

# Semi-direct product of groups, filter banks and sampling

Antonio García García

(joint work with M.A. Hernández-Medina and G. Pérez-Villalón)

---

## Foreword

1. From Paley-Wiener space  $PW_\pi$  to shift-invariant spaces  $V_\varphi^2$  in  $L^2(\mathbb{R})$ : Unitary representation of  $(\mathbb{Z}, +)$  on  $L^2(\mathbb{R})$  by shifts  $\{T_n\}$  with sampling space  $\overline{\text{span}}\{T_n \text{sinc}\}_{n \in \mathbb{Z}}$  or  $\overline{\text{span}}\{T_n \varphi\}_{n \in \mathbb{Z}}$
2.  $U$ -invariant subspaces in abstract Hilbert spaces  $\mathcal{H}$ : Unitary representation of  $(\mathbb{Z}, +)$  (or  $(\mathbb{Z}_N, +)$ ) on  $\mathcal{H}$  as  $\{U^n\}$  with sampling space  $\overline{\text{span}}\{U^n a\}_{n \in \mathbb{Z}}$  (or  $\text{span}\{U^n a\}_{n=0}^{N-1}$ ). Also by using the representation  $\{T^n\}$  with  $T$  bounded invertible operator
3. Sampling at uniform lattices of LCA groups: Unitary representation  $\{U(g)\}_{g \in G}$  of a discrete LCA group  $(G, +)$  on  $\mathcal{H}$  with sampling space  $\overline{\text{span}}\{U(g)a\}_{g \in G}$
4. Sampling associated with finite non abelian groups: knit-product  $G = N \bowtie H$  or semidirect product  $G = N \rtimes H$  of finite groups
5. Sampling associated with infinite non abelian groups: a countable discrete semidirect product  $G = N \rtimes H$  of groups

# Summary

- ▶ Statement of the sampling problem
- ▶ A brief on semi-direct product of groups
- ▶ The filter banks formalism
- ▶ Frame analysis
- ▶ Getting on with sampling
- ▶ Some examples involving crystallographic groups

# Statement of the sampling problem

# Ingredients

- ▶ Let  $(n, h) \mapsto U(n, h)$  be a **unitary representation** on a separable Hilbert space  $\mathcal{H}$  of a group  $G = N \rtimes_{\phi} H$  ( $\#H = L$ ), i.e., homomorphism between  $G$  and  $\mathcal{U}(\mathcal{H})$
- ▶ Fixed  $a \in \mathcal{H}$ , consider the  $U$ -invariant subspace in  $\mathcal{H}$

$$\mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n, h) \underbrace{U(n, h)a}_{\text{Riesz sequence}} : \{\alpha(n, h)\}_{(n,h) \in G} \in \ell^2(G) \right\}$$

- ▶ For each  $x \in \mathcal{A}_a$  consider the data (samples)

$$\mathcal{L}_k x(n) := \langle x, U(n, 1_H) b_k \rangle_{\mathcal{H}}, \quad n \in N, \quad k = 1, 2, \dots, K (\geq L)$$

where  $b_k \in \mathcal{H}$  are fixed not necessarily in  $\mathcal{A}_a$

# Goal

The **stable recovery** of any  $x \in \mathcal{A}_a$  from the data sequence  $\{\mathcal{L}_k x(n)\}_{n \in N; k=1,2,\dots,K}$

- ▶ There exist constants  $0 < A \leq B$  such that

$$A\|x\|^2 \leq \sum_{k=1}^K \sum_{n \in N} |\mathcal{L}_k x(n)|^2 \leq B\|x\|^2, \quad \text{for all } x \in \mathcal{A}_a$$

- ▶ Recovery by means of a sampling formula like

$$x = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H}$$

## Procedure

- ▶ Since there exist  $\mathbf{h}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that

$$\mathcal{L}_k x(n) = \langle \boldsymbol{\alpha}, T_n \mathbf{h}_k \rangle_{\ell^2(G)}, \quad n \in N, \quad k = 1, 2, \dots, K$$

where  $T_n \mathbf{h}_k(m, h) = \mathbf{h}_k(m - n, h)$ ,  $(m, h) \in G$ ,  $k = 1, 2, \dots, K$

## Procedure

- ▶ Since there exist  $\mathbf{h}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that

$$\mathcal{L}_k x(n) = \langle \alpha, T_n \mathbf{h}_k \rangle_{\ell^2(G)}, \quad n \in N, \quad k = 1, 2, \dots, K$$

where  $T_n \mathbf{h}_k(m, h) = \mathbf{h}_k(m - n, h)$ ,  $(m, h) \in G$ ,  $k = 1, 2, \dots, K$

- ▶ Instead of  $x \in \mathcal{A}_a$  proceed to the **stable recovery of  $\alpha \in \ell^2(G)$**  from given data



## Procedure

- ▶ Since there exist  $\mathbf{h}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that

$$\mathcal{L}_k x(n) = \langle \alpha, T_n \mathbf{h}_k \rangle_{\ell^2(G)}, \quad n \in N, \quad k = 1, 2, \dots, K$$

where  $T_n \mathbf{h}_k(m, h) = \mathbf{h}_k(m - n, h)$ ,  $(m, h) \in G$ ,  $k = 1, 2, \dots, K$

