FIRST ORDER EIGENVALUE PERTURBATION THEORY AND THE NEWTON DIAGRAM

Julio Moro
Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain
jmoro@math.uc3m.es

Froilán M. Dopico
Departamento de Matemáticas
Universidad Carlos III de Madrid, Spain
dopico@math.uc3m.es

Abstract First order perturbation theory for eigenvalues of arbitrary matrices is systematically developed in all its generality with the aid of the Newton diagram, an elementary geometric construction first proposed by Isaac Newton. In its simplest form, a square matrix A with known Jordan canonical form is linearly perturbed to $A(\varepsilon) = A + \varepsilon B$ for an arbitrary perturbation matrix B, and one is interested in the leading term in the ε-expansion of the eigenvalues of $A(\varepsilon)$. The perturbation of singular values and of generalized eigenvalues is also covered.

Keywords: eigenvalues, singular values, generalized eigenvalue problems, perturbation theory, asymptotic expansions, Newton diagram, Jordan canonical form.

1. Introduction

Eigenvalue perturbation theory has been an issue in applied mathematics since Lord Rayleigh laid its foundations in [25]. One of his calculations aimed at determining both the eigenfrequencies and eigenmodes of an oscillatory string with constant elasticity modulus and whose mass density was a small deviation of a constant value. This particular problem illustrates perfectly the typical setting of eigenvalue perturbation theory: the matrix or operator under study is assumed to be a slight deviation from some close, simpler matrix or operator for which the spectral problem is completely (and, in most cases, easily) solved. The given operator $\widetilde{A}$ is replaced by a neighboring operator

A whose eigenvalues and eigenvectors are known. Then, the influence of the difference $B = \widetilde{A} - A$ on the spectral objects is analyzed using an appropriate perturbation theory. The usual approach is introducing a perturbation parameter ε and considering the (matrix or) operator

$$A(\varepsilon) = A + \varepsilon B. \tag{1}$$

Its eigenvalues and eigenvectors depend on ε and *are assumed* to converge to the corresponding eigenvalues and eigenvectors as ε goes to zero. In a first stage of the analysis the leading terms of the ε-expansions of the spectral objects of $A(\varepsilon)$ are determined (hence the name of *first order* perturbation theory). In a second stage the convergence of these expansions is justified up to $\varepsilon = 1$. This paper shall be mainly focused on the first stage, although some mention will be made of the appropriate convergence results.

This general framework owes much to the formalism proposed by Schrödinger [33] in his approach to quantum mechanics. He described any observable of a quantum mechanical system as a selfadjoint operator H (the Hamiltonian in the case of the energy) defined in a certain Hilbert space. An isolated eigenvalue λ_0 and its corresponding normalized eigenvector x_0 are interpreted as a bound state x_0 with energy level λ_0. If the system is influenced by some external field, or if some previously neglected interaction has to be taken into account, a perturbation H_1 must be added to the operator H. The question arises of whether there is at least one bound state in the neighborhood of the unperturbed one. One also wants to calculate the bound states and energy levels of $H + H_1$ in the neighborhood of x_0, λ_0. Schrödinger was one of the first to take the approach (1), considering $H + \varepsilon H_1$ and *postulating* the analytic dependence of the perturbed spectral objects on the perturbation parameter ε. The explicit formulas he obtained for eigenvalues and eigenvectors are known as perturbation series in quantum mechanics. Even the splitting of multiple eigenvalues was studied in the first order approximation.

Schrödinger, however, did not prove the convergence of such expansions. That issue was finally settled for isolated eigenvalues of selfadjoint operators in Hilbert space by Rellich in a series of papers [26, 27, 28, 29, 30] which stimulated further advances in similar problems. On one hand, Sz.-Nagy [38] translated Rellich's arguments into the complex domain using the Cauchy-Riesz integral method. This led, in particular, to the extension of Rellich's results to the case of nonselfadjoint operators. On the other hand, Friedrichs [4] developed the perturbation theory of continuous spectra, which turned out to be extremely useful in scattering theory and quantum field theory. Meanwhile, the perturbation theory for one-parameter semigroups was developed by Hille and Phillips [23, 6], and a general framework for the perturbation theory of linear operators was presented by Kato in his well-known treatise [9].

As can be seen from this discussion of the early history of the subject, a great deal of effort was invested in developing eigenvalue perturbation theory for infinite-dimensional operators. In other words, first order eigenvalue perturbation theory began with a strong flavor of functional analysis. The essential ingredients, however, are purely finite-dimensional and can be presented in a strictly finite-dimensional setting without losing any of the mathematical subtleties of the original problem[1]. This is precisely the main goal of the present paper: presenting, in an accessible way and without much technical apparatus, the essentials of first order eigenvalue perturbation theory. It is also our intention to show in §3.2 that the general problem of determining the leading term of the ε-expansion of the eigenvalues of (1) as a function of B is far from being completely solved, even in the finite-dimensional case.

We choose not to present eigenvector perturbation results for two main reasons: the first one is to keep the presentation relatively concise. The second, and more important one, is that the techniques we will employ are able to produce such kind of results only indirectly, and under quite restrictive assumptions.

Making use of a powerful algebraic tool like the Newton diagram method will allow us to present a systematic development of the relevant results by means of purely matrix-analytic techniques. The results we will present are chiefly those obtained by Vishik and Lyusternik [41] and Lidskii [14], together with some consequences and extensions which, to our knowledge, are new (see §3.2.1 and §4 below). The results in [41], intended to be applied on differential operators, were generalized by Lidskii [14] for the finite-dimensional case. He obtained simple explicit formulas for the perturbation coefficients and provided, at the same time, a much more elementary proof. The results in both [41] and [14] were later refined by Baumgärtel (see [1] § 7.4), in the sense of dealing not only with perturbation series for eigenvalues and eigenvectors, but also with the corresponding eigenprojections as functions of ε. Vainberg and Trenogin ([39] § 32), on the other hand, offer a fairly thorough account of similar results, obtained for Fredholm operators by applying the techniques of branching theory. Langer and Najman [10, 11, 12] generalized Lidskii's results to analytic matrix functions $A(\lambda) + B(\lambda, \varepsilon)$, using the local Smith normal form of parameter-dependent matrices (see section 5 for more references).

From the functional analytic bias mentioned above one might get the impression that first order eigenvalue perturbation theory is suited mostly for dealing with infinite-dimensional operators, the finite-dimensional results being just a simple by-product of the former. Also, numerical analysts have not been very keen on first order eigenvalue perturbation results or, as they are also known, on *perturbation expansions*. That is partly true, mainly because perturbation

[1] Actually, the original problem can be reduced to a purely finite-dimensional one via the resolvent integration method (see [1, chapter 10])

bounds, as opposed to perturbation expansions, are more amenable to backward rounding-error analysis, the most widespread tool for analyzing the numerical stability of algorithms for spectral problems. However, first order eigenvalue perturbation results are still extremely useful for the numerical analyst whenever some qualitative information is needed, e.g. on the splitting directions of a multiple eigenvalue, even if perturbation bounds are available (which is not always the case). Borrowing the very words of Stewart and Sun in their book [35, p. 292],

> Although these [perturbation] expansions are usually corollaries of more general results, in many cases the general results themselves were conjectured by looking at first order expansions. The reason is that the expansions often tell ninety percent of the story and yet are free of the clutter that accompanies rigorous upper bounds.

Thus, first order eigenvalue perturbation theory has a role to play in applied mathematics, as well as a definite place among the many different tools-of-the-trade of the applied mathematician.

The paper is organized as follows: the Newton diagram technique is presented in detail in section 2, together with some elementary examples. Section 3 contains the results on perturbation of the standard eigenvalue problem: after setting the stage for using the Newton diagram in this context, the main result (Theorem 2) is proved and discussed in § 3.1. The cases outside the scope of Theorem 2 (what we call the *nongeneric* case) are analyzed in § 3.2. Particular attention is given in § 3.2.1 to the case when the perturbation B has small rank. Section 4 presents first order perturbation results for singular values, which can be easily recovered from the results in section 3. The final section 5 is devoted to a brief review of some of the available results for the perturbation of generalized eigenvalue problems.

2. The Newton Diagram

Although the Newton diagram (also called Newton *polygon*, or Newton-Puiseux diagram) can be applied in much more general settings (including analytic perturbation of infinite-dimensional operators, see [2, § III.8.3], [39, § 1.2] or [1, Appendix A7]), we strictly confine our presentation to the particular context we are dealing with, namely eigenvalues as roots of the characteristic polynomial of a parameter-dependent matrix $A(\varepsilon) = A + \varepsilon B$. Therefore, consider a complex polynomial equation

$$P(\lambda, \varepsilon) = \lambda^n + \alpha_1(\varepsilon)\lambda^{n-1} + \ldots + \alpha_{n-1}(\varepsilon)\lambda + \alpha_n(\varepsilon) = 0 \qquad (2)$$

in λ, with analytic coefficients

$$\alpha_k(\varepsilon) = \hat{\alpha}_k \varepsilon^{a_k} + \ldots, \qquad k = 1, \ldots, n, \qquad (3)$$

where a_k is the *leading exponent* and $\hat{\alpha}_k$ the *leading coefficient* of $\alpha_k(\varepsilon)$ (i.e. $\hat{\alpha}_k \neq 0$ and no term of order lower than a_k appears in the expansion of $\alpha_k(\cdot)$). For our convenience we set $\alpha_0(\varepsilon) \equiv 1$, i.e. $\hat{\alpha}_0 = 1$ and $a_0 = 0$.

