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A link between discrete convolution systems and sampling via frame theory

Antonio García García

uc3m | Universidad **Carlos III** de Madrid
Departamento de **Matemáticas**

Outline

- Statement of the problem
- A brief on discrete frames
- Discrete convolution systems and frames
- The resulting sampling theory
- Some sampling results as particular cases

Statement of the problem

- $(G,+)$ is a countable discrete abelian group
- $G \ni g \mapsto U(g) \in \mathcal{U}(\mathcal{H})$ a unitary representation of G on a separable Hilbert space $\mathcal{H}$
- Given a set $\Phi = \{\varphi_1, \varphi_2, \dots, \varphi_N\}$ in $\mathcal{H}$, consider the U-invariant subspace of $\mathcal{H}$

$$\mathcal{V}_\Phi = \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n \,:\, x_n \in \ell^2(G),\ n = 1, 2, \dots, N \Big\}$$

  where the sequence $\big\{U(g)\varphi_n\big\}_{g\in G;\, n=1,2,\dots,N}$ is a Riesz sequence for $\mathcal{H}$
- We define an average sampling in $\mathcal{V}_\Phi$ as follows:

Given M elements $\psi_m \in \mathcal{H}$, $m = 1, 2, \ldots, M$, which do not belong necessarily to $\mathcal{V}_\Phi$, for any $f \in \mathcal{V}_\Phi$ we define for $m = 1, 2, \ldots, M$

$$\mathcal{L}_m f(g) := \big\langle f, U(g)\psi_m \big\rangle_{\mathcal{H}}, \quad g \in G$$

Motivating fact

These samples can be expressed as the output of a convolution system defined in $\ell^2_N(G)$.

Indeed, for any $f = \sum_{n=1}^{N} \sum_{h \in G} x_n(h) U(h) \varphi_n$ in $\mathcal{V}_\Phi$ one gets

$$\mathcal{L}_m f(g) = \sum_{n=1}^{N} (a_{m,n} * x_n)(g), \;\; g \in G$$

where $a_{m,n}(g) = \langle \varphi_n, U(g)\psi_m \rangle_{\mathcal{H}}$, $g \in G$, belongs to $\ell^2(G)$

In general, we can describe a sampling procedure in $\mathcal{V}_\Phi$ as a **convolution system**

$$\begin{array}{rcl} \mathcal{A} : \ell^2_N(G) & \longrightarrow & \ell^2_M(G) \\ \mathbf{x} = (x_1, x_2, \dots, x_N)^\top & \longmapsto & \mathcal{A}(\mathbf{x}) = A * \mathbf{x} = (\mathcal{L}_1 f, \mathcal{L}_2 f, \dots, \mathcal{L}_M f)^\top \end{array}$$

with an associated matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$

Definition

The above sampling procedure $\mathcal{V}_\Phi \ni f \longmapsto \boldsymbol{\mathcal{L}} f := \mathcal{A}(x) \in \ell^2_M(G)$ is stable if there exist two positive constants $0 < \alpha \leq \beta$ such that

$$\alpha \|f\|^2_{\mathcal{H}} \leq \|\boldsymbol{\mathcal{L}} f\|^2_{\ell^2_M(G)} \leq \beta \|f\|^2_{\mathcal{H}} \quad \textit{for all} \ \ f \in \mathcal{V}_\Phi$$

Goal

- **To characterize** the stable sampling procedures in $\mathcal{V}_\Phi$ obtained by means of a convolution system $\mathcal{A}$
- **To obtain** the reconstruction formulas in $\mathcal{V}_\Phi$ associated with the sampling procedure $\mathcal{A}$, i.e., sampling formulas having the form

$$f = \sum_{m=1}^{M} \sum_{g \in G} \mathcal{L}_m f(g)\, U(g) S_m \quad \text{in } \mathcal{H}$$

for some elements $S_1, S_2, \ldots, S_M \in \mathcal{V}_\Phi$ such that the sequence $\{U(g)S_m\}_{g\in G;\, m=1,2,\ldots,M}$ is a frame for $\mathcal{V}_\Phi$

This is done by linking the convolution system $A * x$ defined in $\ell^2_N(G)$ with a frame of translates $\{T_g \mathbf{c}_m\}_{g\in G;\, m=1,2,\ldots,M}$ in $\ell^2_N(G)$, where $T_g\mathbf{c}_m(h) = \mathbf{c}_m(h-g),\ h \in G$, for $\mathbf{c}_m \in \ell^2_N(G)$