- ▶ Instead of  $x \in \mathcal{A}_a$  proceed to the **stable recovery of  $\alpha \in \ell^2(G)$**  from given data
- ▶ The sequence  $\{T_n \mathbf{h}_k\}_{n \in N; k=1,2,\dots,K}$  should be a **frame** for  $\ell^2(G)$ : characterize these frames

## Procedure

- ▶ Since there exist  $\mathbf{h}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that

$$\mathcal{L}_k x(n) = \langle \alpha, T_n \mathbf{h}_k \rangle_{\ell^2(G)}, \quad n \in N, \quad k = 1, 2, \dots, K$$

where  $T_n \mathbf{h}_k(m, h) = \mathbf{h}_k(m - n, h)$ ,  $(m, h) \in G$ ,  $k = 1, 2, \dots, K$

- ▶ Instead of  $x \in \mathcal{A}_a$  proceed to the **stable recovery of  $\alpha \in \ell^2(G)$**  from given data
- ▶ The sequence  $\{T_n \mathbf{h}_k\}_{n \in N; k=1,2,\dots,K}$  should be a **frame** for  $\ell^2(G)$ : characterize these frames
- ▶ To find its **dual frames**  $\{T_n \mathbf{g}_k\}_{n \in N; k=1,2,\dots,K}$  having the same form

## Procedure

- ▶ Since there exist  $\mathbf{h}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that

$$\mathcal{L}_k x(n) = \langle \alpha, T_n \mathbf{h}_k \rangle_{\ell^2(G)}, \quad n \in N, \quad k = 1, 2, \dots, K$$

where  $T_n \mathbf{h}_k(m, h) = \mathbf{h}_k(m - n, h)$ ,  $(m, h) \in G$ ,  $k = 1, 2, \dots, K$

- ▶ Instead of  $x \in \mathcal{A}_a$  proceed to the **stable recovery of  $\alpha \in \ell^2(G)$**  from given data
- ▶ The sequence  $\{T_n \mathbf{h}_k\}_{n \in N; k=1,2,\dots,K}$  should be a **frame** for  $\ell^2(G)$ : characterize these frames
- ▶ To find its **dual frames**  $\{T_n \mathbf{g}_k\}_{n \in N; k=1,2,\dots,K}$  having the same form
- ▶ Then, in  $\ell^2(G)$

$$\alpha = \sum_{k=1}^K \sum_{n \in N} \langle \alpha, T_n \mathbf{h}_k \rangle_{\ell^2(G)} T_n \mathbf{g}_k = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) T_n \mathbf{g}_k$$

## Procedure

- ▶ Introducing in  $x = \sum_{(m,h) \in G} \alpha(m,h) U(m,h)a$  the expansion for  $\alpha$

$$\begin{aligned}x &= \sum_{(m,h) \in G} \left[ \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) T_n \mathbf{g}_k(m,h) \right] U(m,h)a = \dots \\&= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) \left[ \sum_{(\tilde{n},h) \in G} \mathbf{g}_k(\tilde{n},h) \underbrace{U(n, 1_H) U(\tilde{n}, h) a}_{U(m,h)=U(n+\tilde{n},h)} \right] \\&= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_{k,\mathbf{g}}\end{aligned}$$

## Procedure

- ▶ Introducing in  $x = \sum_{(m,h) \in G} \alpha(m,h) U(m,h)a$  the expansion for  $\alpha$

$$\begin{aligned} x &= \sum_{(m,h) \in G} \left[ \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) T_n \mathbf{g}_k(m,h) \right] U(m,h)a = \dots \\ &= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) \left[ \sum_{(\tilde{n},h) \in G} \mathbf{g}_k(\tilde{n},h) \underbrace{U(n, 1_H) U(\tilde{n},h)}_{U(m,h)=U(n+\tilde{n},h)} a \right] \\ &= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_{k,\mathbf{g}} \end{aligned}$$

- ▶ Next, let's go to do it ...

# A brief on semi direct product of groups

## The semi-direct product $G := N \rtimes_{\phi} H$

Given groups  $(N, \cdot)$  and  $(H, \cdot)$ , and a homomorphism  $\phi : H \rightarrow \text{Aut}(N)$  ( $\phi(h) := \phi_h$ ) their **semi-direct product**  $G := N \rtimes_{\phi} H$  has, in the underlying set  $N \times H$ , the composition law:

$$(n_1, h_1) \cdot (n_2, h_2) := (n_1 \phi_{h_1}(n_2), h_1 h_2), \quad (n_1, h_1), (n_2, h_2) \in G,$$

The homomorphism  $\phi$  is referred as **the action** of the group  $H$  on the group  $N$ . Some basic facts:

- ▶  $(1_N, 1_H)$ , and  $(n, h)^{-1} = (\phi_{h^{-1}}(n^{-1}), h^{-1})$
- ▶  $N \simeq N \times \{1_H\}$  and  $H \simeq \{1_N\} \times H$
- ▶  $G = N \rtimes_{\phi} H$  is **not abelian**, even for abelian  $N$  and  $H$  groups.
- ▶  $N$  is a normal subgroup in  $G$ .