It is well known [1, 9] that the roots λ of (2) are given by expansions in fractional powers of ε. Our goal is to determine both the leading exponents and the leading coefficients of these ε-expansions. To do that, we make the *Ansatz*

$$\lambda(\varepsilon) = \mu\varepsilon^{\beta} + \dots, \tag{4}$$

with μ, β to be determined. Substituting the Ansatz into (2), each $\alpha_k(\varepsilon)\lambda^{n-k}$ produces a term of order $\varepsilon^{a_k+(n-k)\beta}$ plus higher order terms. Hence, $P(\lambda,\varepsilon)$ is an infinite sum of powers of ε, each of them multiplied by an ε-independent coefficient. If, as we assume, $\lambda(\varepsilon)$ is a root of (2), then all the coefficients must be zero. In particular, the lowest order in ε must be present at least twice among the exponents $\{a_k + (n-k)\beta\}_{k=0}^{n}$, i.e. there exist at least two indices $i, j \in \{0, 1, \dots, n\}$ such that

$$a_i + (n-i)\beta = a_j + (n-j)\beta \leq a_k + (n-k)\beta, \qquad k = 0, 1, \dots, n.$$

In order to interpret geometrically this inequality, we plot the values a_k versus k for $k = 0, 1, \dots, n$ on a cartesian grid. Then, the segment S joining (i, a_i) with (j, a_j) has slope β, and, since $\varepsilon^{a_i+(n-i)\beta}$ is the term with the lowest order, no other point (k, a_k) lies below the straight line containing S. Therefore, if we draw the lower boundary of the convex hull of all the points $\{(k, a_k)\}_{k=0}^{n}$, it is clear that S must be on that boundary. Thus, the slopes of the segments on the lower boundary are just the exponents β appearing in (4). By imposing for each β that the coefficient of $\varepsilon^{a_i+(n-i)\beta}$ be zero we obtain that the leading coefficients μ of the eigenvalues of order ε^{β} are the solutions of

$$\sum_{k \in I_s} \mu^{n-k}\hat{\alpha}_k = 0, \tag{5}$$

where the set $I_s = \{k : (k, a_k) \in S\}$ may contain indices other than i, j (e.g. see Fig. 1(a) below, where three points (k, a_k) lie on S). If we denote $k_{\max} = \max I_s$ and $k_{\min} = \min I_s$, then the polynomial equation (5) has $n - k_{\max}$ zero roots, corresponding to the roots of order higher than β, and $k_{\max} - k_{\min}$ nonzero roots which are the leading coefficients we are looking for. As a consequence, the number of roots of (2) with leading exponent β is given by the length of the projection on the horizontal axis of the segment S. Notice that the total sum of the lengths of these projections is n.

Thus, we have derived a quite simple geometrical construction, going back to Newton, which very simply provides us with all the leading powers and leading coefficients we were looking for. The crucial ingredient is the so-called *Newton diagram* or *Newton polygon*.

Definition of the Newton diagram: *Given a polynomial equation of the form (2) with analytic coefficients given by (3), plot* a_k *versus* k *for* $k = 0, 1, \ldots, n$ *(if* $\alpha_k(\cdot) \equiv 0$*, the corresponding point is disregarded). Denote each of these points by* $\pi_k = (k, a_k)$ *and let*

$$\Pi = \{\pi_k : \alpha_k \not\equiv 0\}$$

be the set of all plotted points. Then, the Newton diagram associated with $P(\lambda, \varepsilon)$ *is the lower boundary of the convex hull of the set* Π.

With this terminology, the procedure outlined above for determining the leading terms of the asymptotic expansions (4) goes as follows:

Newton diagram procedure: *Given a polynomial (2) with analytic coefficients given by (3),*

1) *draw the associated Newton diagram.*

2) *The leading exponents* β *of the roots of (2) are the different slopes of the segments forming the Newton diagram.*

3) *The number of roots of order* ε^β *is given by the length of the projection on the horizontal axis of the segment with slope* β.

4) *The leading coefficients* μ *for each root of order* ε^β *are the nonzero roots of equation (5), where* S *is the segment of the Newton diagram with slope* β.

We illustrate the procedure with two specific examples, depicted in Figure 1 below.

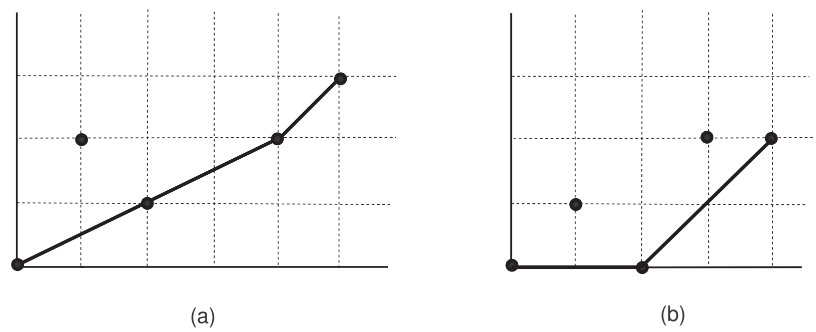

Figure 1. Newton diagrams associated with the polynomials: a) $\lambda^5 + (2\varepsilon^2 - \varepsilon^3)\lambda^4 - \varepsilon\lambda^3 + (-6\varepsilon^2 + 3\varepsilon^5)\lambda + \varepsilon^3 - \varepsilon^4$*; and b)* $\lambda^4 - (\varepsilon + 2\varepsilon^2)\lambda^3 + (\varepsilon^2 - 1)\lambda^2 + (\varepsilon^2 - \varepsilon^3)\lambda + 2\varepsilon^2$

Example 1. Let $P_1(\lambda, \varepsilon) = \lambda^5 + (2\varepsilon^2 - \varepsilon^3)\lambda^4 - \varepsilon\lambda^3 + (-6\varepsilon^2 + 3\varepsilon^5)\lambda + \varepsilon^3 - \varepsilon^4$. Then, $\Pi = \{\pi_0, \pi_1, \pi_2, \pi_4, \pi_5\}$ and the Newton diagram associated with P_1 is the one in Figure 1(a). It consists of two segments, one of slope $1/2$ connecting the points π_0, π_2, π_4, and one of slope 1 connecting π_4 with π_5. Therefore, P_1 has roots of orders $\sqrt{\varepsilon}$ and ε. More precisely, there are four roots of order $\sqrt{\varepsilon}$ whose leading coefficients are the nonzero roots of

$$\hat{\alpha}_0\mu^5 + \hat{\alpha}_2\mu^3 + \hat{\alpha}_4\mu = \mu^5 - \mu^3 - 6\mu = \mu(\mu^2 - 3)(\mu^2 + 2),$$

i.e $\pm\sqrt{3}$ and $\pm\, i\sqrt{2}$. Finally, there is one root of order ε whose leading coefficient is $1/6$, the only root of

$$\hat{\alpha}_4\mu + \hat{\alpha}_5 = -6\mu + 1.$$

Example 2: Let $P_2(\lambda, \varepsilon) = \lambda^4 - (\varepsilon + 2\varepsilon^2)\lambda^3 + (\varepsilon^2 - 1)\lambda^2 + (\varepsilon^2 - \varepsilon^3)\lambda + 2\varepsilon^2$. Now $\Pi = \{\pi_i\}_{i=0}^4$. The corresponding Newton diagram is drawn in Figure 1(b). Hence, there are two roots of $O(1)$ with leading coefficients ± 1, the nonzero roots of

$$\mu^4 + \hat{\alpha}_2\mu^2 = \mu^4 - \mu^2,$$

and two roots of $O(\varepsilon)$, whose leading coefficients $\pm\sqrt{2}$ solve

$$\hat{\alpha}_2\mu^2 + \hat{\alpha}_4 = -\mu^2 + 2 = 0.$$

We stress that the whole argument above, leading to the Newton diagram technique *relies completely on the initial assumption that a convergent asymptotic expansion* (4) *exists*. Otherwise, we might be computing the leading term of a nonexisting quantity. Of course, that was not a concern for Newton: he proposed this method in letters to Leibniz and Oldenburg (see [2, pp. 372-375], [21, pp. 20-42]), and developed it later in his treatises [22], handling infinite series but saying nothing as to their convergence. Only in the 19th century, Puiseux [24] proved, in the course of his investigations on singularities, that the expansions obtained through the Newton diagram converge in a neighborhood of $\varepsilon = 0$. Hence the name of *Newton-Puiseux* diagram.

Once this tool is at our disposal, we set to the task of obtaining first order results on perturbation of matrix eigenvalues

3. First order perturbation bounds for the standard eigenvalue problem

Let λ_0 be an eigenvalue of algebraic multiplicity a of the complex matrix $A \in \mathbb{C}^{n\times n}$, and consider a perturbation

$$A(\varepsilon) = A + \varepsilon\, B$$

for arbitrary $B \in \mathbb{C}^{n\times n}$. It is a well known fact [1, 9] that, for sufficiently small ε, the matrix $A(\varepsilon)$ has a eigenvalues $\lambda_j(\varepsilon)$ with $\lambda_j(0) = \lambda_0$, each of them admitting an expansion in fractional powers of ε. Our goal is to determine the leading term of each expansion applying the Newton diagram technique to the characteristic polynomial $P(\lambda, \varepsilon) = \det(\lambda I - A - \varepsilon B)$ of $A(\varepsilon)$.

In order to prove our main perturbation result (Theorem2 below), it is crucial to carefully determine which points (k, a_k) may appear on the Newton diagram *for a particular given Jordan structure* of the unperturbed matrix A: let

$$\left[\begin{array}{c|c} J & \\ \hline & \widehat{J} \end{array}\right] = \left[\begin{array}{c} Q \\ \hline \widehat{Q} \end{array}\right] A \left[\begin{array}{c|c} P & \widehat{P} \end{array}\right] \tag{6}$$

be a Jordan decomposition of A, i.e.

$$\left[\begin{array}{c} Q \\ \hline \widehat{Q} \end{array}\right] \left[\begin{array}{c|c} P & \widehat{P} \end{array}\right] = I. \tag{7}$$

The matrix J contains all Jordan blocks associated with the eigenvalue of interest λ_0, while $\widehat{J}$ is the part of the Jordan form containing the other eigenvalues. Let

$$J = \Gamma_1^1 \oplus \ldots \oplus \Gamma_1^{r_1} \oplus \ldots \oplus \Gamma_q^1 \oplus \ldots \oplus \Gamma_q^{r_q}, \tag{8}$$

where, for $j = 1, \ldots, q$,

$$\Gamma_j^1 = \ldots = \Gamma_j^{r_j} = \begin{bmatrix} \lambda_0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_0 \end{bmatrix}$$

is a Jordan block of dimension n_j, repeated r_j times, and ordered so that

$$n_1 > n_2 > \ldots > n_q.$$

The n_j are called the *partial multiplicities* for λ_0. The eigenvalue λ_0 is semisimple (nondefective) if $q = n_1 = 1$ and nonderogatory if $q = r_1 = 1$. The algebraic and geometric multiplicities of λ_0 are, respectively,

$$a = \sum_{j=1}^{q} r_j n_j \qquad \text{and} \qquad g = \sum_{j=1}^{q} r_j. \tag{9}$$

We further partition

$$P = \left[\begin{array}{c|c|c|c|c|c|c} P_1^1 & \ldots & P_1^{r_1} & \ldots & P_q^1 & \ldots & P_q^{r_q} \end{array}\right]$$

conformally with (8). The columns of each P_j^k form a right Jordan chain of A with length n_j corresponding to λ_0. If we denote by x_j^k the first column of P_j^k, each x_j^k is a right eigenvector of A associated with λ_0. Analogously, we split

$$Q = \left[\begin{array}{c} Q_1^1 \\ \hline \vdots \\ \hline Q_1^{r_1} \\ \hline \vdots \\ \hline Q_q^1 \\ \hline \vdots \\ \hline Q_q^{r_q} \end{array}\right],$$

also conformally with (8). The rows of each Q_j^k form a left Jordan chain of A of length n_j corresponding to λ_0. Hence, if we denote by y_j^k the last (i.e. n_j-th) row of Q_j^k, each y_j^k is a left eigenvector corresponding to λ_0. With these eigenvectors we build up matrices

$$Y_j = \begin{bmatrix} y_j^1 \\ \vdots \\ y_j^{r_j} \end{bmatrix}, \qquad X_j = [x_j^1, \ldots, x_j^{r_j}],$$

for $j = 1, \ldots, q$,

$$W_s = \begin{bmatrix} Y_1 \\ \vdots \\ Y_s \end{bmatrix}, \qquad Z_s = [X_1, \ldots, X_s],$$

for $s = 1, \ldots, q$, and define square matrices Φ_s and E_s of dimension

$$f_s = \sum_{j=1}^{s} r_j \tag{10}$$

by

$$\Phi_s = W_s B Z_s, \qquad s = 1, \ldots, q,$$

$$E_1 = I, \qquad E_s = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \qquad \text{for } s = 2, \ldots, q,$$

where the identity block in E_s has dimension r_s. Note that, due to the cumulative definitions of W_s and Z_s, every Φ_{s-1}, $s = 2, \ldots, q$, is the upper left block of Φ_s.

