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Bidiagonal decompositions of oscillating systems of vectors ☆

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Abstract

We establish necessary and sufficient conditions, in the language of bidiagonal decompositions, for a matrix V to be an eigenvector matrix of a totally positive matrix. Namely, this is the case if and only if V and V^{-T} are lowerly totally positive.

These conditions translate into easy positivity requirements on the parameters in the bidiagonal decompositions of V and V^{-T}. Using these decompositions we give elementary proofs of the oscillating properties of V. In particular, the fact that the jth column of V has $j-1$ changes of sign.

Our new results include the fact that the Q matrix in a QR decomposition of a totally positive matrix belongs to the above class (and thus has the same oscillating properties).

Keywords: Totally positive matrix; Eigenvectors; Variation diminishing property

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1. Introduction

The matrices with all minors positive are called *totally positive* (TP). We consider the class of matrices that are eigenvector matrices of TP matrices. We denote this class as ETP. The TP matrices are diagonalizable and their eigenvalues are positive and distinct. For convenience we assume that the columns of each ETP matrix are permuted so that the jth column is an eigenvector corresponding to the jth largest eigenvalue. Furthermore, since we are concerned with eigenvector matrices, all matrices in this paper are assumed to be $n \times n$ square matrices.

The utilization of bidiagonal decompositions as means of studying the properties of TP and related matrices has been particularly prominent in recent works [3,4,6,8,9]. Here we take the same approach with the ETP matrices. We establish a classification of the ETP matrices in the language of bidiagonal decompositions. In particular, we prove that a matrix V is ETP if and only if the multipliers needed to eliminate the lower triangular parts of V and V^{-T} in the process of Neville elimination are positive (see Sections 2 and 3 for the formal definitions of these notions).

Using bidiagonal decompositions we give new elementary proofs of the oscillating properties of ETP matrices. In particular, the fact that the jth column of any ETP matrix has exactly $j-1$ changes of sign.

The above characterization of ETP matrices shows that any orthogonal matrix Q in the QR decomposition of a TP matrix is an eigenvector matrix of some symmetric TP matrix. In particular, the jth column of Q has exactly $j-1$ changes of sign.

The paper is organized as follows. In Section 2 we survey the bidiagonal decompositions of TP matrices. We characterize the ETP matrices in terms of their bidiagonal decompositions in Section 3. We give elementary proofs of the oscillating properties of TP and ETP matrices in Sections 4 and 5. Finally, we include two technical proofs in an Appendix.

Notes on notation:

- The matrices with all minors nonnegative are called *totally nonnegative* (TN). The TN matrices some of whose power is TP are called *oscillatory* [5]. Since we are primarily concerned with the oscillating properties of the eigenvectors of TP matrices, we call the ETP matrices "oscillating systems of vectors" in the title in order to avoid possible confusion with the notion of oscillatory matrices.
- We use conventional notation for submatrices: $A(i_1, i_2, \ldots, i_p|j_1, j_2, \ldots, j_q)$ is the submatrix of A consisting of rows $i_1, i_2, \ldots, i_p$ and columns $j_1, j_2, \ldots, j_q$.

2. Background

The idea of using bidiagonal decompositions in the study of TP matrices has been very successful recently. These decompositions are based on a simple idea of eliminating a matrix using only adjacent rows and columns, which can be traced back to Whitney [12]. We review the main facts here and refer the reader to [3,4,8,9] for details.

The bidiagonal decomposition of a matrix A is obtained by eliminating it (in a process called *Neville elimination*) using adjacent rows and columns. Each (row) elimination step is equivalent to factoring out (on the left) a matrix

$$E_i(x) \equiv \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & x\,1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix},$$

which differs from the identity only in the $(i, i-1)$ entry.

The lower triangular part of a matrix (call it A) is eliminated one subdiagonal at a time, starting with the $(n, 1)$ entry. Once the matrix is reduced to an upper triangular form, the same process is applied by columns resulting in the decomposition:

$$A = \left(\prod_{i=1}^{n-1} \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}) \right) D \left(\prod_{1}^{i=n-1} \prod_{n-i+1}^{j=n} E_j^T(b_{i+j-n,j}) \right), \tag{1}$$

where $D = \text{diag}(b_{11}, b_{22}, \ldots, b_{nn})$. In the notation of (1) and throughout this paper, $\prod_1^{i=n-1}$ indicates that the product is taken for i from $n-1$ down to 1. Although somewhat nonstandard, this notation allows us to preserve the symmetry in (1).

According to a result of Gasca and Peña [8, Theorem 4.3], A is TP if and only if (1) exists, it is unique, and $b_{ij} > 0$, $i, j = 1, 2, \ldots, n$.

