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Random versus deterministic two-dimensional traffic flow models

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Deterministic and stochastic cellular automata models available to study two-dimensional traffic flow are compared in this paper. It is shown that a connection between them can be made only when the infinite time and infinite system limits are taken in the appropriate order. We also stress the crucial importance of the choice of boundary conditions in the deterministic model to obtain bulk properties.

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In the last few years there has been a growing interest in the study of cellular automata (CA) models which try to mimic, with simple rules, the features of traffic in highways [1]. Recently some work has also been focused on the behavior of traffic in cities. Along this line two CA models have been proposed [2,3]. Both describe two equal populations of cars moving in perpendicular directions from node to node of a square-lattice-like city with streets pointing only up and right and periodic boundary conditions (BC's). Movement occurs in discrete time steps and traffic lights rule it so that they allow horizontal and vertical movements alternately. The interaction between cars forbids a car to jump to a node if it is occupied by another car *at the same time step.* The model of Ref. [3] includes the ability of cars to turn with probability $\gamma \in [0,1/2]$. When $\gamma=0$ this model reduces to that of Ref. [2]. In both models a phase transition from a freely moving phase to a jammed one occurs above a certain density of cars. Whereas the jammed phase is characterized in both models by a low value of the average velocity v (which approaches 0 as γ goes to 0), there is a drastically different behavior of the average velocity as a function of the density of cars n in the freely moving phase. The deterministic model [2] shows that v remains constant and equal to its maximum value up to the transition density, while the stochastic model [3] exhibits an almost linear decrease with slope $-1/2$ which appears not to depend on γ. A recent mean-field-like study of the latter model [4] confirms this fact. It is the purpose of this paper to make clear the connection between these two models and the origin of this discontinuity of the behavior of $v(n)$ in the parameter γ.

In Ref. [4] a microscopic description of these models was achieved by introducing a set of Boolean variables, namely (i) occupation numbers of site $\mathbf{r}$ and time step t for vertical and horizontal cars, $\mu_{\mathbf{r}}^t$ and $\nu_{\mathbf{r}}^t$, respectively, (ii) turning variables for the μ cars, $\xi_{\mathbf{r}}^t$ ($=1$ with probability γ and $=0$ with probability $1-\gamma$), and for the ν cars, $\eta_{\mathbf{r}}^t$ (the other way around), and (iii) a traffic-light variable, $\sigma^t = t \bmod 2$. With these variables and the constraint $\mu_{\mathbf{r}}^t \nu_{\mathbf{r}}^t = 0\ \forall \mathbf{r},t$, the equation of motion for $\mu_{\mathbf{r}}^{t+1}$ turns out to be

$$\begin{aligned}\mu_{\mathbf{r}}^{t+1} = {} & (\sigma^t \bar{\xi}_{\mathbf{r}}^t + \bar{\sigma}^t \xi_{\mathbf{r}}^t)\mu_{\mathbf{r}}^t + \sigma^t \xi_{\mathbf{r}-\mathbf{x}}^t \mu_{\mathbf{r}-\mathbf{x}}^t + \bar{\sigma}^t \bar{\xi}_{\mathbf{r}-\mathbf{y}}^t \mu_{\mathbf{r}-\mathbf{y}}^t \\ & + \sigma^t \xi_{\mathbf{r}}^t \mu_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{x}}^t + \nu_{\mathbf{r}+\mathbf{x}}^t) - \sigma^t \xi_{\mathbf{r}-\mathbf{x}}^t \mu_{\mathbf{r}-\mathbf{x}}^t (\mu_{\mathbf{r}}^t + \nu_{\mathbf{r}}^t) \\ & + \bar{\sigma}^t \bar{\xi}_{\mathbf{r}}^t \mu_{\mathbf{r}}^t (\mu_{\mathbf{r}+\mathbf{y}}^t + \nu_{\mathbf{r}+\mathbf{y}}^t) - \bar{\sigma}^t \bar{\xi}_{\mathbf{r}-\mathbf{y}}^t \mu_{\mathbf{r}-\mathbf{y}}^t (\mu_{\mathbf{r}}^t + \nu_{\mathbf{r}}^t) , \qquad (1)\end{aligned}$$

and the corresponding one for $\nu_{\mathbf{r}}^{t+1}$ is obtained from the above by exchanging μ and ν, and ξ and η; $\bar{b}$ denotes $1-b$ and subscripts $\mathbf{x}$ and $\mathbf{y}$ denote unit-vector displacements in either direction. The mean velocity at time t is defined as the number of movements per car that occurred at that time step:

$$v(t)=\frac{1}{2N}\sum_{\mathbf{r}}\{(\mu_{\mathbf{r}}^{t+1}-\mu_{\mathbf{r}}^{t})^2+(\nu_{\mathbf{r}}^{t+1}-\nu_{\mathbf{r}}^{t})^2\},$$

$$=1-\frac{1}{n}([\mu^{t+1}\mu^{t}]+[\nu^{t+1}\nu^{t}]), \qquad (2)$$

where N is the total number of cars and $[A^t]\equiv L^{-2}\Sigma_{\mathbf{r}}A_{\mathbf{r}}^t$ stands for the average over the lattice (L is the linear size of the city). Using Eq. (1) and assuming $L\rightarrow\infty$ (so that we can take the random variables out of the brackets replaced by their averages):

$$[\mu^{t+1}\mu^{t}]=\sigma^t\{\bar{\gamma}[\mu^t]+\gamma[\mu^t\mu_{\mathbf{x}}^t]+\gamma[\mu^t\nu_{\mathbf{x}}^t]\}$$
$$+\bar{\sigma}^t\{\gamma[\mu^t]+\bar{\gamma}[\mu^t\mu_{\mathbf{y}}^t]+\bar{\gamma}[\mu^t\nu_{\mathbf{y}}^t]\}, \qquad (3)$$

(and the counterpart for ν) and then $v(t)$ can be written in terms of space correlations rather than time correlations. By breaking up correlations in the steady state ($[\mu^t\mu_{\mathbf{x}}^t]\sim[\mu^t][\mu_{\mathbf{x}}^t]=n^2/4$) it was shown in [4] that, for large cities ($L\rightarrow\infty$), Eq. (2) yields

$$v(t)\sim(1-n)/2 \qquad (4)$$

in the low-density limit ($n\rightarrow 0$), in good agreement with the simulations in the freely moving phase. The decorrelation assumption relies on the fact that the randomness favors the loss of memory of the system between successive car encounters (at least for dilute enough systems).

We face a paradox here when we try to compare this result with the simulations for $\gamma=0$ [2]. On the one hand, the above argument does not hold for this case since it makes explicit use of the randomness to justify the decorrelation; on the other hand, we see that the prediction obtained for the average velocity does not depend on γ, so in principle the argument should also apply to the deterministic case as a limit of vanishing randomness. Anyhow, the simulations of Ref. [2] are in strong disagreement with this result as they unquestionably show that $v=1/2$ ($v=1$ according to the normalization used in Ref. [2]) in the whole freely moving phase. To sort out this question we have to gain a deeper insight into the procedure we are actually following to perform the calculations, and revise the validity of the decorrelation assumption.

In the deterministic model cars can move either vertically or horizontally but they cannot change direction under any circumstance. Randomness is then excluded, so a given initial condition evolves always in the same way. Therefore, for a *finite* L, any initial configuration must end up in one of two different asymptotic regimes: a jammed state in which all cars are stopped, or a periodic state. This is a simple consequence of the following: as the system has a finite number of different states and it evolves in a deterministic way, if the system does not jam, sooner or later it must reach a state in which it was before, and from then on the evolution is necessarily periodic. Clearly, the larger the system, the larger the average period (as illustrated in Fig. 4 of Ref. [2]). Notice that this does not hold for $\gamma\neq 0$ (no matter how small γ is), because a given initial condition may evolve in many different ways, depending on the random variables. Asymptotically periodic states are then ruled out of this case. In the computer simulations reported in Refs. [2, 3] the relevant quantities are computed once the asymptotic regime has been reached, and this makes a big difference between both cases: for $\gamma=0$ this means waiting until the time gets larger than the period of the system, which grows with the system size L; for $\gamma\neq 0$ the time to reach the steady state is independent of L because in this case the system is explicitly diffusive—by virtue of the randomness, which means that there will be a finite relaxation time when $L\rightarrow\infty$.

In deriving Eq. (4) we have performed two operations: *first*, we have taken the limit $L\rightarrow\infty$ ("thermodynamic limit") and *second* we have waited until the steady state is formed (we will henceforth call this procedure the LT limit). Instead, the simulations results have been obtained *first* waiting up to the steady state is formed and *second* extrapolating to the infinite system (we will henceforth call this procedure the TL limit). According to the above discussion, taking the LT or the TL limits makes no difference if $\gamma\neq 0$, but if $\gamma=0$, which limit is being taken may yield different results. It is easy to understand why this is so: in the LT limit, the system will hardly reach a periodic regime (because the average period goes to infinity), whereas in the TL limit we are averaging precisely in the periodic regime (with maximum velocity according to Ref. [2]). With this in mind it is now clear that a connection between the $\gamma\rightarrow 0$ limit of Eq. (4) and the $\gamma=0$ model is to be expected *only* in the LT limit but never in the TL limit; therefore there is no contradiction between Eq. (4) and the results of Ref. [2].