A brief on discrete frames

A sequence $\{x_n\}_{n=1}^{\infty}$ is a **frame** for a separable Hilbert space $\mathcal{H}$ if there exist constants $0 < A \leq B$ (frame bounds) such that

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}$$

Preframe operator T (synthesis) and its **adjoint** T^* (analysis)

$$\begin{array}{rccc} T: & \ell^2(\mathbb{N}) & \longrightarrow & \mathcal{H} \\ & \{c_n\}_{n=1}^{\infty} & \longmapsto & \sum_{n=1}^{\infty} c_n\, x_n \end{array} \quad \rightsquigarrow \quad \begin{array}{rccc} T^*: & \mathcal{H} & \longrightarrow & \ell^2(\mathbb{N}) \\ & x & \longmapsto & \{\langle x, x_n\rangle\}_{n=1}^{\infty} \end{array}$$

Frame operator S

$$S:\mathcal{H} \longrightarrow \mathcal{H}\,, \quad Sx := T\,T^*x = \sum_{n=1}^{\infty} \langle x, x_n\rangle\, x_n\,, \quad x \in \mathcal{H}$$

Some properties

- T is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ is a Bessel sequence
- S is a bounded, invertible, positive, self-adjoint operator and the sequence $\{S^{-1}x_n\}_{n=1}^{\infty}$ is a frame for $\mathcal{H}$ (canonical dual frame)
- For each $x \in \mathcal{H}$ we have,

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle \, S^{-1}x_n = \sum_{n=1}^{\infty} \langle x, S^{-1}x_n \rangle \, x_n$$

- $\{x_n\}_{n=1}^{\infty}$ is a frame $\iff$ T is bounded and onto $\iff$ T^* is injective with a closed range $\iff$ S is bounded and invertible
- The sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are dual frames if for each $x \in \mathcal{H}$ we have

$$x = \sum_{n=1}^{\infty} \langle x, x_n \rangle \, y_n = \sum_{n=1}^{\infty} \langle x, y_n \rangle \, x_n$$

Some properties

- $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ Bessel sequences and $T_y T_x^* = T_x T_y^* = I_{\mathcal{H}}$ $\iff$ $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ dual frames
- $\{x_n\}_{n=1}^{\infty}$ is an exact frame $\iff$ $\{x_n\}_{n=1}^{\infty}$ is a Riesz basis (a unique dual)

  Overcomplete (inexact) frames have an infinity of duals
- In finite dimension: frame $\iff$ spanning set

  The set $\{x_1, x_2, \ldots, x_M\}$ in $\mathbb{C}^N$ is a frame $\iff$ rank $\mathbf{M} = N$, where $\mathbf{M} := (x_1, x_2, \ldots, x_M)$

  Its frame operator is $S = \mathbf{M}\,\mathbf{M}^*$ and $A_{opt} = \sigma_1^2$; $B_{opt} = \sigma_N^2$

  (σ_1 and σ_N are the smallest and largest singular value of $\mathbf{M}$ respectively)

Some applications of frames

1. In **Sampling theory** by identifying the data samples as frame coefficients $\{\langle x, x_n\rangle\}$ and looking for appropriate duals $\{y_n\}$
   This is the aim of this talk
2. In **Signal processing** as a noise suppressing method:

$$\sum_{n=1}^{\infty}\big(\langle x, x_n\rangle + \epsilon_n\big)\, S^{-1}x_n = x + S^{-1}\big(\sum_{n=1}^{\infty}\epsilon_n\, x_n\big)$$

   In case of Riesz or orthonormal bases, error $\propto \|\{\epsilon_n\}\|_2 > 0$
   For frames, it can be suppressed whenever $\{\epsilon_n\} \in \ker T$

3. In **Quantum mechanics**: Gleason functions for Parseval frames

Definition

A Gleason function g of weight W is a function $g : B_d \subset \mathcal{H}_d \longrightarrow K$ such that $\sum_{j \in J} g(x_j) = W$ for all $\{x_j\}_{j \in J}$ Parseval frame for $\mathcal{H}_d$