If we group the factors in the parentheses of (1) into factors L and U then (1) becomes $A = LDU$ – the LDU decomposition of A (which could also be obtained from Gaussian elimination with no pivoting).

The factors L and U in the LDU decomposition of a TP matrix inherit the total positivity properties with respect to their nontrivial minors [6]. This leads us to the following definition.

Definition 1 (*LTP matrix*). A matrix is called lowerly totally positive (denoted as LTP) if it is nonsingular, its LDU decomposition exists, and all nontrivial minors of the L factor in that decomposition are positive (i.e., all minors of the form $\det(L(i_1, i_2, \ldots, i_p | j_1, j_2, \ldots, j_p))$ such that $i_1 \geqslant j_1, i_2 \geqslant j_2, \ldots, i_p \geqslant j_p$).

In other words, a nonsingular matrix $A = LDU$ is LTP if and only if all multipliers b_{ij} for $i > j$ in the decomposition (1) of L are positive. The D factor can have any (nonzero) entries on the diagonal.

This definition differs from the one used by Cryer [2] and Gasca and Peña [7] in that we do not impose any positivity restrictions on the D factor of the LDU decomposition (other than to be nonsingular). The intuition is that the LTP structure of a matrix is unaffected by right diagonal scaling, just like the ETP structure of a matrix is unaffected by such diagonal scaling.

For the LDU decomposition $A = LDU$ of a TP matrix A, we have that L and U^T are LTP and D has a positive diagonal [6].

3. DLTP matrices

Definition 2. We call a matrix V *doubly lowerly totally positive* (denoted as DLTP) if V and V^{-T} are LTP.

In this section we prove that a matrix V is ETP if and only if it is DLTP.

The fact that any ETP matrix is LTP stems from the following result.

Theorem 1 [1]. *Any TP matrix A with eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ is similar to the bidiagonal matrix*

$$B = \begin{bmatrix} \lambda_1 & \lambda_1 - \lambda_2 & & & \\ & \lambda_2 & \lambda_1 - \lambda_3 & & \\ & & \ddots & \ddots & \\ & & & \lambda_{n-1} & \lambda_1 - \lambda_n \\ & & & & \lambda_n \end{bmatrix}$$

via a similarity transformation matrix S that is TN and LTP.

The eigenvector matrix of A is thus LTP as a product of S, which is LTP, and an eigenvector matrix of B, which is upper triangular.

The result of Carnicer and Peña [1] differs slightly from the claim of Theorem 1: the authors prove that S is TN under the weaker assumption that A is oscillatory. A careful examination of their construction reveals that when A is TP, S is also LTP. We thus attribute Theorem 1 to them and defer its elaborate, but straightforward proof to the Appendix.

The following lemma establishes that for a DLTP matrix, the D factors in the LDU decompositions of A and A^{-T} have the same sign patterns. This property is critical in the proof of Theorem 2 below.

Lemma 1. *Let A be DLTP and let $A = LDU$ and $A^{-T} = \bar{L}\bar{D}\bar{U}$ be the LDU decompositions of A and A^{-T}, respectively. Then $D_{ii}\bar{D}_{ii} > 0$ for $i = 1, 2, \ldots, n$.*

Proof. We have $D^{-1}U^{-T} = L^T A^{-T} = (L^T\bar{L}) \cdot (\bar{D}\bar{U}) = L'D'U'\bar{D}\bar{U}$, where $L'D'U'$ is the LDU decomposition of the TP matrix $L^T\bar{L}$. Therefore

$$(D^{-1}U^{-T}D) \cdot D^{-1} \cdot I = L' \cdot (D'\bar{D}) \cdot (\bar{D}^{-1}U'\bar{D}\bar{U}). \tag{2}$$

Since both sides of (2) are LDU decompositions of the same matrix, the corresponding factors must be equal. In particular, $D^{-1} = D'\bar{D}$ and the result follows. □

Theorem 2. *A matrix V is DLTP if and only if it is ETP.*

Proof. If V is the eigenvector matrix of a TP matrix (say) A, then V is LTP according to Theorem 1. On the other side V^{-T}, as an eigenvector matrix of the TP matrix A^T, is also LTP. Thus V is DLTP.

Conversely, if V is DLTP, we follow idea of Gantmacher and Krein [5, Theorem 19, p. 272].

Let $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where the λ_i are such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, but otherwise arbitrary. We will prove that some (high enough) power of $A = V\Lambda V^{-1}$ is TP.