An important observation to be made at this point is that, although the Jordan decomposition of A has been presented in its full generality, and all results presented below are valid for the general case (6), we may assume with no loss of generality that λ_0 is the only eigenvalue of A, i.e that $\widehat{J}$ is empty. The reason is that, *since we are only interested in first order results*, we may disregard quadratic terms in ε. More precisely, if we write the characteristic polynomial of $A(\varepsilon)$ as $P(\lambda, \varepsilon) = \det(\lambda I - \mathrm{diag}(J, \widehat{J}) - \varepsilon \widetilde{B})$, with

$$\widetilde{B} = \left[\begin{array}{c} Q \\ \hline \widehat{Q} \end{array} \right] B \left[\begin{array}{c|c} P & \widehat{P} \end{array} \right] \tag{11}$$

then one can use Schur's formula to factorize P as

$$P(\lambda, \varepsilon) = \det \left(\begin{bmatrix} \lambda I - J - \varepsilon \widetilde{B}_{11} & -\varepsilon \widetilde{B}_{12} \\ -\varepsilon \widetilde{B}_{21} & \lambda I - \widehat{J} - \varepsilon \widetilde{B}_{22} \end{bmatrix} \right) = \widehat{\pi}(\lambda, \varepsilon)\, \pi(\lambda, \varepsilon),$$

with

$$\begin{aligned} \widehat{\pi}(\lambda, \varepsilon) &= \det(\lambda I - \widehat{J} - \varepsilon \widetilde{B}_{22}) \\ \pi(\lambda, \varepsilon) &= \det(\lambda I - J - \varepsilon \widetilde{B}_{11} - \varepsilon^2 \widehat{S}(\lambda, \varepsilon)), \end{aligned}$$

where $\widehat{S}$ is the matrix $\widehat{S}(\lambda, \varepsilon) = \widetilde{B}_{12}(\lambda I - \widehat{J} - \varepsilon \widetilde{B}_{22})^{-1} \widetilde{B}_{21}$. If λ is an eigenvalue of $A(\varepsilon)$ close to λ_0, then it cannot be a root of the polynomial $\widehat{\pi}(\cdot, \varepsilon)$, so it must be a root of the rational function $\pi(\cdot, \varepsilon)$, which depends on $\widehat{J}$ only through terms of the second order in ε. Hence, $\widehat{J}$ has no influence whatsoever on the first order terms and it is sufficient to study $\det(\lambda I - J - \varepsilon \widetilde{B}_{11})$ to characterize the first order behavior. For a rigorous proof, more formal than this plausibility argument, see Lidskii's original paper [14, pp. 83-84] , or [1, § 3.9.1], where spectral projections on the appropriate invariant subspaces are used to completely decouple J from the influence of $\widehat{J}$ up to first order.

A second important simplification is that we may take $\lambda_0 = 0$ at our convenience, since the shift $A \to A - \lambda_0 I$ does not change either the Jordan block structure or the Jordan chains of A. Hence, all results below are invariant under that transformation and λ_0 may be set to zero.

With these simplifying assumptions, one can easily see that the particular form of the leading terms of the eigenvalue expansions will depend mainly on the Jordan structure of A. To see it for the leading exponents, let $J = P^{-1}AP$ be a Jordan form of A, where we assume, as explained above, that A has no eigenvalue other than $\lambda_0 = 0$. In that case, the characteristic polynomial $P(\lambda, \varepsilon) = \det(\lambda I - J - \varepsilon \widetilde{B})$ of $A(\varepsilon)$, with $\widetilde{B} = P^{-1}BP$, is a polynomial of the form (2). If the eigenvalue is semisimple, then J is zero and each $\alpha_k(\varepsilon)$ equals ε^k multiplied by a certain sum of k-dimensional principal minors of $\widetilde{B}$. In this case, the Newton diagram is formed by one single segment of slope $s = 1$. If the eigenvalue is not semisimple, some nontrivial Jordan block appears in J, so, besides the $O(\varepsilon^k)$ terms, each $\alpha_k(\varepsilon)$ contains lower order terms produced by the -1s appearing above the diagonal of J. This clearly shows that the effect of nontrivial Jordan blocks is to introduce in the Newton diagram line segments with slopes less than 1. The smallest possible slope corresponds to a nonderogatory eigenvalue (i.e. one single segment of slope $1/n$) and the largest possible one to the semisimple case. All possible Newton diagrams for the given multiplicity n lie between these two extremal segments.

This said, we are going to determine the *lowest possible Newton diagram compatible with the given Jordan structure* (8). We do it by fixing on the vertical axis an arbitrary height l, ranging between 1 and $f_q = g$. For each height l we look for the rightmost possible point $(k(l), l)$ which may be in the generating set Π of the Newton diagram for some suitable perturbation B. In other words, we are interested in

$$k(l) = \max\{k : \exists B \in \mathbb{C}^{n\times n} \text{ such that } a_k = l\}$$

as a function of $l \in \{1, \ldots, f_q\}$. The following theorem gives us the values of $k(l)$ for the exponents l which are relevant to our argument (recall that f_j is given by formula (10)).

Theorem 1. *For every $l \in \{1, \ldots, f_q\}$ the corresponding $k(l)$ is equal to the sum of the dimensions of the l largest Jordan blocks of J. More precisely, write $f_0 = 0$ and suppose $l = f_{j-1} + \rho$, for some $j = 1, \ldots, q$ and $0 < \rho \leq r_j$. Then,*

$$k(l) = r_1 n_1 + \ldots + r_{j-1} n_{j-1} + \rho n_j$$

and the coefficient of ε^l in $\alpha_{k(l)}$ is equal to $(-1)^l$ multiplied by the sum of all principal minors of Φ_j corresponding to submatrices of dimension l containing the upper left block Φ_{j-1} of Φ_j (if $j = 1$, all *principal minors of dimension l are to be considered). If, in particular, $l = f_j$ for some $j \in \{1, \ldots, q\}$, then the coefficient of ε^{f_j} in $\alpha_{k(f_j)}$ is $(-1)^{f_j} \det \Phi_j$.*

PROOF: As explained above, we may assume that $\lambda_0 = 0$ is the only eigenvalue of the matrix $A \in \mathbb{C}^{n\times n}$. First, recall that given a n by n matrix, the coefficient of λ^{n-k} in its characteristic polynomial is, except for a sign, the sum of all k-dimensional principal minors of the matrix. In our case $P(\lambda, \varepsilon) = \det(\lambda I - A(\varepsilon))$ and each $\alpha_k(\varepsilon)$ in (3) is, up to a sign factor, the sum of all k-dimensional principal minors of $J + \varepsilon\widetilde{B}$. Notice that the only elements of this matrix which are not of order ε are the ones occupying the positions of the 1s above the diagonal of J.

Now, given $l \in \{1, \ldots, f_q\}$, we must determine the largest possible k such that the sum of all k-dimensional principal minors is exactly of order ε^l.

In particular, the minor with the least order must be precisely of that order, implying that the corresponding principal submatrix must contain $k-l$ of the supradiagonal 1s. In other words, maximizing k for the given l is equivalent to including as many 1s as possible in the principal submatrices, while still keeping the order ε^l. Clearly, this is achieved by choosing the rows and columns where the 1s are from the l largest Jordan blocks in J.

If $l = f_j$ for some $j = 1, \ldots, q$, then there is only one way of choosing these blocks. Furthermore, one can easily check our claim on the coefficient of ε^{f_j} in $\alpha_{k(f_j)}$, since the leading term of $\alpha_{k(f_j)}$ is just the determinant of what is left from the k by k principal submatrix once the rows and columns of the chosen $k(f_j) - f_j$ supradiagonal 1s have been removed. The remaining matrix $f_j \times f_j$ matrix is precisely $\varepsilon\Phi_j$.

Finally, if $l = f_j + \rho$ with $\rho < r_{j+1}$, there is more than one way of choosing the l blocks: once the largest f_j Jordan blocks have been exhausted, each one corresponds to a different choice of ρ blocks among the r_{j+1} Jordan blocks of dimension n_{j+1}. As to the coefficient of ε^l, the argument goes much in the same way as above. □

As a consequence of Theorem 1, we conclude that *the lowest possible Newton diagram compatible with the Jordan structure (8) is the concatenation of the segments S_j, $j = 1, \ldots, q$ connecting the points $\mathcal{P}_{j-1}$ and $\mathcal{P}_j$, where $\mathcal{P}_j = (k(f_j), f_j)$ for each $j = 1, \ldots, q$.* This diagram, which has been called [16] the *Newton envelope* associated with the Jordan structure (8), is not only the lowest possible, but also the *most likely* diagram, since the actual diagram corresponding to a specific perturbation B will coincide with the Newton envelope unless some of the $\det \Phi_j$ vanishes, i.e. unless B satisfies an algebraic condition which confines it to an algebraic manifold (i.e. to a set of zero Lebesgue measure) in the set of matrices. In other words, the Newton envelope displays the *generic* behavior of the eigenvalues of A under perturbation, in the sense that it coincides with the Newton diagram in all situations except in those nongeneric cases in which the perturbation B causes one of the Φ_j to be singular. The next subsection is devoted to describe in detail this generic behavior.

