SIAM J. MATRIX ANAL. APPL.
Vol. 30, No. 2, pp. 491–496

A NOTE ON GENERIC KRONECKER ORBITS OF MATRIX PENCILS WITH FIXED RANK*

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Abstract. The set of $m \times n$ complex matrix pencils with rank (normal rank) at most r defines a subset of pencils in a complex $2mn$ dimensional space. For $r = 1, \dots, \min\{m, n\} - 1$, we show that this subset is a closed set, which is the union of $r+1$ irreducible components. Each of these irreducible components is the closure of a certain orbit of strictly equivalent pencils with rank r. The Kronecker canonical forms of these orbits are explicitly described, and their dimensions are counted. These are the Kronecker canonical forms of generic pencils of rank at most r. If $m \neq n$, then each irreducible component has a codimension distinct from the others, and the least of these codimensions is the codimension of the set of matrix pencils with rank at most r. This is $(n-r)(2m-r)$ if $m \geq n$ and $(m-r)(2n-r)$ otherwise.

Key words. Kronecker canonical form, matrix pencils, orbits, closures, irreducible components

AMS subject classifications. 15A21, 15A22, 65F15

DOI. 10.1137/060662538

1. Introduction.

The Kronecker canonical form (KCF) [6, Chapter XII] of matrix pencils may reflect important physical properties of the systems modelled by pencils, such as controllability [4, 10]. Very significant advances in the development of algorithms to compute the KCF have been seen in recent years (see [5] and the references therein). Despite this fact, computing the KCF of matrix pencils is an expensive and delicate task. Therefore, theoretical results that describe generic KCFs of some subsets of matrix pencils, i.e., the KCFs of almost all pencils in the subset, are interesting from an applied point of view.

The *generic KCF* of full rank $m \times n$ complex matrix pencils $A - \lambda B$ with $m \neq n$ was explicitly described in [3, Corollary 7.1] (see also [5, section 3.3]). For $n \times n$ singular matrix pencils, there are n possible generic KCFs, each of them corresponding to an orbit of strictly equivalent matrix pencils of codimension $n+1$. These Kronecker structures were explicitly described in [11, Theorem 1] (see also [3, Corollary 7.2] and [5, section 3.3]). In rigorous mathematical terms, one can say, in the language of algebraic geometry [11], that the set of $n \times n$ singular matrix pencils has exactly n irreducible components of codimension $n+1$, or, in the language introduced in [5], that the set of $n \times n$ singular matrix pencils is the union of the closures of n maximal orbits of strictly equivalent matrix pencils. Another relevant result in this context is that the set of $m \times n$ matrices with rank at most r is a manifold in $\mathbb{C}^{mn}$ of codimension $(m-r)(n-r)$ [3, Lemma 3.3]. However, as far as we know, no similar results exist for pencils with rank at most r. To develop these kinds of results is the purpose of this paper.

*Received by the editors June 9, 2006; accepted for publication (in revised form) by B. T. Kågström January 3, 2008; published electronically May 2, 2008. This research was partially supported by the Ministerio de Educación y Ciencia of Spain through grants BFM-2003-00223, MTM-2006-05361, MTM-2006-06671, and by the PRICIT Program of the Comunidad de Madrid through SIMUMAT Project (Ref. S-0505/ESP/0158).
http://www.siam.org/journals/simax/30-2/66253.html

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We prove that there are exactly $r + 1$ generic KCFs for $m \times n$ pencils with rank r. More precisely, for $r = 1, \ldots, \min\{m, n\} - 1$, we show that the set of matrix pencils with rank *at most* r is the union of the closures of the orbits corresponding to these KCFs, and that these closures are maximal in the sense that they are not contained in the closure of any other orbit of pencils with rank at most r. In addition, the dimensions of these orbits are counted and their KCFs explicitly described. The generic KCFs of the pencils with rank r have no regular part, as it happens for the generic KCFs of full rank $m \times n$ pencils and of $n \times n$ singular pencils [3, Corollaries 7.1 and 7.2], and they have both right and left singular blocks. Each of the generic KCFs with rank r depends on the sum of the right minimal indices [6], which may take values $0, 1, \ldots, r$, or, equivalently, on the sum of the left minimal indices. It is important to note that the orbits corresponding to these $r + 1$ generic KCFs have different dimensions in the case $m \neq n$. To prove these results, we use techniques introduced in [3, 5] (see also [1]). Finally, we present an additional result on the irreducibility of the closures of the orbits of the generic KCFs in the Zariski topology [11, section 1]. Our results include, as a particular case, the KCFs of generic $n \times n$ singular matrix pencils.