In order to check whether Eq. (4) makes a correct prediction in the LT limit when $\gamma=0$ we have performed simulations on the deterministic system at very low density (freely moving phase), where the decorrelation approximation is supposed to hold. In each run we compute $v(t)$ averaged over a large set of randomly chosen initial conditions (each initial condition having the same probability). In Fig. 1 we have plotted, for $L=64$, 128, and 256 and very small n, the function $s(t)\equiv\{\langle v(t)\rangle-1/2\}/n$, which in the limit $n\rightarrow 0$ gives the slope of $v(t)$, as a function of n, for $n=0$ [notice that if Eq. (4) applies to $\gamma=0$, $s(t)$ should be $-1/2$, while if $v(t)=1/2$, $s(t)=0$]. This plot exhibits three striking features: two separated regimes, a discontinuity separating them at $t=2L$, and a jump separating the first two time steps from the rest of the figure. All three deserve some comments.

First of all, we can see that for $2<t<2L$ $s(t)$ is time independent and very accurately equals $-1/4$. As it is different from 0, this result supports what we have been commenting on above, namely, that for $\gamma=0$, the TL and the LT limits lead to different results. In spite of that, the result is also in disagreement with the $-1/2$ prediction of (4). We will comment on this point later. Secondly, there is a big jump at $t=2L$, changing the slope from the value $-1/4$ to almost 0. This feature reveals the strong influence of the periodic BC's in the synchronization of the system for $\gamma=0$ (notice that, due to the traffic lights, a single car will take $2L$ time steps to arrive at the site it started from). This figure is to be compared with Fig. 2, where we have plotted $s(t)$ for $L=64$ and small n also, but for $\gamma=0.05$. Even for such a small value of γ no jump can be appreciated, thus supporting our argument on the existence of a bulk relaxation time when $\gamma\neq 0$. Randomness then largely weakens the influence that the BC's may have on the system. On the other hand, in spite of the fact that the plot is rather noisy, it can be appreciated that

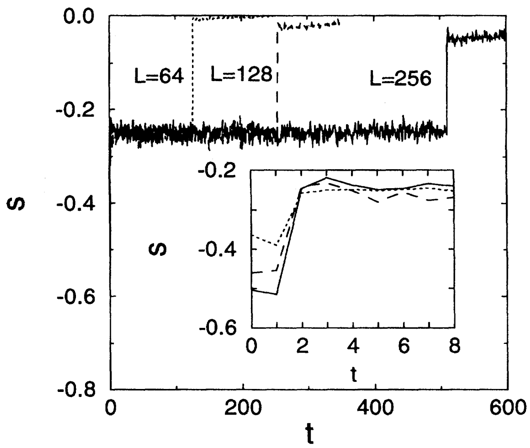

FIG. 1. Plot of $s(t) \equiv [v(t) - 1/2]/n$ as a function of t for three system sizes, L, averaged over initial configurations randomly chosen with equal probability (10^6 for $L=64$, 10^5 for $L=128$, and 2.5×10^4 for $L=256$), for the deterministic system ($\gamma=0$) at density $n=10^{-3}$. The inset shows the same plot at very short times.

the slope has an average value clearly lower than $-1/4$ (around -0.32), but still higher than $-1/2$. As we have said, we will come back to this point later on.

We have performed the same analysis for slightly larger densities. The results appear in Fig. 3, where we have plotted $s(t)$ for $\gamma=0$, a couple of values of n, and three system sizes. We again observe the same three features as in Fig. 1, but this time there is a size-independent transient for $2<t<2L$ rather than a steady state [for $t>2L$ the behavior is different for each size and $s(t)$ eventually reaches the limit $s(t)=0$]. The coincidence of this transient regime for all sizes makes it clear that it can only depend on bulk properties of the system, and accordingly disappears once the BC's come into play. Unfortunately, the slow decay of this transient makes the LT limit numerically accessible only at extremely low densities. At higher densities we cannot determine from these data what the limit of $s(t)$ is or whether it coincides with the TL limit $[s(t)=0]$. In contrast, the results for $\gamma=0.05$ (also shown in Fig. 3) exhibit no difference with respect to Fig. 2 at all.