3.1 The generic case: Lidskii's Theorem

We begin with the main result for generic perturbations, due to Lidskii [14] and, in a more restrictive version, to Vishik and Lyusternik [41].

Theorem 2 (Lidskii [14]). *Let $j \in \{1, \ldots, q\}$ be given, and assume that, if $j > 1$, Φ_{j-1} is nonsingular. Then there are $r_j n_j$ eigenvalues of the perturbed matrix $A + \varepsilon B$ admitting a first-order expansion*

$$\lambda_j^{kl}(\varepsilon) = \lambda_0 + (\xi_j^k)^{1/n_j}\, \varepsilon^{1/n_j} + o(\varepsilon^{1/n_j}) \qquad (12)$$

for $k = 1, \ldots, r_j$, $l = 1, \ldots, n_j$, *where*

(i) *the* ξ_j^k, $k = 1, \ldots, r_j$, *are the roots of equation*

$$\det(\Phi_j - \xi\, E_j) = 0 \tag{13}$$

or, equivalently, the eigenvalues of the Schur complement of Φ_{j-1} *in* Φ_j *(if* $j = 1$, *the* ξ_1^k *are just the* r_1 *eigenvalues of* Φ_1*);*

(ii) *the different values* $\lambda_j^{kl}(\varepsilon)$ *for* $l = 1, \ldots, n_j$ *are defined by taking the* n_j *distinct* n_j*-th roots of* ξ_j^k.

If, in addition, the r_j *solutions* ξ_j^k *of* (13) *are all distinct, then the eigenvalues* (12) *can be expanded locally in power series of the form*

$$\lambda_j^{kl}(\varepsilon) = \lambda_0 + (\xi_j^k)^{1/n_j}\, \varepsilon^{1/n_j} + \sum_{s=2}^{\infty} a_{js}^{kl}\, \varepsilon^{s/n_j}, \tag{14}$$

$k = 1, \ldots, r_j$, $l = 1, \ldots, n_j$.

PROOF: Once the Newton diagram technique is at hand, the proof is a consequence of Theorem 1. First, suppose that both Φ_{j-1} and Φ_j are nonsingular. Then, both $\mathcal{P}_{j-1}$ and $\mathcal{P}_j$ are in the set Π generating the Newton diagram, i.e. the segment S_j of slope $1/n_j$ connecting both points is one of the segments in the diagram (recall that no point (k, a_k) can lie below S_j). This gives us the leading exponent of expansion (12). The leading coefficient comes from carefully examining equation (5). One can check that

$$\sum_{(k,a_k)\in S_j} \mu^{n-k}\hat{\alpha}_k = \mu^{n-k(f_j)} \left[\mu^{n_j r_j}\hat{\alpha}_{k(f_{j-1})} + \sum_{t\in T} \hat{\alpha}_{k(f_j - t)}\mu^{t\, n_j} + \hat{\alpha}_{k(f_j)} \right] = 0, \tag{15}$$

where

$$T = \{t \in \{1, \ldots, r_j - 1\} : \mathcal{Q}_t = (k(f_j - t), f_j - t) \in \Pi\}, \tag{16}$$

i.e. T is the set of indices corresponding to the intermediate points $\mathcal{Q}_t \in \Pi$ eventually lying on S_j. Notice that bracketed expression in (15) depends on μ only through μ^{n_j}. Now, recall from Theorem 1 that for each $l = f_j - t$ with $t \in T$, the corresponding $\hat{\alpha}_{k(l)}$ is (up to the sign) the sum of all principal minors of Φ_j of dimension l containing Φ_{j-1}. This is precisely the way the coefficients of the powers of ξ are obtained in $\det(\Phi_j - \xi E_j)$. Hence, the nonzero solutions of (15) are solutions of $\det(\Phi_j - \mu^{n_j} E_j) = 0$ as well.

Now, suppose that Φ_j is singular. Then, the corresponding point $\mathcal{P}_j$ no longer belongs to the diagram, implying the loss of some of the expansions (12) or, equivalently, the loss of part of the segment S_j. Let β be either $\beta = r_j$ if no point $\mathcal{Q}_t = (k(f_j - t), f_j - t)$ is in Π or $\beta = \min T$ for T defined as in (16). In either case $\mathcal{Q}_\beta = (k(f_j - \beta), f_j - \beta)$ is the rightmost point of Π on S_j. Then, the part of S_j which remains on the Newton diagram is the segment connecting $\mathcal{P}_{j-1}$ with $\mathcal{Q}_\beta$. This accounts for $(r_j - \beta)n_j$ expansions (12), whose leading coefficients are, reasoning as above, the n_j-th roots of the $r_j - \beta$ nonzero solutions of equation (13). As to the βn_j remaining eigenvalues, they correspond to segments whose slope is strictly larger than $1/n_j$. Hence, expansion (12) is still valid, since they correspond to the β null solutions of equation (13).

As to expansion (14), one can check (see the original reference [14] or [16, § 2]) that, after the change of variables

$$z = \varepsilon^{1/n_j}$$
$$\mu = \frac{\lambda}{z}$$

and an appropriate scaling in z, the polynomial equation $P(\lambda, \varepsilon) = \det(\lambda I - A - \varepsilon B) = P(\mu, z) = 0$ has, for any given $z \neq 0$, the same roots as a new polynomial equation $Q(\mu, z) = 0$ with

$$Q(\mu, 0) = \pm\mu^\alpha \det(\Phi_j - \mu^{n_j} E_j)$$

for some suitable $\alpha \geq 0$. Hence, if all r_j roots of equation (13) are known to be distinct, the implicit function theorem can be applied to $Q(\mu, z) = 0$, implying that the $r_j n_j$ roots

$$\mu_j^{kl}(z) = (\xi_j^k)^{1/n_j} + o(1), \qquad k = 1, \ldots, r_j; \quad l = 1, \ldots, n_j,$$

of $Q(\mu(z), z) = 0$ for small enough z are analytic functions of $z = \varepsilon^{1/n_j}$. $\square$

Two special cases of Theorem 2 are well known. If λ_0 is semisimple, i.e. $q = n_1 = 1$ with multiplicity r_1, equation (12) reduces to

$$\lambda_1^{k1}(\varepsilon) = \lambda_0 + \xi_1^k \, \varepsilon + o(\varepsilon), \tag{17}$$

where the ξ_1^k are the eigenvalues of the r_1 by r_1 matrix $Y_1 B X_1$ (cf. [9, § II.2.3]). On the other hand, if λ is nonderogatory, i.e. $q = r_1 = 1$ with multiplicity n_1, equation (12) reduces to

$$\lambda_j^{1l}(\varepsilon) = \lambda_0 + (\xi_1^1)^{1/n_1} \, \varepsilon^{1/n_1} + o(\varepsilon^{1/n_1}),$$

where $\xi_1^1 = y_1^1 B x_1^1$. These two cases coincide when λ is simple.

Theorem 2 does not address either the convergence or the ultimate form of the $o(\varepsilon^{1/n_j})$ term in expansion (12), since these issues are beyond the reach of a purely algebraic tool like the Newton diagram. One can show by other means (see [1, § 9.3.1], [9, § II.1.2]) that whenever both Φ_{j-1} and Φ_j are nonsingular, the $r_j n_j$ eigenvalues (12) acn be written as convergent power series in the variable ε^{1/n_j}. This is no longer true if only Φ_{j-1} is nonsingular, unless some additional information, as in the last part of Theorem2, is available (see, for instance, the perturbation matrix (18) in Example 3 below, for which $\det \Phi_2 = 0$ and two out of the four eigenvalues corresponding to Φ_2 are of order $\varepsilon^{2/3}$)

However, one important special case deserves to be mentioned: if both the unperturbed matrix A and the perturbation matrix B are *normal*, then all eigenvalues of $A + \varepsilon B$ are analytic functions of ε [1, § 7.2], i.e. they have a convergent representation (14) with $n_j = 1$. This property will be crucial in Section 4, when dealing with the perturbation of singular values.

We conclude this subsection by referring the reader interested in eigenvector perturbation results to [14, Theorem 2], the eigenvector perturbation theorem in [14] analogous to the eigenvalue result above, which essentially amounts to replacing (14) in the eigenvalue-eigenvector equation $A(\varepsilon)v(\varepsilon) = \lambda(\varepsilon)v(\varepsilon)$ (the same result appears in [16] as Theorem 2.2).

3.2 Nongeneric perturbations

If the perturbation is nongeneric, i.e. when the matrix B is such that Φ_{j-1} is singular, Theorem 2 does not apply. The question is what can we say about the eigenvalues of $A + \varepsilon B$ in this case. The answer is: not much, at least in such a systematic way as in Theorem 2. Although the Newton diagram is still a powerful instrument which allows us to deal with each particular case, it is not easy to give a clear, global picture of the wide variety of different possible behaviors. To give an idea of the difficulties, consider the following example:

Example 3: Let $A \in \mathbb{R}^{8\times 8}$ be already in Jordan form,

$$A = J = \Gamma_1^1 \oplus \Gamma_2^1 \oplus \Gamma_2^2 \oplus \Gamma_3^1$$

with $n_1 = 3$, $n_2 = 2$, $n_3 = 1$ and $r_1 = r_3 = 1$, $r_2 = 2$, i.e.

$$A = \left[\begin{array}{ccc|cc|cc|c} 0 & 1 & 0 & & & & & \\ 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & & & & & \\ \hline & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ \hline & & & & & 0 & 1 & \\ & & & & & 0 & 0 & \\ \hline & & & & & & & 0 \end{array}\right].$$

According to Theorem 2, a generic perturbation, i.e. any 8 by 8 matrix

$$B = \left[\begin{array}{ccc|cc|cc|c} * & * & * & * & * & * & * & * \\ ** & * & * & * & * & * & * & * \\ *\square & * & * & \clubsuit & * & \clubsuit & * & \heartsuit \\ \hline * & * & * & * & * & * & * & * \\ \clubsuit & * & * & \clubsuit & * & \clubsuit & * & \heartsuit \\ \hline * & * & * & * & * & * & * & * \\ \clubsuit & * & * & \clubsuit & * & \clubsuit & * & \heartsuit \\ \hline \heartsuit & * & * & \heartsuit & * & \heartsuit & * & \heartsuit \end{array}\right]$$

with all three submatrices

$$\Phi_1 = \left[\ \square\ \right], \qquad \Phi_2 = \left[\begin{array}{c|cc} \square & \clubsuit & \clubsuit \\ \hline \clubsuit & \clubsuit & \clubsuit \\ \clubsuit & \clubsuit & \clubsuit \end{array}\right], \qquad \Phi_3 = \left[\begin{array}{c|cc|c} \square & \clubsuit & \clubsuit & \heartsuit \\ \hline \clubsuit & \clubsuit & \clubsuit & \heartsuit \\ \clubsuit & \clubsuit & \clubsuit & \heartsuit \\ \hline \heartsuit & \heartsuit & \heartsuit & \heartsuit \end{array}\right]$$

nonsingular gives rise to a Newton diagram coinciding with the Newton envelope. In that case, Π contains the points $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ and $\mathcal{P}_3$, with $\mathcal{P}_0 = (0,0)$, $\mathcal{P}_1 = (3,1)$, $\mathcal{P}_2 = (7,3)$ and $\mathcal{P}_3 = (8,4)$ (see Figure 2a below). Hence, the eigenvalues of A split typically into three eigenvalues of order $\varepsilon^{1/3}$, four eigenvalues of order $\varepsilon^{1/2}$ and one eigenvalue of order ε.