- For instance, $g(x) = \langle Ax, x \rangle$, $x \in B_d$, with A self-adjoint is a Gleason function of weight $W = \mathrm{tr}\,\mathrm{A}$
- Positive operator valued measurements: $\{E_n\} \subset \mathcal{E}(\mathcal{H})$ such that $I = \sum_n E_n \longleftrightarrow$ Parseval frames for $\mathcal{H}$

$E \in \mathcal{E}(\mathcal{H})$ if E is a self-adjoint operator with $0 \leq E \leq I$

- To obtain generalized probability measures $\nu : \mathcal{E}(\mathcal{H}) \to \mathbb{R} \Rightarrow$ $\nu(E) = \mathrm{tr}(\rho\,\mathrm{E})$, $\mathrm{E} \in \mathcal{E}(\mathcal{H})$, for a density operator ρ (Busch/Gleason theorem)

4. In **Numerical analysis** to computing the best approximation

- Let $\Omega \subset (-1,1)^d$ be a compact domain; the orthonormal basis $\{2^{-d/2} \mathrm{e}^{i\pi n \cdot t}\}_{n \in \mathbb{Z}^d}$ for $L^2(-1,1)^d$ restricted to Ω is a tight, linearly-independent frame with bounds $A = B = 1$ for $L^2(\Omega)$
- The augmented Fourier basis $\{\frac{1}{\sqrt{2}} \mathrm{e}^{in\pi t}\}_{n\in\mathbb{Z}} \cup \{\sqrt{k+1/2}\, P_k\}_{k=1}^{K}$ with K Legendre polynomials is a tight, linearly-independent frame with bounds $A = 1$ and $B = 2$ for $L^2(-1,1)$
- A basis of (Legendre, Chebysev) polynomials plus modified polynomials to approximate functions $f(t) = w(t)g(t) + h(t)$, $t \in [-1,1]$, where g, h are smooth and $w \in L^\infty(-1,1)$ may be singular, oscillatory, etc. Indeed, $\{\varphi_n(t)\}_{n\in\mathbb{N}} \cup \{w(t)\varphi_n(t)_{n\in\mathbb{N}}$ is a frame for $L^2(-1,1)$ with bounds

$$A = 1 + \operatorname*{ess\,inf}_{t\in(-1,1)} |w(t)|^2 \quad \text{and} \quad B = 1 + \operatorname*{ess\,sup}_{t\in(-1,1)} |w(t)|^2$$

- To compute the best approximation of f from $\mathcal{H}_N := \operatorname{span}\{\phi_n\}_{n\in I_N}$

$$P_N f = \sum_{n\in I_N} x_n \phi_n \Leftrightarrow \mathbf{G}_N \mathbf{x} = \mathbf{y}\,, \quad \langle P_N f, \phi_n\rangle = \langle f, \phi_n\rangle\,,\ n \in I_N$$

$$\mathbf{G}_N = \{\langle \phi_m, \phi_n\rangle\} \in \mathbb{C}^{N\times N}\,;\ \mathbf{x} = \{x_n\} \in \mathbb{C}^N\,;\ \mathbf{y} = \{\langle f, \phi_n\rangle\} \in \mathbb{C}^N$$

- Ill-conditioned problem $\Rightarrow$ To regularize by using the SVD of $\mathbf{G}_N = \mathbf{V}\mathbf{\Sigma}\mathbf{V}^*$ to construct a truncated SVD projection

$$P_N^\epsilon f = \sum_{\sigma_n > \epsilon} \frac{\langle f, T_N \mathbf{v}_n\rangle}{\sigma_n} T_N \mathbf{v}_n \quad (T_N^* T_N \mathbf{v}_n = \mathbf{G}_N \mathbf{v}_n = \sigma_n \mathbf{v}_n)$$

$\{T_N \mathbf{v}_n\}$ orthogonal basis for $\mathcal{H}_N$: $\quad \langle T_N \mathbf{v}_n, T_N \mathbf{v}_m\rangle = \sigma_m \delta_{n,m}$

- Finally, to give an estimation for $\|f - P_N^\epsilon f\|$

$$\|f - P_N^\epsilon f\| \le \|f - T_N \mathbf{c}\| + \sqrt{\epsilon}\|\mathbf{c}\|$$