The *rank* of the pencil $A - \lambda B$ is defined in [6, Chapter VI] as the order of its largest minor that is not equal to the zero polynomial in λ. This is also frequently called *normal rank* [1, 5]. We will use the more classical name *rank* throughout this note, because this concept corresponds to the usual rank of matrices if we consider a matrix pencil as a matrix with elements in the field of rational functions in λ.

The fact that the KCFs of generic pencils with rank r depend not only on the rank, but also on the sum of the right (or, equivalently, left) minimal indices is related to a recent result presented in [2]. In [2], the generic change of the KCF of a pencil under low rank perturbations is studied, and it is proved that this change depends on the rank of the perturbation, and also on the sum of its left and right minimal indices.

A different kind of generic singular matrix pencils is considered in [8]. The definition of *genericity* in [8, p. 250] can be useful to study the KCF of very sparse pencils, but it is different from the one that we use in this work. As explained above, our definition of generic KCFs means that the union of the closures of the corresponding orbits is the whole set of pencils with rank at most r, and that these orbits are maximal. This is also the concept used in [5, 11]. This implies, for instance, that most pencils with rank r and generic KCF have all of the entries different from zero.[1] However, in the sense of [8], generic $m \times n$ pencils with rank r do not have all of their entries different from zero [8, Lemma 3.1]; therefore, they are not generic in our sense. We have already remarked that we will describe explicitly the generic KCFs of pencils with rank r; see Theorem 3.2. This cannot be done for the generic pencils with rank r in the sense of [8], where only the sums of the minimal indices of the KCF can be implicitly determined [8, Theorems 7.2 and 7.3].

This paper is organized as follows: In section 2 some background is introduced. The main results—Theorems 3.2, 3.3, and 3.5—are presented in section 3.

2. Previous results.

In this section we briefly summarize the results needed in this note. Simultaneously, the basic notation is introduced.

[1]Note that the set of pencils with rank at most r and all of the entries different from zero is open and dense in the set of pencils with rank at most r. To see this in $r = 1$, note that every pencil with rank at most one can be written as $p(\lambda)q(\lambda)^T$, where $p(\lambda)$ and $q(\lambda)$ are polynomial vectors, one of them of degree 0 and the other one of degree at most 1. Most pencils of this type have all of the entries different from zero.

2.1. Orbits and the Kronecker canonical form. We will use the same notation as in [5]. The *orbit* $\mathcal{O}(\mathcal{M})$ of an $m \times n$ matrix pencil $\mathcal{M}(\lambda) = A - \lambda B$ is the set of matrix pencils strictly equivalent to $\mathcal{M}(\lambda)$:

$$\mathcal{O}(\mathcal{M}) = \{P\mathcal{M}(\lambda)Q : \ P \in \mathbb{C}^{m\times m}, Q \in \mathbb{C}^{n\times n}\ ,\ P\,, Q \text{ nonsingular}\}.$$

These orbits are manifolds in the vector space $\mathbb{C}^{2mn}$, and we will refer to the *codimension* of $\mathcal{O}(\mathcal{M})$ as the codimension in this space. We will denote by $\overline{\mathcal{O}}(\mathcal{M})$ the closure of this orbit.

The most significant element of the orbit $\mathcal{O}(\mathcal{M})$ is the *Kronecker canonical form* (e.g., see [6]) of $\mathcal{M}(\lambda)$. The KCF is the direct sum of the right singular, left singular, and regular structures, consisting of L_k blocks of dimension $k \times (k+1)$ for the *right singular structure* and L_k^T blocks for the *left singular structure*. The *regular structure* consists of Jordan blocks $J_k(\mu)$ corresponding to eigenvalue μ, and N_k corresponding to the infinite eigenvalue. The KCF of $\mathcal{M}(\lambda)$ determines uniquely the orbit $\mathcal{O}(\mathcal{M})$, and, in particular, it fully determines the codimension of $\mathcal{O}(\mathcal{M})$ [3, Theorem 2.2].