## Some examples of semi-direct product of groups

1. The **dihedral group**  $D_{2N}$  is the group of symmetries of a regular  $N$ -sided polygon; it is the semi-direct product  $D_{2N} = \mathbb{Z}_N \rtimes_{\phi} \mathbb{Z}_2$  where  $\phi_{\bar{0}} \equiv Id_{\mathbb{Z}_N}$  and  $\phi_{\bar{1}}(\bar{n}) = -\bar{n}$  for each  $\bar{n} \in \mathbb{Z}_N$ .  
The **infinite dihedral group**  $D_{\infty}$  defined as  $\mathbb{Z} \rtimes_{\phi} \mathbb{Z}_2$  for the similar homomorphism  $\phi$  is the group of isometries of  $\mathbb{Z}$ .
2. The **Euclidean motion group**  $E(d)$  is the semi-direct product  $\mathbb{R}^d \rtimes_{\phi} O(d)$ , where  $O(d)$  is the orthogonal group of order  $d$  and  $\phi_A(x) = Ax$  for  $A \in O(d)$  and  $x \in \mathbb{R}^d$ . It contains as a subgroup any **crystallographic group**  $\mathcal{C}_{M,\Gamma} := M\mathbb{Z}^d \rtimes_{\phi} \Gamma$ , where  $M\mathbb{Z}^d$  denotes a full rank lattice of  $\mathbb{R}^d$  and  $\Gamma$  is any finite subgroup of  $O(d)$  such that  $\phi_{\gamma}(M\mathbb{Z}^d) = M\mathbb{Z}^d$  for each  $\gamma \in \Gamma$ .



- a. Suppose that  $N$  is an locally compact abelian (LCA) group with Haar measure  $\mu_N$  and  $H$  is a locally compact group with Haar measure  $\mu_H$ . Then, the semi-direct product  $G = N \rtimes_{\phi} H$  endowed with the product topology becomes also a topological group
- b. In our case, we will assume that  $G = N \rtimes_{\phi} H$  where  $(N, +)$  is a countable discrete abelian group and  $(H, \cdot)$  is a finite group.

Recall the operational calculus:

- ▶  $(n, h) \cdot (m, l) = (n + \phi_h(m), hl)$
- ▶  $(n, h)^{-1} = (\phi_{h^{-1}}(-n), h^{-1})$
- ▶  $(n, h)^{-1} \cdot (m, l) = (\phi_{h^{-1}}(m - n), h^{-1}l)$
- ▶  $(n, h)^{-1} \cdot (m, 1_H) = (\phi_{h^{-1}}(m - n), h^{-1})$
- ▶  $(n, 1_H)^{-1} \cdot (m, l) = (m - n, l)$

# The filter banks formalism

- The convolution  $\alpha * h$  of  $\alpha, h \in \ell^2(G)$  can be expressed as

$$\begin{aligned} (\alpha * h)(m, l) &= \sum_{(n, h) \in G} \alpha(n, h) h[(n, h)^{-1} \cdot (m, l)] \\ &= \sum_{(n, h) \in G} \alpha(n, h) h(\phi_{h^{-1}}(m - n), h^{-1}l), \quad (m, l) \in G \end{aligned}$$

- $H$ -decimation

$$\begin{aligned} \downarrow_H(\alpha * h)(m) &:= (\alpha * h)(m, 1_H) = \sum_{(n, h) \in G} \alpha(n, h) h(\phi_{h^{-1}}(m - n), h^{-1}) \\ &= \sum_{(n, h) \in G} \alpha(n, h) h[(n - m, h)^{-1}], \quad m \in N \end{aligned}$$

$$\downarrow_H(\alpha * h)(m) = \sum_{h \in H} \sum_{n \in N} \alpha_h(n) h_h(m - n) = \sum_{h \in H} (\alpha_h *_N h_h)(m), \quad m \in N$$

where  $\alpha_h(n) := \alpha(n, h)$  and  $h_h(n) := h[(-n, h)^{-1}]$

For a function  $c : N \rightarrow \mathbb{C}$ , its  $H$ -expander  $\uparrow_H c : G \rightarrow \mathbb{C}$  is

$$(\uparrow_H c)(n, h) = \begin{cases} c(n) & \text{if } h = 1_H \\ 0 & \text{if } h \neq 1_H. \end{cases}$$

In case  $\uparrow_H c$  and  $\mathbf{g}$  belong to  $\ell^2(G)$  we have

$$\begin{aligned} (\uparrow_H c * \mathbf{g})(m, l) &= \sum_{(n, h) \in G} (\uparrow_H c)(n, h) \mathbf{g}[(n, h)^{-1} \cdot (m, l)] \\ &= \sum_{(n, h) \in G} (\uparrow_H c)(n, h) \mathbf{g}(\phi_{h^{-1}}(m - n), h^{-1}l) \\ &= \sum_{n \in N} c(n) \mathbf{g}(m - n, l) = (c *_N \mathbf{g}_l)(m), \quad m \in N, l \in H \end{aligned}$$

where  $\mathbf{g}_l(n) := \mathbf{g}(n, l)$

A  $K$ -channel filter bank with analysis filters  $h_k$  and synthesis filters  $g_k$ ,  $k = 1, 2, \dots, K$

$$\mathbf{c}_k := \downarrow_H (\alpha * h_k), \quad k = 1, 2, \dots, K, \quad \text{and} \quad \beta = \sum_{k=1}^K (\uparrow_H c_k) * g_k,$$