- Ejemplo 1. Let $\Omega \subset (-1,1)^d$ be a compact Lipschitz domain. If $f \in H^{kd}(-1,1)^d$, there exists a constant $C > 0$, independent of f and N, such that

$$\|f - P_N^{\epsilon} f\|_{L^2(\Omega)} \leq C\big(N^{-k} + \sqrt{\epsilon}\big)\|f\|_{H^{kd}(-1,1)^d}$$

- Ejemplo 2. Let $K \in \mathbb{N}$ be fixed. If $f \in H^k(-1,1)$ for $0 \leq k \leq K$, there exists a constant $C > 0$, independent of f and N, such that

$$\|f - P_N^{\epsilon} f\|_{L^2(-1,1)} \leq C\big(N^{-k} + \sqrt{\epsilon}\big)\|f\|_{H^k(-1,1)}$$

Discrete convolution systems and frames

Discrete convolution systems: some needed properties

Let

$$\begin{aligned} \mathcal{A} : \ell^2_N(G) &\longrightarrow \ell^2_M(G) \\ \mathbf{x} &\longmapsto \mathcal{A}(\mathbf{x}) = A * \mathbf{x} \end{aligned}$$

be a convolution system with matrix $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$

- $\mathcal{A}$ is a well defined bounded operator if and only if $\widehat{A} \in \mathcal{M}_{M\times N}\big(L^\infty(\widehat{G})\big)$, where $\widehat{A}(\xi) := \big[\widehat{a}_{m,n}(\xi)\big]$ (transfer matrix)
- Its adjoint operator $\mathcal{A}^* : \ell^2_M(G) \to \ell^2_N(G)$ is also a (bounded) convolution operator with matrix $A^* = \big[a^*_{m,n}\big]^\top \in \mathcal{M}_{N\times M}\big(\ell^2(G)\big)$ where $a^*_{m,n}(g) := \overline{a_{m,n}(-g)}$, $\;g \in G$. Its transfer matrix is $\widehat{A^*}(\xi)$ is the transpose conjugate of $\widehat{A}(\xi)$, i.e., $\widehat{A}(\xi)^*$, a.e. $\xi \in \widehat{G}$
- $\mathcal{A}$ (bounded) is injective with closed range $\iff$ $\mathcal{A}^*\mathcal{A}$ invertible $\iff \delta_A := \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)^*\widehat{A}(\xi)] > 0$

When discrete convolution systems meets frames

- Let $\mathbf{a}_m^*$ denote the m-th column of the matrix A^*, then

$$[\mathcal{A}(\mathbf{x})]_m(g) = [A * \mathbf{x}]_m(g) = \left\langle \mathbf{x}, T_g \mathbf{a}_m^* \right\rangle_{\ell^2_N(G)}$$

  In other words, the operator $\mathcal{A}$ is the analysis operator of the sequence $\left\{T_g \mathbf{a}_m^*\right\}_{g\in G;\, m=1,2,\dots,M}$ in $\ell^2_N(G)$. Consequently:

- $\left\{T_g \mathbf{a}_m^*\right\}_{g\in G;\, m=1,2,\dots,M}$ is a frame for $\ell^2_N(G) \iff \delta_A > 0$
- Assume that $\widehat{A} \in \mathcal{M}_{M\times N}(L^\infty(\widehat{G}))$ and $\widehat{B} \in \mathcal{M}_{N\times M}(L^\infty(\widehat{G}))$, and let $\mathbf{b}_m$ denote the m-th column of the matrix B. Then the sequences $\left\{T_g \mathbf{a}_m^*\right\}_{g\in G;\, m=1,2,\dots,M}$ and $\left\{T_g \mathbf{b}_m\right\}_{g\in G;\, m=1,2,\dots,M}$ form a pair of dual frames for $\ell^2_N(G)$ if and only if

$$\widehat{B}(\xi)\,\widehat{A}(\xi) = I_N\,, \text{ a.e. } \xi \in \widehat{G} \quad \left(\iff \mathcal{B}\,\mathcal{A} = \mathcal{I}_{\ell^2_N(G)} \right)$$