A good number of different possibilities arises whenever some of the matrices Φ_j turn out to be singular. As a first example, consider the perturbation matrix

$$B_1 = \left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right], \tag{18}$$

for which

$$\Phi_3 = \left[\begin{array}{c|cc|c} 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 \end{array}\right].$$

Both Φ_1 and Φ_3 are invertible, but Φ_2 is singular. In this case, the set Π contains the points $\mathcal{P}_0, \mathcal{P}_1$ and $\mathcal{P}_3$, together with the point $\mathcal{Q}_1 = (5,2)$. According to the Newton diagram, shown in Figure 2b, the matrix $A + \varepsilon B_1$

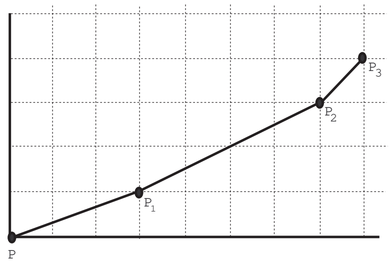

Figure 2a. *Newton diagram associated with the matrix $A + \varepsilon B$ for generic B in Example 3.*

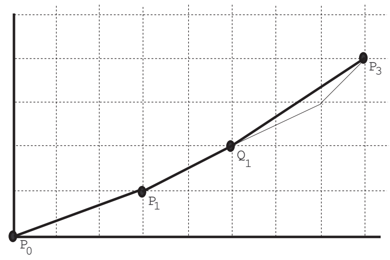

Figure 2b. *Newton diagrams associated with the matrix $A + \varepsilon B_1$ for the matrix B_1 in (18). The Newton envelope is shown as a dashed line, the Newton diagram as a solid one*

has three eigenvalues of order $\varepsilon^{1/3}$, two eigenvalues of order $\varepsilon^{1/2}$ and three eigenvalues of order $\varepsilon^{2/3}$.

Now, take the perturbation

$$B_2 = \left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \qquad (19)$$

producing

$$\Phi_3 = \left[\begin{array}{c|cc|c} 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right].$$

Again, both Φ_1 and Φ_3 are invertible, but now neither $\mathcal{P}_2$ nor $\mathcal{Q}_1$ are on the Newton diagram (see Figure 3a). Thus, the matrix $A + \varepsilon B_2$ has three eigenvalues of order $\varepsilon^{1/3}$ and five eigenvalues of order $\varepsilon^{3/5}$.

If we take as our next nongeneric perturbation

$$B_3 = \left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right], \tag{20}$$

so that

$$\Phi_3 = \left[\begin{array}{c|cc|c} 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

then both Φ_2 and Φ_3 are invertible, but Φ_1 is singular and the set Π contains $\{\mathcal{P}_0, \mathcal{Q}_1, \mathcal{P}_2, \mathcal{P}_3\}$. The corresponding Newton diagram is the one in Figure 3b, and $A + \varepsilon B_3$ has five eigenvalues of order $\varepsilon^{2/5}$, two eigenvalues of order $\varepsilon^{1/2}$ and one eigenvalue of order ε.

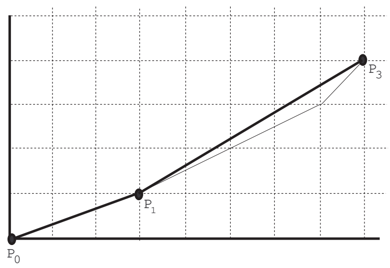

Figure 3a. Newton diagram associated with $A + \varepsilon\, B_2$ for the matrix B_2 in (19). The Newton envelope is shown as a thin solid line, the Newton diagram as a thick one

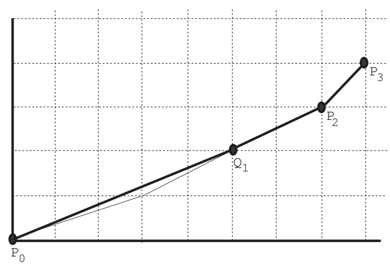

Figure 3b. Newton diagram associated with $A + \varepsilon\, B_3$ for the matrix B_3 in (20). The Newton envelope is shown as a thin solid line, the Newton diagram as a thick one

Now, consider

$$B_4 = \left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \tag{21}$$

with

$$\Phi_3 = \left[\begin{array}{c|cc|c} 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ \hline 2 & 0 & 1 & 0 \end{array}\right].$$

In this case, Φ_1 is singular, while Φ_2 and Φ_3 are not. The point $\mathcal{Q}_1$ is not in the set Π and the Newton diagram is the one in Figure 4a, so $A + \varepsilon B_4$ has seven eigenvalues of order $\varepsilon^{3/7}$ and one eigenvalue of order ε.

Finally, consider

$$B_5 = \left[\begin{array}{ccc|cc|cc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \tag{22}$$

with

$$\Phi_3 = \left[\begin{array}{c|cc|c} 0 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \end{array}\right].$$

In this case, both Φ_1 and Φ_2 are singular and Π contains $\mathcal{P}_0$ and $\mathcal{P}_3$. Again, the point $\mathcal{Q}_1$ does not belong to Π, so the Newton diagram is a single segment of slope $1/2$ (see Figure 4b) and $A + \varepsilon B_5$ has eight eigenvalues of order $\varepsilon^{1/2}$.

We conclude the example here, although the five instances above do not exhaust all possible behaviors: notice that Φ_3 has been assumed to be invertible in all four cases. Other possibilities arise if Φ_3 becomes singular, but we do not pursue the matter further for the sake of brevity. In any case, the example above

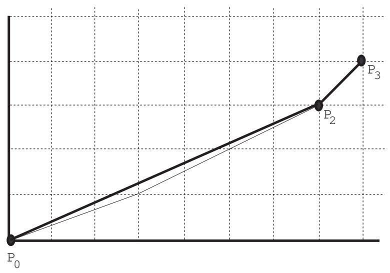

Figure 4a. *Newton diagram associated with $A + \varepsilon B_4$ for the matrix B_4 in (21). The Newton envelope is shown as a thin solid line, the Newton diagram as a thick one*

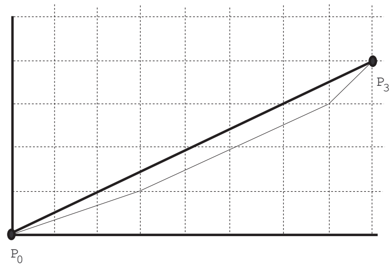

Figure 4b. *Newton diagram associated with $A + \varepsilon B_5$ for the matrix B_5 in (22). The Newton envelope is shown as a thin solid line, the Newton diagram as a thick one*

displays the richness of different behaviors to be expected from nongeneric perturbations to matrices with nontrivial Jordan forms.

In fact, the nongeneric case presents a very important additional difficulty which is absent for the generic case: namely that one can no longer make the simplifying assumption of disregarding the part $\widehat{J}$ of the Jordan canonical form corresponding to eigenvalues other than λ_0. The following example illustrates this:

Example 4: Consider the matrix

$$A = \left[\begin{array}{c|c|c} 0 & & \\ \hline & 0 & \\ \hline & & 1 \end{array}\right] \in \mathbb{R}^{3\times 3}, \tag{23}$$

which is already in Jordan form, but having two different eigenvalues, and take the perturbation

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{24}$$

Obviously, if we disregard the part corresponding to the nonzero eigenvalue 1, the perturbed matrix would reduce to $\begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$ which still has two zero eigenvalues. However, the characteristic polynomial of the 3 by 3 matrix $A + \varepsilon B$ is $P(\lambda, \varepsilon) = \lambda^3 - \lambda^2 - \varepsilon^3$. The corresponding Newton diagram, in Figure 6 below, shows that $A + \varepsilon B$ has one eigenvalues of the order of unity, and two eigenvalues of order $\varepsilon^{3/2}$.

Some attempts have been made to analyze the behavior for certain specific classes of nongeneric perturbations. Some preliminary results may be found

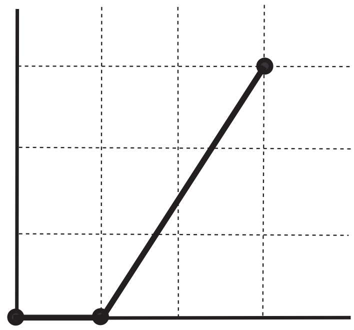

Figure 6. Newton diagram associated with the matrix $A + \varepsilon B$ for the matrices defined in (23) and (24).

in [16, § 3], but the first systematic description of structured perturbations is obtained in Ma and Edelman [15] for upper k-Hessenberg perturbations of Jordan blocks. Following the ideas both in [3] and [16], they show that if a n by n Jordan block is perturbed by an upper k-Hessenberg matrix (i.e. a matrix having the k subdiagonals closest to the main diagonal not zero), then the eigenvalue splits typically into $\lfloor n/k \rfloor$ eigenvalues of order $\varepsilon^{1/k}$ and, if k does not divide n, into r additional eigenvalues of order $\varepsilon^{1/r}$, where $r \equiv n$ mod k. The leading coefficients are also characterized. Again, as in Lidskii's theorem, the typical behavior (in this non-typical situation) is given by certain algebraic conditions which in this case depend only on the elements of the k-th subdiagonal of B.

More recently, Jeannerod [8] has extended Lidskii's results by obtaining explicit formulas for both the leading exponents and leading coefficients of the Puiseux expansions of the eigenvalues of *analytic* perturbations $J + B(\varepsilon)$ of a Jordan matrix J, provided the powers of ε in the perturbation matrix $B(\varepsilon)$ conform in a certain way to the Jordan structure given by J. To be more precise, the least power of ε in either of any two columns of B corresponding to a same Jordan block of J must be the same. Lidskii's Theorem 2 is recovered when that least power is ε for all blocks, but certain nongeneric cases are covered in this more general setting.