In polyphase notation

$$\mathbf{c}_k(m) = \sum_{h \in H} (\alpha_h *_{N} h_{k,h})(m), \quad m \in N, \quad k = 1, 2, \dots, K,$$
$$\beta_l(m) = \sum_{k=1}^K (\mathbf{c}_k *_{N} g_{l,k})(m), \quad m \in N, \quad l \in H,$$

where  $\alpha_h(n) := \alpha(n, h)$ ,  $\beta_l(n) := \beta(n, l)$ ,  $h_{k,h}(n) := h_k[(-n, h)^{-1}]$  and  $g_{l,k}(n) := g_k(n, l)$  are the polyphase components of  $\alpha$ ,  $\beta$ ,  $h_k$  and  $g_k$ ,  $k = 1, 2, \dots, K$

## Our K-channel filter bank

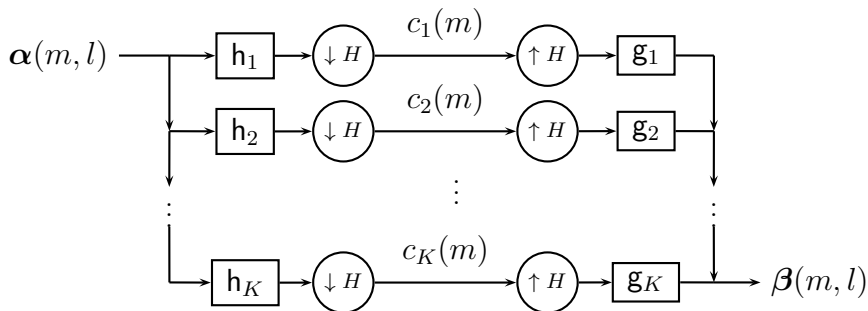


Figure : The K-channel filter bank scheme

Denoting  $H = \{h_1, h_2, \dots, h_L\}$ , assume that  $\mathbf{h}_k, \mathbf{g}_k \in \ell^2(G)$  with  $\widehat{\mathbf{h}}_{k,h_i}, \widehat{\mathbf{g}}_{h_i,k} \in L^\infty(\widehat{N})$  for  $k = 1, 2, \dots, K$  and  $i = 1, 2, \dots, L$ .

Recall,  $\mathbf{h}_{k,h_i}(n) := \mathbf{h}_k[(-n, h_i)^{-1}]$  and  $\mathbf{g}_{h_i,k}(n) := \mathbf{g}_k(n, h_i)$

The  $N$ -Fourier transform in

$$\mathbf{c}_k(m) = \sum_{h \in H} (\boldsymbol{\alpha}_h *_{N} \mathbf{h}_{k,h})(m) \rightsquigarrow \widehat{\mathbf{c}}_k(\gamma) = \sum_{h \in H} \widehat{\mathbf{h}}_{k,h}(\gamma) \widehat{\boldsymbol{\alpha}}_h(\gamma) \text{ a.e. } \gamma \in \widehat{N}$$

In matrix notation,

$$\mathbf{C}(\gamma) = \mathbf{H}(\gamma) \mathbf{A}(\gamma) \text{ a.e. } \gamma \in \widehat{N}$$

$$\mathbf{C}(\gamma) = (\widehat{\mathbf{c}}_1(\gamma), \widehat{\mathbf{c}}_2(\gamma), \dots, \widehat{\mathbf{c}}_K(\gamma))^{\top},$$

$$\mathbf{A}(\gamma) = (\widehat{\boldsymbol{\alpha}}_{h_1}(\gamma), \widehat{\boldsymbol{\alpha}}_{h_2}(\gamma), \dots, \widehat{\boldsymbol{\alpha}}_{h_L}(\gamma))^{\top}, \text{ and } \mathbf{H}(\gamma) \text{ is the } K \times L \text{ matrix}$$

$$\mathbf{H}(\gamma) = \left( \widehat{\mathbf{h}}_{k,h_i}(\gamma) \right)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}, \quad \gamma \in \widehat{N}$$

$$\beta_l(m) = \sum_{k=1}^K (\mathbf{c}_k *_{N} \mathbf{g}_{l,k})(m) \rightsquigarrow \hat{\beta}_l(\gamma) = \sum_{k=1}^K \hat{\mathbf{g}}_{l,k}(\gamma) \hat{\mathbf{c}}_k(\gamma) \text{ a.e. } \gamma \in \hat{N}$$

In matrix notation,

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{C}(\gamma) \text{ a.e. } \gamma \in \hat{N}$$

$$\mathbf{B}(\gamma) = (\hat{\beta}_{h_1}(\gamma), \hat{\beta}_{h_2}(\gamma), \dots, \hat{\beta}_{h_L}(\gamma))^{\top},$$

$$\mathbf{C}(\gamma) = (\hat{\mathbf{c}}_1(\gamma), \hat{\mathbf{c}}_2(\gamma), \dots, \hat{\mathbf{c}}_K(\gamma))^{\top}$$

$\mathbf{G}(\gamma)$  is the  $L \times K$  matrix

$$\mathbf{G}(\gamma) = \left( \hat{\mathbf{g}}_{h_i,k}(\gamma) \right)_{\substack{i=1,2,\dots,L \\ k=1,2,\dots,K}}, \quad \gamma \in \hat{N}$$

$\hat{\mathbf{g}}_{h_i,k} \in L^2(\hat{N})$  is the Fourier transform of  $\mathbf{g}_{h_i,k}(n) := \mathbf{g}_k(n, h_i) \in \ell^2(N)$