When discrete convolution systems meets frames and sampling

- $\{T_g\mathbf{a}_m^*\}$ frame $\iff \delta_A > 0$, and there exist a dual frame $\{T_g\mathbf{b}_m\}$ via the Moore-Penrose pseudo inverse $\widehat{A}(\xi)^\dagger = \big[\widehat{A}(\xi)^*\widehat{A}(\xi)\big]^{-1}\widehat{A}(\xi)^*$
- Thus, for each $\mathbf{x} \in \ell^2_N(G)$ we have

$$\mathbf{x} = \sum_{m=1}^{M}\sum_{g\in G}\big\langle \mathbf{x}, T_g\mathbf{a}_m^*\big\rangle_{\ell^2_N(G)} T_g\mathbf{b}_m = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\, T_g\mathbf{b}_m$$

- Applying the natural isomorphism $\mathcal{T}_{U,\Phi} : \ell^2_N(G) \to \mathcal{V}_\Phi$ which maps the standard orthonormal basis $\{\boldsymbol{\delta}_{g,n}\}_{g\in G;\, n=1,2,\dots,N}$ for $\ell^2_N(G)$ onto the Riesz basis $\big\{U(g)\varphi_n\big\}_{g\in G;\, n=1,2,\dots,N}$ for $\mathcal{V}_\Phi$, for each $f = \mathcal{T}_{U,\Phi}\mathbf{x} \in \mathcal{V}_\Phi$ we get

$$f = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\, \underbrace{\mathcal{T}_{U,\Phi}\big(T_g\mathbf{b}_m\big)}_{U(g)\big(\mathcal{T}_{U,\Phi}\mathbf{b}_m\big)} = \sum_{m=1}^{M}\sum_{g\in G}\mathcal{L}_m f(g)\, U(g)\,\underbrace{\big(\mathcal{T}_{U,\Phi}\mathbf{b}_m\big)}_{S_m}$$

The resulting sampling theory

Theorem

Let $A = [a_{m,n}] \in \mathcal{M}_{M\times N}\big(\ell^2(G)\big)$ *be the matrix defining the sampling vector* $\mathcal{L}f$ *for each* $f \in \mathcal{V}_\Phi$, *and assume that* $\widehat{A} \in \mathcal{M}_{M\times N}\big(L^\infty(\widehat{G})\big)$.

The following statements are equivalent:

(a) *The constant* $\delta_A := \operatorname*{ess\,inf}_{\xi\in\widehat{G}} \det[\widehat{A}(\xi)^*\widehat{A}(\xi)] > 0$

(b) *There exist constants* $0 < \alpha \leq \beta$ *such that*

$$\alpha\|f\|^2 \leq \sum_{m=1}^{M}\sum_{g\in G} |\mathcal{L}_m f(g)|^2 \leq \beta\|f\|^2, \quad f \in \mathcal{V}_\Phi$$

(c) *There exists* $\widehat{B} \in \mathcal{M}_{N\times M}\big(L^\infty(\widehat{G})\big)$ *such that* $\widehat{B}(\xi)\,\widehat{A}(\xi) = I_N$, *a.e.* $\xi \in \widehat{G}$

Theorem

(d) *There exists a matrix* $B \in \mathcal{M}_{N\times M}(\ell^2(G))$ *with* $\widehat{B} \in \mathcal{M}_{N\times M}(L^\infty(\widehat{G}))$*, such that*

$$\mathbf{x} = B * \boldsymbol{\mathcal{L}} f \quad \text{and} \quad f = \sum_{n=1}^{N} \sum_{g\in G} x_n(g) U(g)\varphi_n\,, \quad f \in \mathcal{V}_\Phi$$

That is, the bounded convolution system $\mathcal{B} : \ell^2_M(G) \to \ell^2_N(G)$ *satisfies* $\mathcal{B}\,\mathcal{A} = \mathcal{I}_{\ell^2_N(G)}$

(e) *There exist* M *elements* $S_m \in \mathcal{V}_\Phi$ *such that the sequence* $\{U(g)S_m\}_{g\in G;\, m=1,2,\dots,M}$ *is a frame for* $\mathcal{V}_\Phi$*, and the formula*

$$f = \sum_{m=1}^{M} \sum_{g\in G} \mathcal{L}_m f(g)\, U(g)S_m \quad \text{in } \mathcal{H}$$

holds in $\mathcal{V}_\Phi$

Theorem

(f) *There exists a frame* $\{S_{g,m}\}_{g\in G;\, m=1,2,\dots,M}$ *for* $\mathcal{V}_\Phi$ *such that for each* $f \in \mathcal{V}_\Phi$ *we have the expansion*

$$f = \sum_{m=1}^{M} \sum_{g\in G} \mathcal{L}_m f(g)\, S_{g,m} \quad \text{in } \mathcal{H}$$