Nonetheless, in both cases [15, 8] the particular structure of the perturbations to the Jordan blocks is not preserved by undoing the change of basis leading to the Jordan form. Hence, not much information is provided for nongeneric perturbations of *arbitrary* matrices. Therefore, it is safe to say that, so far, the problem of categorizing all possible behaviors of the eigenvalues of $A + \varepsilon B$ as a function of the perturbation matrix B is a very difficult open problem, which is still quite far from being solved.

However, there is still another specific class of nongeneric perturbations for which quite some information can be obtained

3.2.1 Low rank perturbations. A relevant class of nongeneric perturbations are perturbations of low rank, which frequently arise when a system has to be controlled using less parameters than the number of degrees of freedom defining the system [40]. We focus on how do the eigenvalues and the Jordan structure of a matrix change when it is perturbed by a matrix of low rank, with particular attention to how many Jordan blocks are destroyed for each eigenvalue.

Although some of the used techniques have a similar flavor to the Newton diagram technique, an important difference with the rest of the paper has to be highlighted: *the perturbations studied here are not infinitesimal. Thus we will be able to obtain absolute, instead of first order, results holding for perturbations of any magnitude.* This said, we begin with an example which illustrates how a small rank perturbation typically behaves.

Example 5: Let us denote by $J_s(\lambda_0)$ a Jordan block of dimension s corresponding to the eigenvalue λ_0. Consider the following matrix

$$\begin{aligned} J &= J_3(1) \oplus J_3(1) \oplus J_3(1) \oplus J_1(1) \oplus J_3(-1) \oplus J_2(-1) \oplus J_2(-1) \oplus \\ &\quad J_3(2) \oplus J_3(2). \end{aligned}$$

We have generated 200 random matrices B with rank 2 and computed the Jordan canonical form [2] of $A+B$. The maximum and minimum values obtained for the ratios $\|B\|/\|J\|$ have been 305 and 130, i.e. the matrices $A+B$ are not by any means small perturbations of A. Surprisingly enough, in 171 out of the 200 cases analysed the Jordan canonical form of $A+B$ had one block $J_3(1)$, one block $J_1(1)$ and one block $J_2(-1)$. The rest of the blocks were one-dimensional blocks corresponding to eigenvalues different from $1, -1, 2$. This experiment suggests the following *generic* behavior: *If λ_0 is an eigenvalue of A having g (geometric multiplicity) Jordan blocks and B is a matrix with rank $m < g$, then λ_0 is also an eigenvalue of $A+B$ having exactly $g-m$ Jordan blocks equal to the $g-m$ smallest Jordan blocks of λ_0 in A*, i.e., when perturbing a matrix by a generic rank m perturbation, the m largest Jordan blocks for each eigenvalue dissapear and the rest of the Jordan blocks remain. This “usual" behavior has been observed in many other experiments, but notice that it is easy to build

[2] This was done with the command `'jordan'` of the Symbolic Math Toolbox in MATLAB 5.3. To work properly this command requires the elements of the matrix argument to be integers or ratios of small integers. Therefore, the random matrices B had moderately large integer elements, and are not as random as one might think. This is the reason why about 15% of the generated matrices did not follow the generic behavior described in the example.

m-rank perturbations which do not follow this rule. If $m \geq g$, the eigenvalue λ_0 is typicaly no longer an eigenvalue of $A+B$, as expected.

Now we will try to understand this generic behavior. The next Lemma gives necessary and sufficient algebraic conditions for it.

Lemma 1. *Let A be a matrix with Jordan form (6), i.e. having an eigenvalue λ_0 with Jordan blocks of dimensions $n_1 > n_2 > \ldots > n_q$ repeated $r_1, r_2, \ldots, r_q$ times. Let $m \in \{1, \ldots, g\}$ be an integer number, where g is the geometric multiplicity (9). Suppose $m = f_{j-1} + \rho$ for some $j = 1, \ldots, q$, $0 < \rho \leq r_j$ with $f_0 = 0$. Then λ_0 is an eigenvalue of the matrix $A+B$ with Jordan blocks of dimensions $n_j > n_{j+1} \ldots > n_q$ repeated $r_j - \rho, r_{j+1}, \ldots, r_q$ times (if $\rho = r_j$ there are no blocks of dimension n_j) if and only if*

a) $\operatorname{rank}(A+B-\lambda_0 I)^k = \operatorname{rank}(A-\lambda_0 I)^k + k\,m$, *for all* $k = 1, 2, \ldots, n_j$, *and*

b) $\operatorname{rank}(A+B-\lambda_0 I)^k = \operatorname{rank}(A+B-\lambda_0 I)^{n_j}$ *for all* $k \geq n_j$.

PROOF: Just remember [7, pag. 127] that for an arbitrary matrix C the number $d_k(C, \lambda) = \operatorname{rank}(C-\lambda I)^{k-1} - \operatorname{rank}(C-\lambda I)^k$ is the total number of Jordan blocks of C of dimensions greater than or equal to k associated with the eigenvalue λ. Therefore $d_k(C, \lambda) - d_{k+1}(C, \lambda)$ is the number of Jordan blocks of C associated to λ with dimension equal to k. Thus Conditions a) and b) of the Lemma are equivalent to $d_k(A+B, \lambda_0) = d_k(A, \lambda_0) - m$ for $k = 1, \ldots, n_j$ and $d_k(A+B, \lambda_0) = 0$ for $k > n_j$. The rest is trivial. □

Notice that Lemma 1 does not force B to be a matrix of rank m. In fact, our goal is to justify why most matrices of rank m fulfill the previous equations a) and b) on the ranks. The following plausibility argument shows, under mild restrictions on a matrix B with rank $m = f_{j-1} + \rho$, that something very unlikely has to happen for conditions a) and b) not to be fulfilled. In the first place, notice that for any k the matrix $(A-\lambda_0 I+B)^k$ can be expanded as a sum of $(A-\lambda_0 I)^k$ plus products of two matrices, with the left factor being of the form $(A-\lambda_0 I)^s B$ with s ranging from 0 to $k-1$. Secondly, elementary linear algebra results show that if $\operatorname{col}(B) \cap \ker(A-\lambda_0 I)^{n_j-1} = \{0\}$ (here col and ker stand for column and null spaces respectively) then $\operatorname{rank}(A-\lambda_0 I)^s B = \operatorname{rank}(B) = m$ for all $s = 0, 1, \ldots, n_j - 1$. Therefore, the equalities of item a) in Lemma 1 hold unless some "rank cancellation" occurs. It is well known that rank cancellations are very uncommon when the involved matrices have small rank, as in our case. With regard to the assumption $\operatorname{col}(B) \cap \ker(A-\lambda_0 I)^{n_j-1} = \{0\}$, it is really very mild, since $\dim\ \operatorname{col}(B) = m$ and $\dim\ \ker(A-\lambda_0 I)^{n_j-1} \leq a-m$. Hence, it follows that the set of matrices B of rank m not satisfying this condition has zero Lebesgue measure.

No need to say that the previous argument does not prove any rigorous result, but it sugests that the problem of understanding why the behavior described in

Example 5 is the most likely one is a problem of determining the Lebesgue measure of the set of matrices B of rank m not fulfilling the conditions of Lemma 1. The most convenient way to deal with this problem is to rewrite conditions a), b) in a simpler but equivalent form. All of these questions are beyond the scope of this survey because different techniques than the ones used here are required. However an argument much in the same spirit of the Newton diagram approach allows us to prove the following theorem which is an important step towards the complete solution of this question. More on this will be said in [17].

Theorem 3. *Let A be a $n \times n$ matrix with Jordan form (6), i.e. having an eigenvalue λ_0 with Jordan blocks of dimensions $n_1 > n_2 > \ldots > n_q$ repeated $r_1, r_2, \ldots, r_q$ times, and algebraic and geometric multiplicities given by (9). Let B be a $n \times n$ matrix with rank $m = f_{j-1} + \rho$ for some $j = 1, \ldots, q$, $0 < \rho \leq r_j$ with $f_0 = 0$. Then the characteristic polynomial of $A + B$ is of the form*

$$p(\lambda) = (\lambda - \lambda_0)^{\tilde{a}}\, t(\lambda - \lambda_0),$$

where

$$\tilde{a} = (r_j - \rho)n_j + r_{j+1}n_{j+1} + \ldots + r_q n_q \tag{25}$$

and $t(\lambda - \lambda_0)$ is a monic polynomial of degree $n - \tilde{a}$. Moreover the constant coefficient of t is

$$t(0) = (-1)^{m+n-a}\, C_0 \, \det(\widehat{Q}(A+B)\widehat{P} - \lambda_0 I), \tag{26}$$

where $\widehat{Q}$, $\widehat{P}$ are as in (6) and C_0 is the sum of all principal minors of Φ_j corresponding to submatrices of dimension m containing the upper left block Φ_{j-1} of Φ_j (if $j = 1$, all *principal minors of dimension m are to be considered). If, in particular, $m = f_j$ for some $j \in \{1, \ldots, q\}$, then C_0 is simply* $\det \Phi_j$.

PROOF: This proof resembles that of Theorem 1, but is more involved, since B is not a small perturbation of A. Thus, the simplifying assumption that A has only one eigenvalue cannot be made and the block $\widehat{J}$ of the Jordan canonical form of A containing the eigenvalues different from λ_0 plays also a role. Similarly, all four blocks of the matrix $\widetilde{B}$ defined in (11) have to be taken into account.

We begin by writing the characteristic polynomial of $A + B$ as

$$p(\lambda) = \det((\lambda - \lambda_0)I - \mathrm{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) - \widetilde{B}).$$

For the sake of simplicity we define $\tilde{\lambda} \equiv \lambda - \lambda_0$ and $p_0(\tilde{\lambda}) \equiv p(\lambda)$, so the coefficient of $\tilde{\lambda}^{n-k}$ in $p_0(\tilde{\lambda})$ is $(-1)^k$ times the sum of all k-dimensional principal minors of $\mathrm{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) + \widetilde{B}$. Notice that all principal minors having more than m rows containing *only* elements of $\widetilde{B}$ are zero because

$\mathrm{rank}(\widetilde{B}) = \mathrm{rank}(B) = m$. This simple remark is the key to prove the Theorem. The next step is to find the largest dimension of principal minors having m rows which contain only elements of $\widetilde{B}$. If the dimension of these minors is denoted by $k_{\max}$ then

$$p_0(\tilde{\lambda}) = \tilde{\lambda}^{n-k_{\max}}\, t(\tilde{\lambda}),$$

with t a monic polynomial of degree $k_{\max}$.