## Perfect reconstruction

In terms of the **polyphase matrices**  $\mathbf{G}(\gamma)$  and  $\mathbf{H}(\gamma)$

$$\mathbf{B}(\gamma) = \mathbf{G}(\gamma) \mathbf{H}(\gamma) \mathbf{A}(\gamma) \quad \text{a.e. } \gamma \in \widehat{N}$$

### Theorem

A  $K$ -channel filter bank with  $h_k, g_k$  belong to  $\ell^2(G)$  and  $\widehat{h}_{k,h_i}, \widehat{g}_{h_i,k}$  belong to  $L^\infty(\widehat{N})$  for  $k = 1, 2, \dots, K$  and  $i = 1, 2, \dots, L$ , satisfies the **perfect reconstruction property** if and only if  $\mathbf{G}(\gamma) \mathbf{H}(\gamma) = \mathbf{I}_L$  a.e.  $\gamma \in \widehat{N}$ , where  $\mathbf{I}_L$  denotes the identity matrix of order  $L$ .

$\ell^2(G) \ni \alpha \mapsto \mathbf{A} \in L_L^2(\widehat{N})$  is a unitary operator

# Frame analysis

The **translation**  $T_m$  ( $m \in N$ ) and the **involution** operators are defined in  $\ell^2(G)$  as

$$T_m \alpha(n, h) := \alpha((m, 1_H)^{-1} \cdot (n, h)) = \alpha(n - m, h)$$

$$\tilde{\alpha}(n, h) := \overline{\alpha((n, h)^{-1})}, \quad (n, h) \in G$$

For a  $K$ -channel filter bank we have

$$\mathbf{c}_k(m) = \downarrow_H (\alpha * \mathbf{h}_k)(m) = \langle \alpha, T_m \tilde{\mathbf{h}}_k \rangle_{\ell^2(G)},$$

$$(\uparrow_H \mathbf{c}_k * \mathbf{g}_k)(m, h) = \sum_{n \in N} \mathbf{c}_k(n) \mathbf{g}_k(m - n, h) = \sum_{n \in N} \langle \alpha, T_n \tilde{\mathbf{h}}_k \rangle_{\ell^2(G)} T_n \mathbf{g}_k(m, h)$$

In the perfect reconstruction setting, for any  $\alpha \in \ell^2(G)$  we have

$$\alpha = \sum_{k=1}^K \sum_{n \in N} \langle \alpha, T_n \tilde{\mathbf{h}}_k \rangle_{\ell^2(G)} T_n \mathbf{g}_k \quad \text{in } \ell^2(G)$$

## Theorem

For  $f_k$  in  $\ell^2(G)$ ,  $k = 1, 2, \dots, K$ , denote  $f_{k,h_i}(n) := f_k(n, h_i)$ ,  $h_i \in H$ , and consider the associated matrix  $\mathbf{H}(\gamma)$ . Then,

1. The sequence  $\{T_n f_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  is a Bessel sequence for  $\ell^2(G)$  if and only if  $B_{\mathbf{H}} < \infty$  (if and only if the function  $\widehat{f}_{k,h_i} \in L^\infty(\widehat{N})$  for each  $k = 1, 2, \dots, K$  and  $i = 1, 2, \dots, L$ )
2. The sequence  $\{T_n f_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  is a frame for  $\ell^2(G)$  if and only if  $0 < A_{\mathbf{H}} \leq B_{\mathbf{H}} < \infty$

where  $\mathbf{H}(\gamma)$  is the  $K \times L$  matrix (taking analysis filters  $h_k = \widetilde{f}_k$ )

$$\mathbf{H}(\gamma) = \left( \widehat{f}_{k,h_i}(\gamma) \right)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}, \quad \gamma \in \widehat{N}$$

and its associated constants

$$A_{\mathbf{H}} := \operatorname{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min} [\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)]; \quad B_{\mathbf{H}} := \operatorname{ess\,sup}_{\gamma \in \widehat{N}} \lambda_{\max} [\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)]$$

## Theorem

Let  $\{T_n \mathbf{f}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  and  $\{T_n \mathbf{g}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  be two **Bessel sequences** for  $\ell^2(G)$  with associated matrices  $\mathbf{H}(\gamma)$  and  $\mathbf{G}(\gamma)$ . Then,

- (a) The sequences  $\{T_n \mathbf{f}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  and  $\{T_n \mathbf{g}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  are **dual frames** for  $\ell^2(G)$  if and only if condition  $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$  a.e.  $\gamma \in \widehat{N}$  holds.
- (b) The sequences  $\{T_n \mathbf{f}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  and  $\{T_n \mathbf{g}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  are **dual Riesz bases** for  $\ell^2(G)$  if and only if  $K = L$  and  $\mathbf{G}(\gamma) = \mathbf{H}(\gamma)^{-1}$  a.e.  $\gamma \in \widehat{N}$

where the matrices  $\mathbf{H}(\gamma)$  and  $\mathbf{G}(\gamma)$  are:

(taking **analysis filters**  $h_k = \widetilde{\mathbf{f}}_k$  and **synthesis filters**  $\mathbf{g}_k$ )

$$\mathbf{H}(\gamma) = \left( \overline{\widehat{\mathbf{f}}_{k,h_i}(\gamma)} \right)_{\substack{k=1,2,\dots,K \\ i=1,2,\dots,L}}; \quad \mathbf{G}(\gamma) = \left( \widehat{\mathbf{g}}_{h_i,k}(\gamma) \right)_{\substack{i=1,2,\dots,L \\ k=1,2,\dots,K}}, \quad \gamma \in \widehat{N}$$

## Some final remarks

Whenever  $\mathbf{f}_k \in \ell^1(G)$ ,  $k = 1, 2, \dots, K$ , conditions are easier...