The reconstruction elements $\{S_m\}_{m=1,2,\dots,M}$ in $\mathcal{V}_\Phi$ are necessarily obtained from the columns $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_M$ of a matrix B satisfying (c)

$$S_m = \mathcal{T}_{U,\Phi}\mathbf{b}_m = \sum_{n=1}^{N} \sum_{g\in G} b_{n,m}\, U(g)\varphi_n\,, \quad m = 1, 2, \dots, M$$

If $M = N$ and $\operatorname*{ess\,inf}_{\xi\in\widehat{G}} |\det[\widehat{A}(\xi)]| > 0$ we are in the Riesz bases setting

Sampling results as particular cases

Average sampling in a shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$

$\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big[U(p)f\big](t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$

The average samples

$$\mathcal{L}_m f(p) := \big\langle f, \psi_m(\cdot - p)\big\rangle_{L^2(\mathbb{R}^d)}, \quad p \in \mathbb{Z}^d$$

have the matrix $A = [a_{m,n}]$, with $a_{m,n}(p) = \big\langle \varphi_n, \psi_m(\cdot - p)\big\rangle_{L^2(\mathbb{R}^d)}$. Under the hypotheses in the sampling theorem, for the classical shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$ described as

$$V_\Phi^2 = \Big\{ \sum_{n=1}^{N} \sum_{p\in\mathbb{Z}^d} x_n(p)\, \varphi_n(t-p) \ : \ x_n \in \ell^2(\mathbb{Z}^d),\, n = 1, 2, \dots, N \Big\}$$

we obtain

$$f(t) = \sum_{m=1}^{M} \sum_{p\in\mathbb{Z}^d} \big\langle f, \psi_m(\cdot - p)\big\rangle_{L^2(\mathbb{R}^d)} S_m(t-p) \quad \text{in } L^2(\mathbb{R}^d)$$

As V_Φ^2 is a RKHS, we have also pointwise convergence

Sampling associated with a crystallographic group

P is a non-singular $d \times d$ matrix, $\Gamma = \{\gamma_1 = I, \gamma_2, \dots, \gamma_N\}$ a finite subgroup of $O(d)$ of order N such that $\gamma(P\mathbb{Z}^d) = P\mathbb{Z}^d$ for each $\gamma \in \Gamma$.

Consider the **crystallographic group** is $\mathcal{C}_{P,\Gamma} := P\mathbb{Z}^d \rtimes_\sigma \Gamma$ (Recall that $(x,\gamma)\cdot(x',\gamma') = (x+\gamma x', \gamma\gamma')$) and its quasi regular representation on $L^2(\mathbb{R}^d)$

$$U(p,\gamma)f(t) = f[\gamma^\top(t-p)]\,, \quad p \in P\mathbb{Z}^d,\, \gamma \in \Gamma \text{ and } f \in L^2(\mathbb{R}^d)$$

Consider the U-invariant subspace in $L^2(\mathbb{R}^d)$

$$\mathcal{V}_\varphi = \Big\{ \sum_{(p,\gamma)\in\mathcal{C}_{P,\Gamma}} x(p,\gamma)\, \underbrace{\varphi[\gamma^\top(t-p)]}_{U(p,\gamma)\varphi(t)} \;:\; \{x(p,\gamma)\} \in \ell^2(\mathcal{C}_{P,\Gamma}) \Big\}$$

$$= \Big\{ \sum_{n=1}^{N} \sum_{p\in P\mathbb{Z}^d} x_n(p)\, \underbrace{U(p,I)\, \underbrace{U(0,\gamma_n)\varphi}_{\varphi_n}}_{\varphi_n(t-p)} \;:\; x_n \in \ell^2(P\mathbb{Z}^d),\; n = 1,2,\dots,N \Big\}$$

Choosing M functions $\psi_m \in L^2(\mathbb{R}^d)$, $m = 1, 2, \ldots, M$, we consider the average sampling in $\mathcal{V}_\varphi$

$$\mathcal{L}_m f(p) = \langle f, U(p, I)\psi_m \rangle = \langle f, \psi_m(\cdot - p) \rangle , \quad p \in P\mathbb{Z}^d .$$