Let $F = A^m = V\Lambda^m V^{-1}$. From the Cauchy–Binet identity we have for any minor of F:

$$\det(F(i_1, \ldots, i_p|k_1, \ldots, k_p)) = \sum_{1\leqslant\alpha_1<\cdots<\alpha_p\leqslant n} \lambda_{\alpha_1}^m \cdots \lambda_{\alpha_p}^m$$
$$\times \det(V(i_1, \ldots, i_p|\alpha_1, \ldots, \alpha_p)) \cdot \det(V^{-1}(\alpha_1, \ldots, \alpha_p|k_1, \ldots, k_p)), \quad (3)$$

where $1 \leqslant i_1 < \cdots < i_p \leqslant n$ and $1 \leqslant k_1 < \cdots < k_p \leqslant n$. For large enough m, the sign of this minor is dominated by its leading term,

$$\lambda_1^m \cdots \lambda_p^m \cdot \det(V(i_1, \ldots, i_p|1, \ldots, p)) \cdot \det(V^{-T}(k_1, \ldots, k_p|1, \ldots, p)), \quad (4)$$

which is positive (since V and V^{-T} are LTP and the D factors in the LDU decompositions of V and V^{-T} have the same sign pattern on the diagonal—see Lemma 1). □

The following theorem describes the properties of the upper triangular factor of a DLTP matrix.

Theorem 3. *If $A = LDU$ is DLTP and D has a positive diagonal, then U^{-T} is LTP.*

Proof. With the notation as in Lemma 1, $D^{-1}U^{-T} = L'D'U'\bar{D}\bar{U}$. By comparing the lower triangular factors we get $D^{-1}U^{-T}D = L'$ is LTP. Since D has positive diagonal, the result follows. □

Note: A matrix $A = LDU$ such that L and U^{-T} are LTP and D has a positive diagonal is called a *γ-matrix* [7]. The γ-matrix property is necessary, but not sufficient to characterize the eigenvector matrices or the Q factors of TP matrices. In addition, the γ-matrix property requires the eigenvector matrices and the Q factors to be appropriately scaled.

Corollary 1. *An orthogonal matrix Q is LTP if and only if it is a Q factor in a QR decomposition of a TP matrix. In this case Q is DLTP.*

Proof. If $A = QR$ and R is upper triangular, then A and Q share the L factor in their respective LDU decompositions. Therefore, if A is TP, then Q is LTP. Conversely, if $Q = LDU$ is orthogonal and LTP, then Q is the Q factor in the QR decomposition of any TP matrix $L\bar{D}\bar{U}$, where $\bar{D}$ is diagonal with positive entries on the diagonal and $\bar{U}$ is unit upper triangular and *upperly totally positive* (i.e., $\bar{U}^T$ is LTP).

Since Q is orthogonal, $Q^{-T} = Q$. Thus Q is DLTP. □

In particular, Q has the same oscillating properties as the eigenvector matrices of TP matrices and the number of sign changes in the jth column of Q is exactly $j - 1$ (see Section 5).

Theorem 2 implies the following about the structure of orthogonal LTP matrices.

Corollary 2. *Every LTP orthogonal matrix Q is an eigenvector matrix of some symmetric TP matrix. Therefore, if we choose the first nonzero entry in each eigenvector to be positive, the orthogonal eigenvector matrices Q of the symmetric TP matrices are parameterized by the $n(n-1)/2$ positive multipliers $b_{ij} > 0, i > j$, in the bidiagonal decomposition of Q.*

We finish this section with a characterization of those matrices A (not necessarily TP or with positive entries) that have real eigenvalues and DLTP eigenvector matrices. As explained in Section 5 this property implies that the eigenvectors have oscillating properties.

Corollary 3. *Let A be a real matrix with real distinct eigenvalues ordered as $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$, and V be any eigenvector matrix of A corresponding to this ordering. Then V is DLTP if and only if there exists a positive integer m_0 such that A^{2m} is TP for $m \geqslant m_0$.*

Proof. We follow the arguments in (3) and (4) with A^{2m}. $\square$

4. Variation-diminishing properties of TP matrices

The bidiagonal decompositions of nonsingular TN matrices can be used to obtain easy proofs of certain well known (see, e.g., [5]) variation-diminishing properties for these matrices. In particular, a multiplication by a nonsingular TN matrix does not increase the number of sign changes in a vector. This result is mentioned without proof in [3, Section 4.4], and we are unaware of a proof (using bidiagonal decompositions) anywhere else. For completeness, we include one here.

First, we recall the definition of number of sign changes in a vector.

Definition 3 (*Number of sign changes* [5, p. 86]). Let $u_1, u_2, \ldots, u_n$ be a sequence of numbers. If some of the numbers are zero, then we can assign them arbitrary signs. The *number of sign changes* of the above sequence is then the number of instances for $i = 1, 2, \ldots, n-1$ where u_i and u_{i+1} have different signs. Depending on our choice of signs of the zero components we can have a different number of sign changes. We define S_u^- and S_u^+ to be the minimum and maximum number of sign changes among all possible choices of signs of the zero components.