- ▶ Since  $\widehat{\mathbf{f}}_{k,h_i}$  are continuous on the compact  $\widehat{N}$ , the sequence  $\{T_n \mathbf{f}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  is always a Bessel sequence
- ▶ Frame condition to  $\{T_n \mathbf{f}_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  reduces to:

$$\text{rank } \mathbf{H}(\gamma) = L \quad \text{for all } \gamma \in \widehat{N}$$

or

$$\min_{\gamma \in \widehat{N}} |\det[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)]| > 0$$

# Getting on with sampling

For each  $x \in \mathcal{A}_a$  in the  $U$ -invariant subspace in  $\mathcal{H}$

$$\mathcal{A}_a = \left\{ \sum_{(n,h) \in G} \alpha(n,h) U(n,h)a : \{\alpha(n,h)\}_{(n,h) \in G} \in \ell^2(G) \right\}$$

we consider for  $k = 1, 2, \dots, K$  its **generalized samples** at  $m \in N$

$$\begin{aligned} \mathcal{L}_k x(m) &:= \langle x, U(m, 1_H) b_k \rangle_{\mathcal{H}} \\ &= \sum_{(n,h) \in G} \alpha(n,h) \langle a, U[(n,h)^{-1} \cdot (m, 1_H)] b_k \rangle \\ &= \downarrow_H (\alpha * \mathbf{h}_k)(m) = \langle \alpha, T_m \tilde{\mathbf{h}}_k \rangle_{\ell^2(G)}, \end{aligned}$$

where

$$\mathbf{h}_k(n,h) := \langle a, U(n,h) b_k \rangle_{\mathcal{H}}, \quad (n,h) \in G$$

belongs to  $\ell^2(G)$ ,  $k = 1, 2, \dots, K$



Suppose that there exists a **perfect reconstruction  $K$ -channel filter-bank** with **analysis filters** the above  $h_k$  and **synthesis filters**  $g_k$ ,  $k = 1, 2, \dots, K$ . Then, we have

$$\alpha = \sum_{k=1}^K \sum_{n \in N} \downarrow_H (\alpha * h_k)(n) T_n g_k = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) T_n g_k \quad \text{in } \ell^2(G).$$

We consider the natural isomorphism  $\mathcal{T}_{U,a} : \ell^2(G) \rightarrow \mathcal{A}_a$

$$\mathcal{T}_{U,a} : \delta_{(n,h)} \mapsto U(n,h)a \quad \text{for each } (n,h) \in G.$$

For each  $m \in N$ ,  $\mathcal{T}_{U,a}$  possesses the following **shifting property**

$$\mathcal{T}_{U,a}(T_m f) = U(m, 1_H)(\mathcal{T}_{U,a} f), \quad f \in \ell^2(G).$$

Applying the isomorphism  $\mathcal{T}_{U,a}$ , for each  $x \in \mathcal{A}_a$  we get the expansion

$$\begin{aligned}x &= \mathcal{T}_{U,a}\alpha = \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) \mathcal{T}_{U,a}(T_n \mathbf{g}_k) \\&= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H)(\mathcal{T}_{U,a} \mathbf{g}_k) \\&= \sum_{k=1}^K \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H) c_{k,\mathbf{g}} \quad \text{in } \mathcal{H},\end{aligned}$$

where  $c_{k,\mathbf{g}} = \mathcal{T}_{U,a} \mathbf{g}_k$ ,  $k = 1, 2, \dots, K$ .

Notice that the sequence  $\{U(n, 1_H) c_{k,\mathbf{g}}\}_{n \in N; k=1,2,\dots,K}$  is a frame for  $\mathcal{A}_a$  is a frame for  $\mathcal{A}_a$ .

In fact, the following result holds:

For  $h_k$  such that  $h_k(n, h) := \langle a, U(n, h) b_k \rangle_{\mathcal{H}}$ ,  $(n, h) \in G$ , consider the  $K \times L$  matrix

$$\mathbf{H}(\gamma) = \left( \overline{\widehat{f}_{k, h_i}(\gamma)} \right)_{\substack{k=1, 2, \dots, K \\ i=1, 2, \dots, L}}, \quad \gamma \in \widehat{N}$$

where  $f_k := \widetilde{h}_k$  and  $f_{k, h_i}(n) = f_k(n, h_i)$ ,  $k = 1, 2, \dots, K$ ,  $i = 1, 2, \dots, L$ . Consider also its associated constants

$$A_{\mathbf{H}} := \operatorname{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)]; \quad B_{\mathbf{H}} := \operatorname{ess\,sup}_{\gamma \in \widehat{N}} \lambda_{\max} [\mathbf{H}^*(\gamma) \mathbf{H}(\gamma)]$$

Assume that the matrix  $\mathbf{H}(\gamma)$  has all its entries in  $L^\infty(\widehat{N})$

# The sampling result

## Theorem

The following statements are equivalent:

1. The constant  $A_{\mathbf{H}} = \operatorname{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min}[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)] > 0$ .
2. There exist  $g_k$  in  $\ell^2(G)$ ,  $k = 1, 2, \dots, K$ , such that the associated polyphase matrix  $\mathbf{G}(\gamma)$  has all its entries in  $L^\infty(\widehat{N})$ , and it satisfies  $\mathbf{G}(\gamma)\mathbf{H}(\gamma) = \mathbf{I}_L$  a.e.  $\gamma \in \widehat{N}$ .
3. There exist  $K$  elements  $c_k \in \mathcal{A}_a$  such that the sequence  $\{U(n, 1_H)c_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  is a **frame** for  $\mathcal{A}_a$  and for each  $x \in \mathcal{A}_a$  we have the sampling formula

$$x = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n, 1_H) c_k \quad \text{in } \mathcal{H}$$

# The sampling result

## Theorem

3. There exist  $K$  elements  $c_k \in \mathcal{A}_a$  such that the sequence  $\{U(n, 1_H)c_k\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  is a frame for  $\mathcal{A}_a$  and for each  $x \in \mathcal{A}_a$  we have the sampling formula

$$x = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) U(n, 1_H)c_k \quad \text{in } \mathcal{H}$$

4. There exists a frame  $\{C_{k,n}\}_{n \in \mathbb{N}; k=1,2,\dots,K}$  for  $\mathcal{A}_a$  such that for each  $x \in \mathcal{A}_a$  we have the expansion

$$x = \sum_{k=1}^K \sum_{n \in \mathbb{N}} \mathcal{L}_k x(n) C_{k,n} \quad \text{in } \mathcal{H}$$

Notice that  $K \geq L$  where  $L$  is the order of the group  $H$ . In case  $K = L$ , we obtain:

## Corollary

In the case  $K = L$ , assume that the matrix  $\mathbf{H}(\gamma)$  has all entries in  $L^\infty(\widehat{N})$ . The following statements are equivalents:

1. The constant  $A_{\mathbf{H}} = \operatorname{ess\,inf}_{\gamma \in \widehat{N}} \lambda_{\min}[\mathbf{H}^*(\gamma)\mathbf{H}(\gamma)] > 0$ .
2. There exist  $L$  unique elements  $c_k, k = 1, 2, \dots, L$ , in  $\mathcal{A}_a$  such that the associated sequence  $\{U(n, 1_H)c_k\}_{n \in N; k=1,2,\dots,L}$  is a **Riesz basis** for  $\mathcal{A}_a$ , and for each  $x \in \mathcal{A}_a$  we have the sampling formula

$$x = \sum_{k=1}^L \sum_{n \in N} \mathcal{L}_k x(n) U(n, 1_H)c_k \quad \text{in } \mathcal{H}$$

Moreover, the **interpolation property**  $\mathcal{L}_k c_{k'}(n) = \delta_{k,k'} \delta_{n,0_N}$ , where  $n \in N$  and  $k, k' = 1, 2, \dots, L$ , holds.

## An interesting particular case

Denote the subgroup  $H$  as  $H = \{h_1 = 1_H, h_2, \dots, h_L\}$

Given a fixed  $b \in \mathcal{H}$ , for any  $x \in \mathcal{A}_a$  we consider its samples

$$\mathcal{L}_k x(m) := \langle x, U(m, h_k)b \rangle, \quad m \in N \text{ and } k = 1, 2, \dots, L$$

Since  $(m, h_k) = (m, 1_H)(0_N, h_k)$

$$U(m, h_k)b = U(m, 1_H) \underbrace{U(0_N, h_k)b}_{b_k} = U(m, 1_H)b_k, \quad k = 1, 2, \dots, L$$

Here  $K = L$ , and we are in the Riesz bases setting

In this case, the matrix  $\mathbf{H}(\gamma)$  is determined by its first column

# Some examples involving crystallographic groups



The **euclidean motion group**  $E(d) = \mathbb{R}^d \rtimes_{\phi} O(d)$ : for the homomorphism  $\phi : O(d) \rightarrow \text{Aut}(\mathbb{R}^d)$  given by  $\phi_A(x) = Ax$ , we have the composition law  $(x, A) \cdot (x', A') = (x + Ax', AA')$

We consider the **crystallographic group**  $\mathcal{C}_{M, \Gamma} := M\mathbb{Z}^d \rtimes_{\phi} \Gamma$  where  $M$  be a non-singular  $d \times d$  matrix and  $\Gamma$  a finite subgroup of  $O(d)$  of order  $L$  such that  $A(M\mathbb{Z}^d) = M\mathbb{Z}^d$  for each  $A \in \Gamma$  and its **quasi regular representation on  $L^2(\mathbb{R}^d)$** : for  $n \in M\mathbb{Z}^d$ ,  $A \in \Gamma$  and  $f \in L^2(\mathbb{R}^d)$

$$U(n, A)f(t) = f[A^{\top}(t - n)], \quad t \in \mathbb{R}^d$$

For a fixed  $\varphi \in L^2(\mathbb{R}^d)$  such that the sequence  $\{U(n, A)\varphi\}_{(n, A) \in \mathcal{C}_{M, \Gamma}}$  is a **Riesz sequence for  $L^2(\mathbb{R}^d)$**  we consider the  $U$ -invariant subspace in  $L^2(\mathbb{R}^d)$

$$\mathcal{A}_{\varphi} = \left\{ \sum_{(n, A) \in \mathcal{C}_{M, \Gamma}} \alpha(n, A) \varphi[A^{\top}(t - n)] : \{\alpha(n, A)\} \in \ell^2(\mathcal{C}_{M, \Gamma}) \right\}$$

