M} \sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\, \alpha_\lambda[K_{h_m}] = \sum_{m=1}^{M} \sum_{\lambda\in\Lambda} s_{T,m}(\lambda)\, \alpha_\lambda(H_m)$$

where $H_m = K_{h_m}$ and $h_m = \mathcal{T}_{\sigma_S}(\mathbf{b}_m)$
We have used that

$$\mathcal{T}_{\sigma_S}(T_\lambda \mathbf{b}_m) = T_\lambda(\mathcal{T}_{\sigma_S}\mathbf{b}_m) = T_\lambda(h_m)$$

Observe that $\mathbf{b}_m = \big(b_{1,m}(\lambda), b_{2,m}(\lambda), \dots, b_{N,m}(\lambda)\big)^\top$ is the m-th column of B, and

$$H_m = \sum_{n=1}^{N} \sum_{\lambda\in\Lambda} b_{n,m}(\lambda)\, \alpha_\lambda(S_n)\,, \quad m = 1, 2, \dots, M$$

- Under the above hypotheses, the **average sampling formula** reads

$$T = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}} \alpha_\lambda(H_m) \quad \text{in } \mathcal{HS}\text{-norm}$$

- Since convergence in $\mathcal{HS}$-norm implies convergence in operator norm, for each $f \in L^2(\mathbb{R}^d)$ we get the pointwise expansion:

$$T(f) = \sum_{m=1}^{M} \sum_{\lambda \in \Lambda} \left\langle T, \alpha_\lambda(Q_m) \right\rangle_{\mathcal{HS}} [\alpha_\lambda(H_m)](f) \quad \text{in } L^2\text{-norm}$$

Consequences, comments and a final example

Some consequences

- Whenever $M=N$ the sequence $\{\alpha_\lambda(H_n)\}_{\lambda\in\Lambda;\, n=1,2,\dots,N}$ is a Riesz basis for $V^2_{\mathbf{S}}$, and the interpolatory property

$$\langle H_m\pi(\lambda)g_n, \pi(\lambda)\widetilde{g}_n\rangle = \delta_{m,n}\delta_{\lambda,0}$$

  where $\lambda\in\Lambda$ and $m,n=1,2,\dots,N$, holds
- Assume that the sequence $\mathbf{a}=\{a(\lambda)\}_{\lambda\in\Lambda}$ satisfies

$$0<\operatorname*{ess\,inf}_{\xi\in\widehat{\Lambda}}|\mathcal{F}^\Lambda_s(\mathbf{a})(\xi)|\le\operatorname*{ess\,sup}_{\xi\in\widehat{\Lambda}}|\mathcal{F}^\Lambda_s(\mathbf{a})(\xi)|<\infty$$

  where $a(\lambda)=\langle S\pi(\lambda)g, \pi(\lambda)\widetilde{g}\rangle_{L^2(\mathbb{R}^d)}$, $\lambda\in\Lambda$, with $g,\widetilde{g}\in L^2(\mathbb{R}^d)$. Then, there exists a unique $H\in V^2_s$ such that the sequence $\{\alpha_\lambda(H)\}_{\lambda\in\Lambda}$ is a Riesz basis for V^2_s and the sampling formula

$$T=\sum_{\lambda\in\Lambda}\langle T\pi(\lambda)g, \pi(\lambda)\widetilde{g}\rangle_{L^2(\mathbb{R}^d)}\,\alpha_\lambda(H) \quad \text{in } \mathcal{HS}\text{-norm}$$

  holds for each $T\in V^2_s$.

Some comments

- In case $S = \varphi \otimes \psi$ operators in V_S^2 are *Gabor multipliers*:

$$\sum_{\lambda\in\Lambda} c(\lambda)\,\alpha_\lambda(S)(\eta) = \sum_{\lambda\in\Lambda} c(\lambda)\,\big(\pi(\lambda)\varphi \otimes \pi(\lambda)\psi\big)(\eta)$$
$$= \sum_{\lambda\in\Lambda} c(\lambda)\,V_\psi\,\eta(\lambda)\pi(\lambda)\varphi = \mathcal{G}_{\mathbf{c}}^{\psi,\varphi}(\eta)\,,\ \eta \in L^2(\mathbb{R}^d)$$