If $S_u^- = S_u^+$, we say that the (exact) number of sign changes of the vector u is $S_u = S_u^- = S_u^+$.

Next, we establish that the simplest nontrivial TN matrix, $E_i(x)$, $x > 0$, does not diminish the maximum number of sign changes in a vector.

Lemma 2. *Let u be an n-vector and $w = E_i(x) \cdot u$ or $w = E_i(x)^T \cdot u$, where $x > 0$. Then $S_w^+ \leqslant S_u^+$.*

Proof. We first prove the case $w = E_i(x) \cdot u$. We have $w_j = u_j$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, and $w_i = u_i + x u_{i-1}$.

We assume that $u_{i-1} \neq 0$, otherwise $w = u$ and the claim is trivially true. Without loss of generality we can also assume that $u_{i-1} > 0$, since $S_{-u}^+ = S_u^+$.

Consider first the case $u_i \geqslant 0$. Then we have $w_i > 0$. We assign the same signs to the entries of u as the corresponding signs assigned to the entries of w in the computation of S_w^+. (We can always do this. We have $u_j = w_j$, $j \neq i$. The ith entry of w, $w_i > 0$, is counted with a positive sign in S_w^+. We thus assign the same (positive) sign to $u_i \geqslant 0$.) With signs thus assigned u has S_w^+ sign changes. Therefore $S_w^+ \leqslant S_u^+$.

Alternatively, we consider the case $u_i < 0$. We can assume that w_i is counted in S_w^+ with a positive sign (otherwise $S_u^+ = S_w^+$). For $j = 1, 2, \ldots, i-1, i+1, \ldots, n$, we assign to u_j the same sign as the one assigned to w_j in the computation of S_w^+. Denote by S the thus obtained number of sign changes in u. We claim that $S_w^+ \leqslant S \leqslant S_u^+$. The second inequality is obvious. To see that the first one is true, we consider the sequences

$$\operatorname{sign}(u_{i-1}, u_i, u_{i+1}) = (+, -, *) \quad \text{and} \quad \operatorname{sign}(w_{i-1}, w_i, w_{i+1}) = (+, +, *),$$

where the "$*$" stands for the *same* sign ("+" or "−") assigned to both $u_{i+1} = w_{i+1}$. If $\operatorname{sign}(u_{i+1}) = \operatorname{sign}(w_{i+1}) = +$, then $S_w^+ + 2 = S$. If $\operatorname{sign}(u_{i+1}) = \operatorname{sign}(w_{i+1}) = -$, then $S_w^+ = S$.

We are done.

The case $w = E_i(x)^T \cdot u$ is analogous. □

By applying the above lemma repeatedly, we obtain the following corollary.

Corollary 4 (*TN variation diminishing property*). *For any real vector u and a nonsingular TN matrix A*, $S_{Au}^+ \leqslant S_u^+$ *and* $S_{Au}^- \leqslant S_u^-$.

Proof. According to Theorem 4.2 in [8], A can be uniquely factored in the form (1) with $b_{ii} > 0, i = 1, 2, \ldots, n$, and $b_{ij} \geqslant 0, i \neq j$.[3] Then the first assertion follows by repeatedly applying Lemma 2.

To prove the second, let $J = \operatorname{diag}(-1, 1, -1, \ldots, (-1)^n)$. If $w = Au$ then $Ju = BJw$, where $B = JA^{-1}J$ is the *re-signed inverse* of A, which is also nonsingular and TN (see, e.g., [5, Proposition 5°, p. 75]).

From Lemma 2, $S_{Ju}^+ = S_{BJw}^+ \leqslant S_{Jw}^+$. Since for any n-vector v, $S_v^+ + S_{Jv}^- = n - 1$, we have $S_u^- \geqslant S_w^- = S_{Au}^-$. □

When A is TP a much stronger result is true: $S_{Au}^+ \leqslant S_u^-$. To prove it, we need the following lemma.

Lemma 3. *If A is TP*, *then for every i*, $2 \leqslant i \leqslant n$, *there exist TP matrices B and C*, *and positive numbers x and y*, *such that*

$$A = B \cdot E_i(x) \quad \textit{and} \quad A = C \cdot E_i^T(y).$$

Proof. This lemma is nearly obvious if we use a limiting argument: From Cauchy–Binet, any minor of $B = A \cdot E_i(-x)$ is a linear function of x, say $ax + b$, where $b > 0$ is the value of the same minor in A. Clearly, for small enough x all minors of B are positive. The second claim follows analogously. □

However, in the spirit of the rest of this paper, in the Appendix we give a second, constructive proof of Lemma 3 based on bidiagonal decompositions, by providing an explicit way to factor an $E_i(x)$ out of A.