Here, the matrix $A = [a_{m,n}]$ has entries

$$a_{m,n}(p) = \big\langle \varphi(t), \psi_m(\gamma_n t - p) \big\rangle_{L^2(\mathbb{R}^d)}$$

Under the hypotheses in the sampling theorem, there exist $M \geq N$ sampling functions $S_m \in \mathcal{V}_\varphi$ for $m = 1, 2, \ldots, M$, such that the sequence $\{S_m(\cdot - p)\}_{p \in P\mathbb{Z}^d;\, m=1,2,\ldots,M}$ is a frame for $\mathcal{V}_\varphi$ and, for each $f \in \mathcal{V}_\varphi$, we have

$$f(t) = \sum_{m=1}^{M} \sum_{p \in P\mathbb{Z}^d} \big\langle f, \psi_m(\cdot - p) \big\rangle_{L^2(\mathbb{R}^d)} S_m(t - p) \quad \text{in } L^2(\mathbb{R}^d)$$

The case of pointwise sampling

G is a countable discrete subgroup of a LCA group $\widetilde{G}$

$t \in \widetilde{G} \mapsto U(t) \in \mathcal{U}(L^2(\widetilde{G}))$ a unitary representation of $\widetilde{G}$ on $\mathcal{H} = L^2(\widetilde{G})$

Let $\mathcal{V}_\Phi$ be the corresponding U-invariant subspace of $L^2(\widetilde{G})$; for any $f \in \mathcal{V}_\Phi$ we define, from M fixed points $t_m \in \widetilde{G},\ m = 1, 2, \dots, M$, its samples as

$$\mathcal{L}_m f(g) := \big[U(-g)f\big](t_m), \quad g \in G,\ m = 1, 2, \dots, M$$

The associated matrix A has entries $a_{m,n}(g) = \big[U(-g)\varphi_n\big](t_m),\ g \in G$.
If the functions $[U(g)\varphi_n](t)$, $g \in G$ and $n = 1, 2, \dots, N$, are continuous on $\widetilde{G}$, and the condition

$$\sup_{t \in \widetilde{G}} \sum_{g \in G} \big|[U(g)\varphi_n](t)\big|^2 < +\infty\,, \quad n = 1, 2, \dots, N$$

holds, then the subspace $\mathcal{V}_\Phi$ is a reproducing kernel Hilbert space of continuous bounded functions in $L^2(\widetilde{G})$

▶ Whenever $\mathcal{H} = L^2(\mathbb{R}^d)$ and $[U(p)f](t) := f(t-p)$, $t \in \mathbb{R}^d$, $p \in \mathbb{Z}^d$, the samples read

$$\mathcal{L}_m f(p) = \big[U(-p)f\big](t_m) = f(p+t_m)\,, \quad p \in \mathbb{Z}^d \text{ and } m = 1,2,\dots,M$$

Under hypotheses in the sampling theorem on the matrix $A = [a_{m,n}]$, where $a_{m,n}(p) = \varphi_n(t_m + p)$, we obtain in the shift-invariant subspace V_Φ^2 of $L^2(\mathbb{R}^d)$ the sampling formula

$$f(t) = \sum_{m=1}^{M} \sum_{p \in \mathbb{Z}^d} f(p+t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d$$

holds, for some functions $S_m \in V_\Phi^2$, $m = 1,2,\dots,M$. The convergence of the series in $L^2(\mathbb{R}^d)$-norm implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$

► In the example of the quasi regular representation of the crystallographic group $\mathcal{C}_{P,\Gamma} = P\mathbb{Z}^d \rtimes_\sigma \Gamma$, for each $f \in \mathcal{V}_\varphi$ the defined samples read

$$\mathcal{L}_m f(p) = \big[U(-p,I)f\big](t_m) = f(p+t_m)\,,\ p \in P\mathbb{Z}^d \text{ and } m = 1,2,\dots,M$$

Under hypotheses in the sampling theorem on the matrix $A = [a_{m,n}]$, where $a_{m,n}(p) = \varphi[\gamma_n^\top(t_m - p)]$, there exist M functions $S_m \in \mathcal{V}_\varphi$, $m = 1,2,\dots,M$, such that, for each $f \in \mathcal{V}_\varphi$, the sampling formula

$$f(t) = \sum_{m=1}^{M} \sum_{p \in P\mathbb{Z}^d} f(p+t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$.