## Average samples

Choosing  $K$  functions  $b_k \in L^2(\mathbb{R}^d)$  we consider the **average samples** of  $f \in \mathcal{A}_\varphi$

$$\mathcal{L}_k f(n) = \langle f, U(n, I)b_k \rangle = \langle f, b_k(\cdot - n) \rangle, \quad n \in M\mathbb{Z}^d.$$

Under the hypotheses in our sampling theorem, there exist  $K \geq L$  **sampling functions**  $\psi_k \in \mathcal{A}_\varphi$  for  $k = 1, 2, \dots, K$ , such that the sequence  $\{\psi_k(\cdot - n)\}_{n \in M\mathbb{Z}^d; k=1,2,\dots,K}$  is a frame for  $\mathcal{A}_\varphi$ , and we get the sampling expansion

$$f(t) = \sum_{k=1}^K \sum_{n \in M\mathbb{Z}^d} \langle f, b_k(\cdot - n) \rangle_{L^2(\mathbb{R}^d)} \psi_k(t - n) \quad \text{in } L^2(\mathbb{R}^d)$$

If the generator  $\varphi \in C(\mathbb{R}^d)$  and the function  $t \mapsto \sum_n |\varphi(t - n)|^2$  is bounded on  $\mathbb{R}^d$ , then  $\mathcal{A}_\varphi$  is a **reproducing kernel Hilbert space** (RKHS) of continuous functions in  $L^2(\mathbb{R}^d) \rightsquigarrow$  **pointwise convergence**

## Pointwise samples

$\{U(n, h)\}_{(n, h) \in G}$  a unitary representation of  $G = N \rtimes_{\phi} H$  on  $L^2(\mathbb{R}^d)$

If the generator  $\varphi \in L^2(\mathbb{R}^d)$  of  $\mathcal{A}_{\varphi}$  satisfies

- ▶ For each  $(n, h) \in G$ , the function  $U(n, h)\varphi$  is continuous on  $\mathbb{R}^d$
- ▶  $\sup_{t \in \mathbb{R}^d} \sum_{(n, h) \in G} |[U(n, h)\varphi](t)|^2 < +\infty$

Then the subspace  $\mathcal{A}_{\varphi} = \left\{ \sum_{(n, h) \in G} \alpha(n, h) U(n, h) \varphi \right\}$  is a RKHS of bounded continuous functions in  $L^2(\mathbb{R}^d)$ .

For  $K$  fixed points  $t_k \in \mathbb{R}^d$ ,  $k = 1, 2, \dots, K$ , we consider for each  $f \in \mathcal{A}_{\varphi}$  the new samples given by

$$\mathcal{L}_k f(n) := [U(-n, 1_H)f](t_k), \quad n \in N \text{ and } k = 1, 2, \dots, K.$$

For any  $f = \sum_{(m,h) \in G} \alpha(m,h) U(m,h) \varphi$  in  $\mathcal{A}_\varphi$  we have

$$\begin{aligned} \mathcal{L}_k f(n) &= \left[ \sum_{(m,h) \in G} \alpha(m,h) U[(-n, 1_H) \cdot (m,h)] \varphi \right] (t_k) \\ &= \sum_{(m,h) \in G} \alpha(m,h) [U(m-n, h) \varphi] (t_k) = \langle \alpha, T_n \mathbf{f}_k \rangle_{\ell^2(G)}, \quad n \in N \end{aligned}$$

where  $\mathbf{f}_k(m,h) := \overline{[U(m,h)\varphi](t_k)}$ ,  $(m,h) \in G$ , belongs to  $\ell^2(G)$ ,  $k = 1, 2, \dots, K$ . Under the hypotheses in sampling theorem (on the new  $\mathbf{h}_k := \tilde{\mathbf{f}}_k \in \ell^2(G)$ ,  $k = 1, 2, \dots, K$ ) we will get a sampling formula for the new data sequence  $\{\mathcal{L}_k f(n)\}_{n \in N; k=1,2,\dots,K}$

In the particular case of the quasi regular representation of a crystallographic group  $\mathcal{C}_{M,\Gamma} = M\mathbb{Z}^d \rtimes_{\phi} \Gamma$ , for each  $f \in \mathcal{A}_{\varphi}$  these new samples read

$$\mathcal{L}_k f(n) = [U(-n, I)f](t_k) = f(t_k + n), \quad n \in M\mathbb{Z}^d; \quad k = 1, 2, \dots, K$$

Thus, under the hypotheses in our sampling theorem, there exist  $K$  functions  $\psi_k \in \mathcal{A}_{\varphi}$ ,  $k = 1, 2, \dots, K$ , such that for each  $f \in \mathcal{A}_{\varphi}$  the sampling formula

$$f(t) = \sum_{k=1}^K \sum_{n \in M\mathbb{Z}^d} f(t_k + n) \psi_k(t - n), \quad t \in \mathbb{R}^d$$

holds. The convergence of the series in the  $L^2(\mathbb{R}^d)$ -norm sense implies pointwise convergence which is absolute and uniform on  $\mathbb{R}^d$

That's all!