Theorem 4 (*TP variation diminishing property*). *If A is TP and u is a nonzero n-vector*, *then* $S_{Au}^+ \leqslant S_u^-$.

Proof. One way to compute S_u^- is to count any zero (say u_i) in u with the same sign as the sign of the first nonzero component in u following u_i. The trailing zero components of u can be counted with the same sign as the last nonzero in u.

[3] Additionally, $b_{ij}, i \neq j$, must satisfy $b_{ij} = 0$ if $b_{i-1,j} = 0$ and $i > j$, and $b_{ij} = 0$ if $b_{i,j-1} = 0$ and $i < j$. However, these conditions are unimportant here.

The main idea here is to factor appropriate $E_i(x)$ and $E_i^T(y)$ out of A and factor them into u to make the zero components of u nonzeros with the same signs as in the above computation of S_u^-.

The construction is straightforward.

If $u_{i-1} = 0$ but $u_i \neq 0$, we use Lemma 3 to write $A = F \cdot E_i^T(y)$ for some $y > 0$, where F is TP. If $v = E_i^T(y)u$, then $v_{i-1} = yu_i$ (the magnitude of y is unimportant; its only purpose is to make v_{i-1} have the same sign as u_i). We continue this process until we obtain a vector (call it p) which can only have zero components at the end, say $p_k = \cdots = p_n = 0$, but $p_{k-1} \neq 0$. By factoring out $E_j(x_j)$, $j = k, k+1, \ldots, n$ and factoring them into p we obtain a new vector (call it q) which has no zero components and $S_q = S_q^+ = S_u^-$. If $A = BC$ where $q = Cu$, then $w = Au = BCu = Bq$. By applying Corollary 4 we obtain $S_w^+ \leqslant S_q^+ = S_u^-$. □

5. Oscillating properties of ETP matrices

In this section we use the bidiagonal decompositions of ETP matrices to establish their oscillating properties.

Theorem 5. *Let the $n \times n$ matrix A be LTP and let*

$$u = c_1 a_1 + c_2 a_2 + \cdots + c_k a_k,$$

where $c_1, c_2, \ldots, c_k$ are arbitrary real constants such that $c_k \neq 0$, and $a_1, a_2, \ldots, a_n$ are the columns of A. Then

$$S_u^+ \leqslant k - 1.$$

Proof. We have $u = Ac$, where $c = (c_1, c_2, \ldots, c_k, 0, \ldots, 0)^T$. Let $A = LDU$ be the LDU decomposition of A, since L is LTP and unit lower triangular, from (1), we get

$$L = \prod_{i=1}^{n-1} \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}). \tag{5}$$

The matrices $E_i(x)$ satisfy $E_i(x)E_j(y) = E_j(y)E_i(x)$, unless $|i - j| = 1$. Thus we can re-order the factors in (5) to obtain for A:

$$A = \left(\prod_{i=1}^{n-1} \prod_{i+1}^{j=n} E_j(b_{ji}) \right) \cdot D \cdot U \equiv LDU. \tag{6}$$

We can think of (6) as having been obtained by performing Neville elimination one *column* at a time. Clearly the same multipliers would be used.

Let $v = DUc$. Since $c_k \neq 0$ and $c_i = 0$ for $i > k$, we have

$$v = (v_1, v_2, \ldots, v_k, 0, 0, \ldots, 0),$$

where $v_k \neq 0$ and $v_i = 0$ for $i > k$. In turn, the condition $v_i = 0$ for $i > k$ implies $E_i(x) \cdot v = v$ for $i > k + 1$. Thus

$$u = Ac = Lv = \left(\prod_{i=1}^{n-1} \prod_{i+1}^{j=n} E_j(b_{ji}) \right) \cdot v = \left(\prod_{i=1}^{k} \prod_{i+1}^{j=n} E_j(b_{ji}) \right) \cdot v.$$

Let

$$w \equiv \left(\prod_{k+1}^{j=n} E_j(b_{jk})\right) \cdot v.$$

We have $w_i = v_i$ for $i = 1, 2, \ldots, k$ and $w_i = b_{ik} b_{i-1,k} \cdots b_{k+1,k} v_k \neq 0$ for $i > k$. Since $b_{ij} > 0$ for $i > j$, the entries $w_k, w_{k+1}, \ldots, w_n$ have the same sign (and are nonzero). Therefore $S_w^+ \leqslant k - 1$.