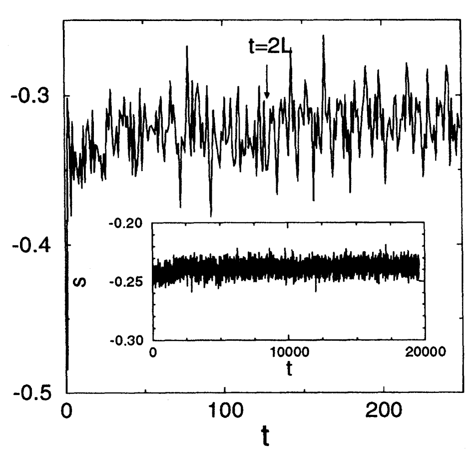

FIG. 2. Plot of $s(t)$ for $L=64$, $n=3\times10^{-3}$, and $\gamma=0.05$. Notice the absence of any jump at $t=2L$. The inset shows $s(t)$ for $\gamma=0$, $n=1.5\times10^{-3}$ and $L=64$, for a run performed with the "entangled" boundary conditions. Averages are performed over 10^6 initial configurations.

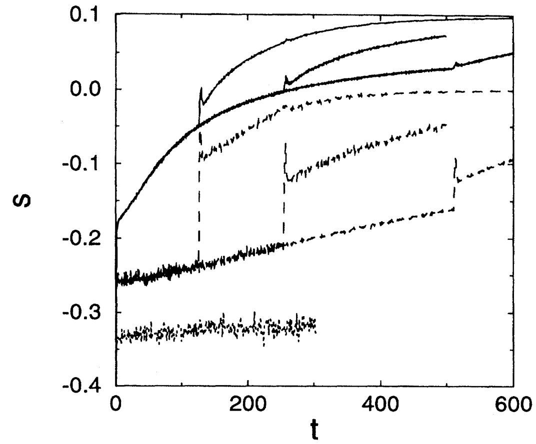

FIG. 3. Same as Fig. 1 for two larger densities: $n=10^{-2}$ (dashed lines) and $n=5\times10^{-2}$ (solid lines). The latter are shifted 0.1 units up. The dotted line (the lowest curve in the figure) shows the same as Fig. 2 for $n=10^{-2}$.

Let us return now to the discrepancies between the slopes measured from Figs. 1 and 2 and the value $s(t)=-1/2$ given by (4). To clarify what is going on we need to be more careful in the analysis. For $\gamma=0$, we can trace back the argument that led to Eq. (4) in order to see what is wrong in the decorrelation assumption: remember that the evolution is such that a car does not move if the site it wants to move to is occupied by another car *at the same time step*. This means that if, at time t, we have μ-type cars (which move only vertically) at sites $\mathbf{r}$ and $\mathbf{r}+\mathbf{y}$, at the next time step at which traffic lights allow vertical movement, the car at $\mathbf{r}+\mathbf{y}$ will move to $\mathbf{r}+2\mathbf{y}$, while the car at $\mathbf{r}$ will remain there. In successive time steps both cars will move, maintaining that separation between them. This situation will last until a third car comes into play. It is clear that the lower the density, the less probable are these three-car encounters, so in the $n\rightarrow0$ limit, $[\mu^t\mu_y^t]\sim0$, and similarly $[\nu^t\nu_x^t]\sim0$. For $\gamma=0$ (and $L\rightarrow\infty$) Eq. (3) and its counterpart become

$$[\mu^{t+1}\mu^t]\sim\sigma^t\frac{n}{2}+\bar{\sigma}^t\{[\mu^t\mu_y^t]+[\mu^t\nu_y^t]\}\,,$$

$$[\nu^{t+1}\nu^t]\sim\bar{\sigma}^t\frac{n}{2}+\sigma^t\{[\nu^t\nu_x^t]+[\nu^t\mu_x^t]\}\,;$$

with the result we have just obtained and maintaining the decorrelation assumption for the other two correlations, this couple of equations together with Eq. (2) lead to the expression (in the limit $n\rightarrow0$) $v\sim(2-n)/4$, now in perfect agreement with Fig. 1. By the way, the above argument does not hold for the first two time steps (while the above synchronization mechanism is taking over) because the initial configurations are chosen at random with equal probability and this gives as a result that all correlations have the same value, $n^2/4$. As we have seen in the beginning, this unavoidably