Sampling in a subgroup H of G

Let H be a subgroup of G with finite index L. Then,

$$G = (g_1 + H) \cup (g_2 + H) \cup \cdots \cup (g_L + H)$$

with $(g_l + H) \cap (g_{l'} + H) = \varnothing$, $l \neq l'$, for a fix set $\{g_0, g_1, \ldots, g_L\}$ of representatives of the cosets of H. For $\mathcal{V}_\Phi$ we have

$$\begin{aligned} \mathcal{V}_\Phi &= \Big\{ \sum_{n=1}^{N} \sum_{g\in G} x_n(g)\, U(g)\varphi_n \;:\; x_n \in \ell^2(G) \Big\} \\ &= \Big\{ \sum_{n=1}^{N} \sum_{l=1}^{L} \sum_{h\in H} \underbrace{x_n(g_l + h)}_{x_{nl}(h)} \underbrace{U(g_l + h)\varphi_n}_{U(h)\underbrace{U(g_l)\varphi_n}_{\varphi_{nl}}} \Big\} \end{aligned}$$

$$\big(x_{11}(h), \ldots, x_{1L}(h), x_{21}(h), \ldots, x_{2L}(h), \ldots, x_{N1}(h), \ldots, x_{NL}(h)\big)^\top \in \ell^2_{NL}(H)$$

(here we consider a new index nl, from 11 to NL)

- In $\mathcal{H} := L^2(\mathbb{R}^d)$, $G := \mathbb{Z}^d$ and $\big[U(p)f\big](t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$, consider $H := P\mathbb{Z}^d$ a sublattice in $\mathbb{Z}^d$ where P denotes a $d \times d$ matrix of integer entries with positive determinant $L := \det P$.

Average sampling in the shift-invariant subspace V_Φ^2 has the matrix $A = [a_{m,nl}]$, where $a_{m,nl}(p) = \big\langle \varphi_n(t), \psi_m(t-p+g_l)\big\rangle_{L^2(\mathbb{R}^d)}$, $p \in P\mathbb{Z}^d$

The corresponding sampling formula reads

$$f(t) = \sum_{m=1}^{M} \sum_{p \in P\mathbb{Z}^d} \langle f, \psi_m(\cdot - p)\rangle_{L^2(\mathbb{R}^d)}\, S_m(t-p)\,, \quad t \in \mathbb{R}^d$$

for some sampling functions $S_m \in V_\Phi^2$, $m = 1, 2, \ldots, M$. Moreover, the sampling sequence $\{S_m(t-p)\}_{p \in P\mathbb{Z}^d;\, m=1,2,\ldots,M}$ is a frame for V_Φ^2.

Convergence in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$

▶ Whenever $\mathcal{H} = L^2(\mathbb{R}^d)$ $G := \mathbb{Z}^d$ and $[U(p)f](t) := f(t-p)$, $t \in \mathbb{R}^d$ and $p \in \mathbb{Z}^d$, for M fixed points $t_m \in \mathbb{R}^d$, we consider in the shift-invariant subspace V_Φ^2 the samples

$$\mathcal{L}_m f(p) := [U(-p)f](t_m) = f(p+t_m),\ p \in P\mathbb{Z}^d,\ m = 1,2,\ldots,M$$

This pointwise sampling has matrix $A = [a_{m,nl}]$, where $a_{m,nl}(p) = \varphi_n(t_m + p - g_l)$, $p \in P\mathbb{Z}^d$, for $m = 1,2,\ldots,M$, $n = 1,2,\ldots,N$ and $l = 1,2,\ldots,L$.

The corresponding sampling formula reads

$$f(t) = \sum_{m=1}^{M} \sum_{p \in P\mathbb{Z}^d} f(p+t_m)\, S_m(t-p)\,, \quad t \in \mathbb{R}^d\,.$$

for some sampling functions $S_m \in V_\Phi^2$, $m = 1,2,\ldots,M$. Moreover, the sampling sequence $\{S_m(t-p)\}_{p \in P\mathbb{Z}^d;\, m=1,2,\ldots,M}$ is a frame for V_Φ^2.

Convergence of the series in the $L^2(\mathbb{R}^d)$-norm sense implies pointwise convergence which is absolute and uniform on $\mathbb{R}^d$

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