Now

$$u = \left(\prod_{i=1}^{k-1} \prod_{i+1}^{j=n} E_j(b_{ji})\right) \cdot w$$

and Lemma 2 implies $S_u^+ \leqslant S_w^+ = k - 1$. □

For our next result we recall the notions of a converse matrix and that of a re-resigned matrix. If $A = [a_{ij}]_{i,j=1}^n$, then its *converse*

$$A^\# \equiv [a_{n-i+1,n-j+1}]_{i,j=1}^n$$

is obtained by reversing the rows and columns of A. Also, we *re-sign* a matrix by reversing the signs of the entries in a checkerboard pattern:

$$A^* \equiv [(-1)^{i+j} a_{ij}]_{i,j=1}^n.$$

The matrices $A^\#$ and A^* are similar to A and one can trivially verify that A^{-T} is LTP if and only if $A^{*\#}$ is LTP (using, e.g., Cauchy–Binet).

Theorem 6. *Let the matrix A be DLTP and let a_j, $j = 1, 2, \ldots, n$, be its columns. If $c_i, c_{i+1}, \ldots, c_j$, $\left(\sum_{k=i}^{j} c_k^2 \neq 0\right)$ is a sequence of real numbers, then the number of sign changes of*

$$u = c_i a_i + c_{i+1} a_{i+1} + \cdots + c_j a_j$$

lies between $i - 1$ and $j - 1$, i.e.,

$$i - 1 \leqslant S_u^- \leqslant S_u^+ \leqslant j - 1.$$

In particular, the number of sign changes in a_j is exactly $j - 1$.

Proof. Since A is LTP, Theorem 5 implies that $S_u^+ \leqslant j - 1$.

Now consider the matrix $A^{*\#}$. Let its columns be $\bar{a}_1, \ldots, \bar{a}_n$. Since $A^{*\#}$ is LTP, Theorem 5 implies that the maximum number of sign changes of the vector

$$w = c_j \bar{a}_{n-j+1} - c_{j-1} \bar{a}_{n-j+2} + \cdots + (-1)^{i-j} c_i \bar{a}_{n-i+1}$$

does not exceed $n - i$, i.e., $S_w^+ \leqslant n - i$. From $S_u^- + S_w^+ = n - 1$ we have $S_u^- \geqslant i - 1$. □

Theorem 6 allows us to also prove that the nodes of two successive eigenvectors alternate (see, e.g., [5, p. 90]), which is another fundamental oscillating property of ETP matrices. Additionally, Corollary 1 implies that the columns of the Q factor of a TP matrix also have this property.

Appendix A

In this appendix we prove Theorem 1 and provide an alternative, constructive proof of Lemma 3. The proofs are elaborate, but straightforward manipulations of the underlying bidiagonal decompositions.

Before we continue, we extend our notation from (1).

The matrices

$$L^{(i)} \equiv \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}) = \begin{bmatrix} 1 & & & & & \\ & \ddots & & & & \\ & b_{n-i+1,1} & 1 & & & \\ & & b_{n-i+2,2} & 1 & & \\ & & & \ddots & \ddots & \\ & & & & b_{n,i} & 1 \end{bmatrix} \tag{A.1}$$

and

$$U^{(i)} \equiv \prod_{n-i+1}^{j=n} E_j^T(b_{i+j-n,j}) = \begin{bmatrix} 1 & & & & & \\ & \ddots & b_{1,n-i+1} & & & \\ & & 1 & b_{2,n-i+2} & & \\ & & & 1 & \ddots & \\ & & & & \ddots & b_{i,n} \\ & & & & & 1 \end{bmatrix}$$

are $n \times n$ lower and $n \times n$ upper bidiagonal, respectively. The decomposition (1) now becomes

$$A = L^{(1)} \cdots L^{(n-1)} \cdot D \cdot U^{(n-1)} \cdots U^{(1)}.$$

Proof of Theorem 1. We follow the construction of Carnicer and Peña [1].[4]

In the first part of the proof the authors construct unit lower triangular TN matrices R and P such that $T = (RP^{\#})^{-1} \cdot A \cdot (RP^{\#})$ is tridiagonal.

The matrix R is constructed as a product of $E_i(b_{ij})$ with exactly one such factor for every entry in the lower triangular part of A that is set to zero. The order in which the zeros are created is the same as that of Neville elimination applied by columns, thus

$$R = \prod_{i=1}^{n-2} \prod_{i+2}^{j=n} E_j(b_{ji}).$$

The matrices $E_i(x)$ satisfy $E_i(x)E_j(y) = E_j(y)E_i(x)$, unless $|i - j| = 1$, thus we can re-order the factors of R to get

$$R = \prod_{i=1}^{n-2} \prod_{j=n-i+1}^{n} E_j(b_{j,i+j-n}).$$

Using the notation of (A.1) we have

$$R = L^{(1)} L^{(2)} \cdots L^{(n-2)},$$

4 Note that the authors of [1] call the TN matrices totally positive and the TP matrices strictly totally positive.

where all nontrivial entries b_{ij}, $i > j$, in $L^{(1)}, L^{(2)}, \dots, L^{(n-2)}$ are nonzero. Similarly we write

$$P^{\#} = U^{(n-2)}U^{(n-1)} \cdots U^{(1)}$$

and thus

$$V \equiv RP^{\#} = L^{(1)}L^{(2)} \cdots L^{(n-2)}U^{(n-2)}U^{(n-1)} \cdots U^{(1)}.$$

We are clearly missing the last factor $L^{(n-1)}$ in order to claim that the transformation matrix is LTP. This factor will come from the step of bidiagonalization of T.