Let α be an index set included in $\{1, 2, \ldots, n\}$ and denote by $(\mathrm{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) + \widetilde{B})(\alpha, \alpha)$ the principal matrix of $\mathrm{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) + \widetilde{B}$ that lies in the rows and columns indexed by α. In order to construct the largest principal minors having m rows with only elements of $\widetilde{B}$, the set α has to be of the form

$$\alpha = \{i_1, \ldots, i_s, a+1, a+2, \ldots, n\} \quad \text{with} \quad 1 \leq i_1 < i_2 < \ldots < i_s \leq a,$$

because the diagonal elements of $\widehat{J} - \lambda_0 I + \widetilde{B}_{22}$, which have indices $a + 1, a+2, \ldots, n$, are all of them different from the diagonal elements of $\widetilde{B}_{22}$ and the largest admissible size is desired. Suppose now that $i_1, \ldots, i_s$ are chosen among the indices corresponding to r Jordan blocks of $J - \lambda_0 I$. The row with the largest index chosen from a given block, say the i_b-th row, contributes to the principal minor only with elements of $\widetilde{B}$, either because it is the bottom row of the block or because $i_b + 1$ does not belong to α, and thus the element in the position $(i_b, i_b + 1)$ where $J - \lambda_0 I$ has a superdiagonal 1 is not in the minor. This imposes the restriction $r \leq m$ on r. Hence, to obtain the maximum number of elements in α, i.e $k_{\max}$, the indices $i_1 < \ldots < i_s$ have to correspond to a set of m complete largest Jordan blocks of $J - \lambda_0 I$. The number of possible choices is $r_j!/(\rho!(r_j - \rho)!)$, which is simply one when $m = f_j$. In any case

$$k_{\max} = r_1 n_1 + \ldots + r_{j-1} n_{j-1} + \rho n_j + n - a,$$

$\tilde{a} = n - k_{\max}$ which is equation (25).

Now we prove (26). Remember that $t(0)$ is $(-1)^{k_{\max}}$ times the sum of all $k_{\max}$-dimensional principal minors of $\mathrm{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) + \widetilde{B}$. Moreover the only non-zero $k_{\max}$-dimensional principal minors are of the kind described in the previous paragraph. Consider one of these minors and call it M. Let us denote by $1 = j_1 < j_2 < \ldots < j_h$ ($h = k_{\max} - (n - a) - m$) the indices of the rows of this minor where $J - \lambda_0 I$ has superdiagonal 1s. The j_k-th row of this minor is the sum of two rows: one is the $j_k + 1$-th row e_{j_k+1} of the identity matrix, the other a piece of a row of $\widetilde{B}$. Using this fact, we can expand M as a sum of determinants whose j_k-th row is either e_{j_k+1} or a row with only elements of $\widetilde{B}$. With the exception of the determinant with all the vectors $e_{j_1+1}, e_{j_2+1}, \ldots, e_{j_h+1}$, the rest of these determinants are zero because each contains more than m rows with elements of $\widetilde{B}$. The cofactor expansion of the remaining determinant along the rows $1 = j_1 < j_2 < \ldots < j_h$ leads to a value for M equal to $(-1)^h \det(\widehat{J} - \lambda_0 I + \widetilde{B}_{22})$ times a minor of Φ_j corresponding to a submatrix of dimension m containing the upper left block Φ_{j-1}. Extending this argument to all non-zero $k_{\max}$-dimensional principal minors of $\operatorname{diag}(J - \lambda_0 I, \widehat{J} - \lambda_0 I) + \widetilde{B}$ leads to (26). □

Notice that Theorem 3 is valid for any matrix B of rank m. If the following two additional restrictions are imposed on B: $t(0) \neq 0$ and $\operatorname{rank}(A - \lambda_0 I + B) = \operatorname{rank}(A - \lambda_0 I) + m$, then $A + B$ has m Jordan blocks less than A for λ_0, and the sum of the dimensions of the remaining blocks is precisely the sum of the dimensions of the $g - m$ smallest Jordan blocks of A for λ_0. It only remains to prove that changes between the dimensions of theses smallest blocks do not happen. This seems intuitively clear because the rank of B has already been used in imposing $t(0) \neq 0$. More on this will be said in [17].

In any case, the question raised in this section is, as far as we know, a new one in the literature. The only two references we are aware of [31, 32] are still unpublished work. In [31] the problem of rank one perturbations is adressed and it is proved that the condition $t(0) \neq 0$ is necessary and sufficient for $A + B$ to have one Jordan block less than A for λ_0, and for the dimensions of the remaining blocks to be precisely the dimensions of the $g-1$ smallest Jordan blocks of A for λ_0. A similar behavior appears when adding one new row and one new column to a given matrix. This has been studied in [32, Section 2].

4. First order perturbation bounds for singular values

Given an arbitrary matrix $A \in \mathbb{C}^{m \times n}$, one can take advantage of its singular values being eigenvalues of an associated Hermitian matrix in order to obtain first order singular value perturbation results via Theorem 2. The key is using the so-called *Jordan-Wielandt matrix*

$$C = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)} \tag{27}$$

associated with A. One can easily check [35, § I.4.1] that if $m \geq n$ and $A = U\Sigma V^*$ is a singular value decomposition with $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m\times n}$, then the Hermitian matrix C has $2n$ eigenvalues $\pm\sigma_1, \ldots, \pm\sigma_n$ with corresponding normalized eigenvectors

$$\frac{1}{\sqrt{2}} \left[\frac{u_i}{\pm v_i} \right],$$

where u_i is the i-th column of U and v_i the i-th column of V. In addition, C has $m-n$ zero eigenvalues with eigenvectors $\left[\begin{array}{c|c} u_i^T & 0 \end{array}\right]^T$, $i = n+1, \ldots, m$.

If the matrix A is perturbed to $A(\varepsilon) = A + \varepsilon B$ as in section 3, the Jordan-Wielandt matrix C is correspondingly perturbed to $C(\varepsilon) = C + \varepsilon D$ with

$$D = \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}, \tag{28}$$

i.e. to the Jordan-Wielandt matrix of $A(\varepsilon)$ provided the perturbation parameter ε is real. Recall that both C and D are Hermitian and, consequently, normal, so according to our remark right after the proof of Theorem 2, the eigenvalues of $C(\varepsilon)$ are analytic expansions of ε of the form (14) with $n_j = 1$. Of course, this is not necessarily true for the singular values of $A(\varepsilon)$, due to the nonnegativity restriction. However, the loss of analyticity can only be caused by some transversal crossing $\sigma(0) = 0$ of the two eigenvalues $\lambda(\varepsilon) = \pm\sigma(\varepsilon)$ of $C(\varepsilon)$ through the ε-axis at $\varepsilon = 0$. Thus, to recover the singular values all we have to do is to take the *nonnegative* branch. In other words, to obtain the leading term of a singular value expansion of $A(\varepsilon)$ we compute the expansion of the eigenvalue $\lambda(\varepsilon) = \xi\,\varepsilon + O(\varepsilon^2)$ of $C(\varepsilon)$. Then, the corresponding singular value of $A(\varepsilon)$ is just $\sigma(\varepsilon) = |\xi|\varepsilon + O(\varepsilon^2)$.

Some results on singular value perturbation expansions have been obtained by Stewart [34] via the ε-expansions for the eigenvalues $\sigma(\varepsilon)^2$ of $A(\varepsilon)^*A(\varepsilon)$. Sun, on the other hand, deals in [36] with the case of simple nonzero singular values, while the case of zero and multiple singular values is treated in [37]. Both cases are completely described in the following theorem, which, to our knowledge, is new.

Theorem 4. *Let $A \in \mathbb{C}^{m\times n}$, $m \geq n$ be a matrix of rank r, and let σ_0 be a singular value of A.*

If $\sigma_0 > 0$, let $U_0 \in \mathbb{C}^{m\times k}$ and $V_0 \in \mathbb{C}^{n\times k}$ be matrices whose columns span simultaneous bases of the respective left and right singular subspaces of A associated with σ_0. Then, for any $B \in \mathbb{C}^{m\times n}$, the matrix $A(\varepsilon) = A + \varepsilon B$ has k singular values analytic in ε which can be expanded as

$$\sigma_j(\varepsilon) = \sigma_0 + \xi_j\ \varepsilon + O(\varepsilon^2), \tag{29}$$

where the ξ_j, $j = 1, \ldots, k$ are the eigenvalues of the matrix

$$\frac{1}{2}(U_0^* B V_0 + V_0^* B^* U_0).$$

If $\sigma_0 = 0$ let

$$A = \left[\begin{array}{c|c|c} U_r & U_0 & U_z \end{array}\right] \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \left[\begin{array}{c} V_r^* \\ \hline V_0^* \end{array}\right]$$

be a singular value decomposition of A with $U_r \in \mathbb{C}^{m\times r}$, $U_0 \in \mathbb{C}^{m\times(n-r)}$, $U_z \in \mathbb{C}^{m\times(m-n)}$, $V_r \in \mathbb{C}^{n\times r}$ and $V_0 \in \mathbb{C}^{n\times(n-r)}$. Then, for any $B \in \mathbb{C}^{m\times n}$, the matrix $A(\varepsilon) = A + \varepsilon B$ has $n-r$ singular values analytic in ε which can be expanded as

$$\sigma_j(\varepsilon) = \xi_j\ \varepsilon + O(\varepsilon^2), \tag{30}$$

where the ξ_j, $j = 1, \ldots, k$ are the singular values of the $(m-r)\times(n-r)$ matrix

$$\left[\begin{array}{c} U_0^* \\ \hline U_z^* \end{array}\right] B\, V_0.$$

PROOF: As previously observed, we view the singular values of $A(\varepsilon) = A + \varepsilon B$ as the nonnegative eigenvalues of its Jordan-Wielandt matrix $C(\varepsilon) = C + \varepsilon D$ with C and D given, respectively, by (27) and (28).

In the simplest case when the singular value σ_0 is not zero, the columns of

$$\frac{1}{\sqrt{2}}\left[\begin{array}{c} U_0 \\ \hline V_0 \end{array}\right] \in \mathbb{C}^{(m+n)\times k}$$

form an orthonormal basis of the space of eigenvectors associated with the semisimple eigenvalue σ_0 of the Hermitian matrix C. Hence, formula (17) with $Y_1 = X_1^*$ applied to the perturbed matrix $C + \varepsilon D$ leads to expansion (29).