  $\mathcal{G}_{\mathbf{c}}^{\psi,\varphi}$ is the Gabor multiplier with windows ψ, φ and mask $\mathbf{c}$ in $\ell^2(\Lambda)$ used in time-frequency analysis

- The *convolution of a function f and an operator S* is formally defined by the operator-valued integral $f * S = \int_{\mathbb{R}^{2d}} f(z)\,\alpha_z(S)dz$. In particular $\mathbf{c} *_\Lambda S := S *_\Lambda \mathbf{c} := \sum_{\lambda\in\Lambda} c(\lambda)\,\alpha_\lambda(S)$. Thus,

$$V_S^2 = \ell^2(\Lambda) *_\Lambda S$$

- Average sampling can be expressed as a convolution of two operators:

$$\left\langle T, \alpha_\lambda(Q) \right\rangle_{\mathcal{HS}} = \operatorname{tr}[T\alpha_\lambda(Q)^*] = \operatorname{tr}[T\alpha_\lambda(Q^*)]$$
$$= T *_\Lambda \check{Q}^*(\lambda) = T *_\Lambda \widetilde{Q}(\lambda), \quad \lambda \in \Lambda$$

where $\widetilde{Q} = \check{Q}^*$. Recall that

$$S * T(z) := \operatorname{tr}[S\alpha_z(\check{T})], \quad z \in \mathbb{R}^{2d}$$

where $\check{T} = PTP$ and P denotes the *parity operator* $(P\varphi)(t) = \varphi(-t)$ for $\varphi \in L^2(\mathbb{R}^d)$.
Replacing $\mathbb{R}^{2d}$ by a lattice $\Lambda \subset \mathbb{R}^{2d}$ we obtain the convolution of two operators S, T at the lattice Λ

An illustrative example

- Assume $V_{\mathbf{S}}^2$ with N stable generators of the form $S_n = \varphi_n \otimes \widetilde{\varphi}_n$ with $\varphi_n, \widetilde{\varphi}_n \in \mathcal{S}_0(\mathbb{R}^d)$, $n = 1, 2, \ldots, N$. In this regard,

$$\mathcal{F}_W(\varphi_n \otimes \widetilde{\varphi}_n)(z) = \mathrm{e}^{\pi i x\cdot\omega}\, V_{\widetilde{\varphi}_n}\varphi_n(z)\,, \quad z = (x, \omega) \in \mathbb{R}^{2d}$$

- For each $T \in V_{\mathbf{S}}^2$ we consider the diagonal channel samples

$$\big\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)}\,, \quad \lambda \in \Lambda$$

with $g_m, \widetilde{g}_m \in \mathcal{S}_0(\mathbb{R}^d)$, $m = 1, 2, \ldots, M$. In this case,

$$\begin{aligned} a_{m,n}(\lambda) &= \big\langle \big(\varphi_n \otimes \widetilde{\varphi}_n\big)\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m\big\rangle_{L^2(\mathbb{R}^d)} \\ &= \overline{V_{g_m}\widetilde{\varphi}_n(\lambda)}\, V_{\widetilde{g}_m}\varphi_n(\lambda)\,, \quad \lambda \in \Lambda \end{aligned}$$

- It is known that the sequences $\big\{a_{m,n}(\lambda)\big\}_{\lambda\in\Lambda}$ belong to $\ell^1(\Lambda)$ and, as a consequence, the entries of $\widehat{A}$ are continuous functions on the compact $\widehat{\Lambda}$

- Thus, the sampling conditions in the definition of generalized stable sampling procedure reduce to

$$\det[\widehat{A}(\xi)^*\widehat{A}(\xi)] \neq 0 \quad \text{for all } \xi \in \widehat{\Lambda}$$

- Under the above circumstances:

Any $T = \sum_{n=1}^{N} \mathcal{G}_{\mathbf{c}_n}^{\widetilde{\varphi}_n,\varphi_n} \in V_{\mathbf{S}}^2$ can be recovered, in a stable way, from its diagonal channel samples $\left\langle T\pi(\lambda)g_m, \pi(\lambda)\widetilde{g}_m \right\rangle_{L^2(\mathbb{R}^d)}$

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