The construction from here on is involved but the idea is simple. The bidiagonalization of T consists of $n-1$ steps of updating V by multiplying on the right by matrices of the type $D^{(k)} \cdot E_n(1)E_{n-1}(1) \cdots E_k(1)$ for $k = 2, \dots, n$, where the matrices $D^{(k)}$ are diagonal with positive entries. Thus the first step is

$$\begin{aligned} V_1 &= V \cdot D^{(2)} \cdot E_n(1)E_{n-1}(1) \cdots E_2(1) \\ &= RP^{\#} \cdot D^{(2)} \cdot E_n(1)E_{n-1}(1) \cdots E_2(1). \end{aligned}$$

Using the techniques of [10, Section 4.2], we can propagate each of the factors $D^{(2)}$, $E_n(1), E_{n-1}(1), \dots, E_3(1)$ into the decomposition $RP^{\#}$ changing the bidiagonal factors accordingly, but not affecting their nonzero structure. The last factor, $E_2(1)$, we only propagate up to the left of the diagonal factor, obtaining

$$V_1 = L_1^{(1)}L_1^{(2)} \cdots L_1^{(n-2)}E_2(x_1)D_1U_1^{(n-2)}U_1^{(n-3)} \cdots U_1^{(1)}.$$

After $n-1$ such steps we have

$$\begin{aligned} V_{n-1} = {} & L_{n-1}^{(1)}L_{n-1}^{(2)} \cdots L_{n-1}^{(n-2)} \cdot (E_2(x_1)E_3(x_2) \cdots E_n(x_{n-1})) \\ & \times D_{n-1}U_{n-1}^{(n-2)}U_{n-1}^{(n-3)} \cdots U_{n-1}^{(1)}. \end{aligned}$$

Setting $L_{n-1}^{(n-1)} = E_2(x_1)E_3(x_2) \cdots E_n(x_{n-1})$ we obtain the decomposition

$$V_{n-1} = L_{n-1}^{(1)}L_{n-1}^{(2)} \cdots L_{n-1}^{(n-1)}D_{n-1}U_{n-1}^{(n-2)}U_{n-1}^{(n-3)} \cdots U_{n-1}^{(1)}.$$

The similarity transformation matrix is thus TN and LTP (but not TP), since $x_i > 0$, for $i = 1, \dots, n-1$, according to [10, Section 4.2]. □

Constructive Proof of Lemma 3. Let $A = UDL$ be the UDL decomposition of A. It suffices to prove that we can factor L as $L = \bar{L} \cdot E_i(x)$ for some $x > 0$, where $\bar{L}$ is again unit lower triangular and LTP.[5]

We will use the reverse process of the one used in the last part of the proof of Theorem 4.3 in [10]; these transformations ((4.5)–(4.7) in [10]) produced the bidiagonal decomposition of the product of a lower triangular matrix and a matrix $E_j(x)$. We are now in the reverse situation – factoring an $E_j(x)$ out of a lower triangular LTP matrix.

We start with the bidiagonal decomposition of L

$$L = L^{(1)}L^{(2)} \cdots L^{(n-1)}.$$

[5] The UDL decomposition of A and the bidiagonal decompositions of the factors L, D, and U in that decomposition can be easily obtained from the bidiagonal decomposition of A as follows. If the LDU decomposition of $A^{\#}$ is $A^{\#} = \widetilde{L}\widetilde{D}\widetilde{U}$, then $U = \widetilde{L}^{\#}$, $D = \widetilde{D}^{\#}$, $L = \widetilde{U}^{\#}$. The bidiagonal decompositions of converse matrices are easily obtained as described in [11, Section 5.4].