The situation is slightly more complicated when $\sigma_0 = 0$, since one has to keep track of the $m-n$ additional null eigenvalues of C: the multiplicity of zero as an eigenvalue of C is now $m - n + 2(n-r) = m + n - 2r$, and the columns of

$$\mathcal{Z} = \frac{1}{\sqrt{2}}\begin{bmatrix} U_0 & U_0 & \sqrt{2}U_z \\ V_0 & -V_0 & 0 \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n-2r)}$$

form an orthonormal basis of the null space of C. Again, applying Theorem 2 to the perturbation $C + \varepsilon D$ leads to the expansion (30), where the ξ_j are the nonnegative eigenvalues of the matrix

$$\begin{aligned} \mathcal{M} &= \mathcal{Z}^*\begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}\mathcal{Z} \\ &= \frac{1}{2}\begin{bmatrix} M_0 + M_0^* & M_0 - M_0^* & \sqrt{2}M_z \\ M_0^* - M_0 & -(M_0 + M_0^*) & -\sqrt{2}M_z \\ \sqrt{2}M_z^* & -\sqrt{2}M_z^* & 0 \end{bmatrix}, \end{aligned} \tag{31}$$

with $M_0 = V_0^* B^* U_0 \in \mathbb{C}^{(n-r)\times(n-r)}$ and $M_z = V_0^* B^* U_z \in \mathbb{C}^{(n-r)\times(m-n)}$. The proof is completed once we realize that the matrix $\mathcal{M}$ in (31) is unitarily similar to the Jordan-Wielandt matrix of

$$\widetilde{M} = \left[\begin{array}{c|c} M_0 & M_z \end{array}\right] = V_0^* B^* \left[\begin{array}{c|c} U_0 & U_z \end{array}\right].$$

One can easily check that the unitary matrix

$$\mathcal{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} I_{n-r} & I_{n-r} & 0 \\ -I_{n-r} & I_{n-r} & 0 \\ 0 & 0 & \sqrt{2} I_{m-n} \end{bmatrix}$$

satisfies that

$$\mathcal{Q}^* \mathcal{M} \mathcal{Q} = \begin{bmatrix} 0 & \widetilde{M} \\ \widetilde{M}^* & 0 \end{bmatrix},$$

i.e. $m - n$ eigenvalues of $\mathcal{M}$ are zero, and the remaining $2(n - r)$ ones are plus/minus the $n - r$ singular values of $\widetilde{M}$. □

5. First order perturbation bounds for generalized eigenvalue problems

In this final section we will just hint the close connections of the results in section 3 with their natural extension for generalized eigenvalue problems obtained by Najman [18, 19], and Langer and Najman [10, 11, 12] in a series of papers devoted to first order eigenvalue perturbation theory for perturbed analytic matrix functions. We focus on these results not only because of their great generality, but also because the main tool in their proofs is the Newton diagram technique, the common umbrella for most of the results in the present survey. This reinforces the resemblance of both the content of the results and the leading ideas in their proofs.

Following the presentation of Langer and Najman, we will analyze the behavior of the eigenvalues $\lambda(\varepsilon)$ of a square $n \times n$ matrix function

$$A(\lambda) + B(\lambda, \varepsilon) \tag{32}$$

for small ε in the neighborhood of an eigenvalue λ_0 of the unperturbed matrix function $A(\lambda)$, i.e. a value such that $\det A(\lambda_0) = 0$. The matrix function is assumed to be analytic around λ_0 with $\det A(\lambda) \not\equiv 0$. The perturbation $B(\lambda, \varepsilon)$ is assumed to be analytic in a neighborhood of $(\lambda_0, 0)$ with $B(\lambda, 0) = 0$ for every λ.

This general framework includes most of the usual spectral problems: taking $A(\lambda) = A - \lambda I$, $B(\lambda, \varepsilon) = \varepsilon B$ leads to the standard eigenvalue problem, the choice $A(\lambda) = \lambda^2 M + \lambda C + K$ with Hermitian positive definite matrices M, K and positive semidefinite C corresponding to perturbed quadratic matrix polynomials appearing in vibrational systems. Many other perturbed generalized eigenvalue problems can be fitted into this framework.

Najman began studying in [18] the case when $A(\lambda)$ is Hermitian and $B(\lambda, \varepsilon) = \varepsilon B(\lambda)$, not necessarily Hermitian. His results were an extension of those in Gohberg, Lancaster and Rodman [5] for the case when both $A(\lambda)$ and

$B(\lambda)$ are Hermitian and $B(\lambda)$ is positive definite. The results in [18] were precised and improved in [10], which already dealt with the general formulation (32). Reference [10] describes the *generic* behavior of the eigenvalues of (32) in the sense of Theorem 2 above, i.e. the behavior which is to be expected unless the perturbation lies on a certain algebraic manifold of zero measure related both with the perturbation and the spectral structure of A. Further work was devoted to exploring nongeneric cases [11, 12], as well as to the study of certain specific quadratic matrix polynomials appearing in damped vibrational systems [13, 19].

We will only state the analogous of Lidskii's Theorem 2 as stated in [11]. As in section 3, this requires some preliminary notations. Also as in section 3 we simplify the presentation *by assuming henceforth that* $\lambda_0 = 0$: let $A(\lambda)$ be an analytic $n \times n$ matrix function with $\det\, A(0) = 0$ and $\det\, A(\lambda) \not\equiv 0$. The *geometric multiplicity* of $\lambda_0 = 0$ is

$$g = \dim \ker\, A(0).$$

To define its partial multiplicities we make use of the Smith local form: one can show (see [20]) that there exist $n \times n$ matrix functions $E(\lambda)$, $F(\lambda)$ analytic and invertible close to $\lambda_0 = 0$ such that

$$A(\lambda) = E(\lambda) D(\lambda) F(\lambda), \tag{33}$$

with $D(\lambda) = \text{diag}(\lambda^{\nu_1}, \ldots, \lambda^{\nu_n})$. Notice that since the ranks of $A(0)$ and $D(0)$ coincide, $n - g$ exponents ν_i must be zero. With no loss of generality we may assume that

$$D(\lambda) = \text{diag}(\lambda^{n_1}, \ldots, \lambda^{n_1}, \lambda^{n_2}, \ldots, \lambda^{n_2}, \ldots, \lambda^{n_q}, \ldots, \lambda^{n_q}, 1, \ldots, 1),$$

where $n_1 < n_2 < \ldots < n_q$, each exponent n_j is repeated r_j times for $j = 1, \ldots, q$, and $r_1 + \ldots + r_q = g$, the geometric multiplicity of $\lambda_0 = 0$. The exponents n_j are called the *partial multiplicities* for $\lambda_0 = 0$ and

$$a = \sum_{j=1}^{q} r_j n_j$$

is its *algebraic multiplicity* as an eigenvalue of $A(\lambda)$. As in section 3, we define the auxiliary quantities

$$f_j = r_1 + \ldots + r_j, \quad j = 1, \ldots, q.$$

Now, consider the perturbed matrix function $A(\lambda) + B(\lambda, \varepsilon)$, with $B(\lambda, \varepsilon)$ analytic around $(0, 0)$ and $B(\lambda, 0) \equiv 0$. Then , using (33), $\lambda(\varepsilon)$ is an eigenvalue of $A(\lambda) + B(\lambda, \varepsilon)$ if and only if it is an eigenvalue of $D(\lambda) + \widehat{B}(\lambda, \varepsilon)$

for $\widehat{B}(\lambda,\varepsilon) = E(\lambda)^{-1}B(\lambda,\varepsilon)F(\lambda)^{-1}$. Now, partition

$$D(\lambda) = \begin{bmatrix} D_1(\lambda) & 0 \\ 0 & I \end{bmatrix}, \qquad \widehat{B}(\lambda,\varepsilon) = \begin{bmatrix} \widehat{B}_{11}(\lambda,\varepsilon) & \widehat{B}_{12}(\lambda,\varepsilon) \\ \widehat{B}_{21}(\lambda,\varepsilon) & \widehat{B}_{22}(\lambda,\varepsilon) \end{bmatrix},$$

where both $D_1(\lambda)$ and $\widehat{B}_{11}(\lambda,\varepsilon)$ are $g \times g$, and denote

$$H = \frac{\partial \widehat{B}_{11}}{\partial \varepsilon}(0,0). \tag{34}$$

For each $j = 1,\ldots,q,\ k = 1,\ldots,r_j - 1$ we define Δ_{jk} as the sum of all the $(g - f_j + k)$-dimensional principal minors of the submatrix $H(\alpha_{j-1},\alpha_{j-1})$ of H which contain the submatrix $H(\alpha_j,\alpha_j)$, where

$$\alpha_{j-1} = \{f_{j-1}+1, f_{j-1}+2,\ldots,g\}, \qquad \alpha_j = \{f_j+1, f_j+2,\ldots,g\}$$

and we have used the same notation as in the proof of Theorem 3 to represent submatrices. We also define

$$\Delta_j = \det\, H(\alpha_{j-1},\alpha_{j-1})$$

for each $j = 1,\ldots,q$, and for convenience we set $\Delta_{q+1} = 1$.

Theorem 5 (Langer & Najman [11]). *If for some $j \in \{1,\ldots,q\}$ both Δ_j and Δ_{j+1} do not vanish, then there are $r_j n_j$ eigenvalues of the perturbed matrix polynomial $A(\lambda) + B(\lambda,\varepsilon)$ near $\lambda_0 = 0$ admitting a first-order expansion*

$$\lambda_j^{kl}(\varepsilon) = (\xi_j^k)^{1/n_j}\,\varepsilon^{1/n_j} + o(\varepsilon^{1/n_j}) \tag{35}$$

for $k = 1,\ldots,r_j$, $l = 1,\ldots,n_j$, where the ξ_j^k, $k = 1,\ldots,r_j$, are the roots of the equation

$$\Delta_j + \sum_{k=1}^{r_j-1} \Delta_{jk}\xi^k + \Delta_{j+1}\xi^{r_j} = 0. \tag{36}$$

The resemblance of Theorems 2 and 5 is obvious. Disregarding some notational changes, like the partial multiplicities n_j being ordered increasingly, or the shift of one in the subindex j, it is clear that the quantities Δ_j play exactly the same role in this case as $\det \Phi_{j-1}$ in section 3. In particular, the generic behavior corresponds to nonvanishing Δ_j, $j = 1,\ldots,q$. The expansion (35) is exactly the same as (12), and the equation (36) determining the leading coefficients is the analogue of (15) or, equivalently, of (13) in this more general context. The surprising thing is that essentially the same result is obtained in this much more general context without much additional complication, just by replacing the Jordan canonical form with the Smith local form.

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