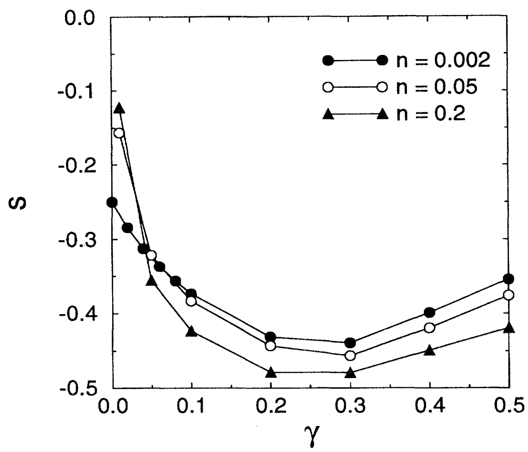

FIG. 4. Plot of the steady value of $s(t)$ as a function of γ for different values of n and $L=128$. To reduce noise, the values for $\gamma \neq 0$ have been obtained averaging along 10^6 time steps once the steady state has been reached. The only point for $\gamma=0$ has been obtained averaging over time for $t<2L$.

leads to Eq. (4) for v [5]. This is the origin of the jump in Fig. 1 at $t=2$ (the last of the features of this figure that still remained to be explained).

For $\gamma \neq 0$ the above argument is not valid, but neither does Eq. (4) seem to describe the behavior appropriately when γ is close to 0. For this reason we have computed the (in the LT limit) steady value of $s(t)$ for different γ's and several densities. The results appear in Fig. 4. Surprisingly, this figure clearly shows that the value $s=-1/2$ is nothing but a rough estimate of the actual slope—and only for γ's not too close to 0. The estimate improves as n increases but it can never be considered an accurate prediction for the slope [6]. The value for $\gamma=0$ could only be computed for the smallest density because larger densities do not reach the steady state before $t=2L$. Anyhow, it can be guessed from the figure that the limit $\gamma \rightarrow 0$ seems to exist and to have a nonzero value, although this value approaches 0 as n increases. Densities above ≈ 0.2 cannot be reached because the system for $\gamma=0.01$ already undergoes the jamming transition. It is worthwile to remark that the results of Fig. 4 depend only very slightly on the system size (up to the largest size we have studied, $L=256$).

To summarize, the main conclusion that can be drawn from this work is that, in spite of the apparent difference between the deterministic and the stochastic models, there is a regime in which both can be connected, before the BC's have any influence. After that point both systems separate because the BC's strongly favor the synchronization in the deterministic model, while their influence is discarded in the stochastic model due to the random dynamics. Then the choice of BC's turns out to be crucial in the deterministic model when one is interested in the bulk properties of the system. We can illustrate this point further with a preliminary simulation we have performed in the deterministic model with a different choice of BC's. In the periodic BC's all the streets getting out of the system through an edge are connected with themselves at the other edge. In the new BC's the streets are randomly entangled before the connection is made. This is done once at the beginning and the entanglement is kept for all the simulations, so that the system is still *deterministic*. With these new BC's we expect that the synchronization of the system is made more difficult. We have plotted $s(t)$ for one of these runs in the inset of Fig. 2. The jump at $t=2L$ has disappeared and, in spite of the length of the run, no influence of the system size can be appreciated. It is very interesting that, after a transient of about 3000 time steps, $s(t)$ reaches a value very close to $-1/4$, the limit $\gamma \rightarrow 0$ of the stochastic model, and *not* to 0, the value obtained with the periodic BC's. It seems clear that this result opens many questions. Given the strong influence of the BC's in the freely moving phase of the deterministic model, one may wonder how they will influence the transition to the jammed phase (both its location in the phase diagram and the structure of the jam). On the other hand, the results obtained with the "entangled" BC's as well as those pointed out by the short-time regime with the periodic BC's are compatible with the results obtained with the stochastic model in the limit $\gamma \rightarrow 0$, so this might be an alternative way to reach the bulk properties of the deterministic model.

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[5] Actually, for finite size simulations, the value of the correlations is $\sim(N/2)[(N/2)-1]L^{-4}=(n^2/4)-(n/2L^2)$, so for small n (of the order $1/L^2$), deviations of the value $n^2/4$ are noticeable, as can be seen in Fig. 1.

[6] Anyhow, Eq. (4) seems to be an overall good approximation for $v(n)$ in the freely moving phase (see Ref. [4] for more details). The reason is that, although the difference between a function and an approximation to it may be small, the difference between their derivatives will almost certainly be amplified.