Next, we split the $(n, n-1)$ entry in $L^{(i-1)}$ in two and factor $L^{(i-1)}$ accordingly. Let $x_1 = \frac{1}{2}(L^{(i-1)})_{n,n-1}$. Then $L^{(i-1)} = \bar{L}^{(i-1)} E_n(x_1)$, where $\bar{L}^{(i-1)}$ equals $L^{(i-1)}$ with the exception of its $(n, n-1)$ entry which is $(\bar{L}^{(i-1)})_{n,n-1} = x_1$. We move the "bulge" $E_n(x_1)$ all the way to the right

$$
\begin{aligned}
L &= L^{(1)}L^{(2)} \cdots \underline{L^{(i-1)}} \cdot L^{(i)} \cdot L^{(i+1)} \cdots L^{(n-1)} \\
&= L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \underline{E_n(x_1) \cdot L^{(i)}} \cdot L^{(i+1)} \cdots L^{(n-1)} \\
&= L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \underline{E_{n-1}(x_2) \cdot L^{(i+1)}} \cdots L^{(n-1)} \\
&= L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}_{(i)} \cdot \bar{L}_{(i+1)} \cdot \underline{E_{n-2}(x_3) \cdots} L^{(n-1)} \\
&= \ldots \\
&= L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdots \underline{E_{i+1}(x_{n-i}) L^{(n-1)}} \\
&= L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdots \bar{L}^{(n-1)} E_i(x_{n-i+1}) \\
&= \bar{L} \cdot E_i(x_{n-i+1})
\end{aligned}
\tag{A.2}
$$

(the matrices that are transformed on each step are underlined), where

$$\bar{L} \equiv L^{(1)}L^{(2)} \cdots \bar{L}^{(i-1)} \cdot \bar{L}^{(i)} \cdot \bar{L}^{(i+1)} \cdots \bar{L}^{(n-1)}.$$

Each transformation step in (A.2) is performed using the relationship

$$E_{n-j}(x_{j+1}) L^{(i+j)} = \bar{L}^{(i+j)} E_{n-j-1}(x_{j+2}) \tag{A.3}$$

for $j = 0, 1, \ldots, n-i-1$.

Let the offdiagonal entries of $L^{(i+j)}$ and $\bar{L}^{(i+j)}$ in (A.3) be $l_1, \ldots, l_{n-1}$ and $\bar{l}_1, \ldots, \bar{l}_{n-1}$, respectively. Then by comparing entries of both sides of (A.3) we have that $L^{(i+j)}$ equals $\bar{L}^{(i+j)}$ with the exception of:

$$\bar{l}_{n-j-1} = l_{n-j-1} + x_{j+1}, \quad \bar{l}_{n-j-2} = \frac{l_{n-j-2} l_{n-j-1}}{\bar{l}_{n-j-1}}, \quad x_{j+2} = \frac{l_{n-j-2} x_{j+1}}{\bar{l}_{n-j-1}},$$

(this is analogous to (4.13)–(4.16) in [10]).

The key observation here is that the nonzero patterns of $L^{(i+j)}$ and $\bar{L}^{(i+j)}$ are the same, the nontrivial entries of $\bar{L}^{(i+j)}$ remain positive, the zeroes remain zero, and $x_{j+2} > 0$.

Therefore, at the end, $\bar{L}$ is LTP and $x \equiv x_{n-i+1} > 0$.

Note that this method is particularly fit for numerical computations. Since it does not involve subtractions, it will not suffer from roundoff-induced subtractive cancellation and loss of relative accuracy. □

References

[1] J.M. Carnicer, J.M. Peña, Bidiagonalization of oscillatory matrices, Linear Multilinear Algebra, 42 (4) (1997) 365–376.

[2] C. Cryer, Some properties of totally positive matrices, Linear Algebra Appl. 15 (1976) 1–25.

[3] S.M. Fallat, Bidiagonal factorizations of totally nonnegative matrices, Amer. Math. Monthly 108 (8) (2001) 697–712.

[4] S. Fomin, A. Zelevinsky, Total positivity: tests and parametrizations, Math. Intelligencer 22 (1) (2000) 23–33.

[5] F. Gantmacher, M. Krein, Oscillation Matrices and Kernels and Small Vibrations of Mechanical Systems, revised ed., AMS Chelsea, Providence, RI, 2002.

[6] M. Gasca, J.M. Peña, Total positivity and Neville elimination, Linear Algebra Appl. 165 (1992) 25–44.

[7] M. Gasca, J.M. Peña, Total positivity, *QR* factorization, and Neville elimination, SIAM J. Matrix Anal. Appl. 14 (4) (1993) 1132–1140.

[8] M. Gasca, J.M. Peña, On factorizations of totally positive matrices, Total Positivity and Its Applications, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996, pp. 109–130.
[9] C.R. Johnson, D.D. Olesky, P. van den Driessche, Elementary bidiagonal factorizations, Linear Algebra Appl. 292 (1–3) (1999) 233–244.
[10] P. Koev, Accurate eigenvalues and SVDs of totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 27 (1) (2005) 1–23.
[11] P. Koev, Accurate computations with totally nonnegative matrices, SIAM J. Matrix Anal. Appl. 29 (2007) 731–751.
[12] A.M. Whitney, A reduction theorem for totally positive matrices, J. Anal. Math. 2 (1952) 88–92.