2.2. Inclusion relationships between orbit closures. The *dominance ordering* in the set of sequences of nonnegative integers specifies that $(a_1, a_2, \ldots) \geq (b_1, b_2, \ldots)$ if $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i$ for $i = 1, 2, \ldots$. We say that $(a_1, a_2, \ldots) > (b_1, b_2, \ldots)$ if $(a_1, a_2, \ldots) \geq (b_1, b_2, \ldots)$ and $(a_1, a_2, \ldots) \neq (b_1, b_2, \ldots)$ [5, section 2.1].

For every matrix pencil $\mathcal{M}(\lambda)$ with rank r, we consider the following three sequences defined in [5]:

$$\mathcal{R}(\mathcal{M}) + r = (r_0 + r, r_1 + r, r_2 + r, \ldots);$$

where r_i is the number of right singular blocks L_j in the KCF of $\mathcal{M}(\lambda)$ with $j \geq i$;

$$\mathcal{L}(\mathcal{M}) + r = (l_0 + r, l_1 + r, l_2 + r, \ldots),$$

where l_i is the number of left singular blocks L_j^T in the KCF of $\mathcal{M}(\lambda)$ with $j \geq i$; and, for every $\mu \in \mathbb{C} \cup \{\infty\}$,

$$\mathcal{J}_\mu(\mathcal{M}) + p = (w_1(\mu) + p, w_2(\mu) + p, \ldots),$$

where $w_i(\mu)$ is the number of Jordan blocks associated with the eigenvalue μ of dimension greater than or equal to i in the regular structure of the KCF of $\mathcal{M}(\lambda)$, and p is the number of right singular blocks in the KCF of $\mathcal{M}(\lambda)$. These sequences allow us to obtain inclusion relationships between the closures of the orbits of two different matrix pencils. This is presented in Theorem 2.1 below, obtained in [9], and later reformulated in [1] and [5]. We state the theorem as it appears in [5].

THEOREM 2.1 (see [5, Theorem 3.1]). *Let $\mathcal{M}_1, \mathcal{M}_2$ be two $m \times n$ complex matrix pencils with $p\,(\mathcal{M}_1)$ and $p\,(\mathcal{M}_2)$ right singular blocks in their KCFs, respectively. Then $\overline{\mathcal{O}}(\mathcal{M}_1) \supseteq \overline{\mathcal{O}}(\mathcal{M}_2)$ if and only if the following relations hold:*

(i) $\mathcal{R}(\mathcal{M}_1) + \operatorname{rank}(\mathcal{M}_1) \geq \mathcal{R}(\mathcal{M}_2) + \operatorname{rank}(\mathcal{M}_2)$,
(ii) $\mathcal{L}(\mathcal{M}_1) + \operatorname{rank}(\mathcal{M}_1) \geq \mathcal{L}(\mathcal{M}_2) + \operatorname{rank}(\mathcal{M}_2)$,
(iii) $\mathcal{J}_\mu(\mathcal{M}_1) + p\,(\mathcal{M}_1) \leq \mathcal{J}_\mu(\mathcal{M}_2) + p\,(\mathcal{M}_2)$

for all $\mu \in \mathbb{C} \cup \{\infty\}$.

3. The set of singular pencils of rank at most r. If we restrict ourselves to the set of $m \times n$ matrix pencils with fixed rank equal to r, then the conditions in Theorem 2.1 simplify significantly. In this case, $\overline{\mathcal{O}}(\mathcal{M}_1) \supseteq \overline{\mathcal{O}}(\mathcal{M}_2)$ if and only if the following conditions hold: (i) $\mathcal{R}(\mathcal{M}_1) \geq \mathcal{R}(\mathcal{M}_2)$, (ii) $\mathcal{L}(\mathcal{M}_1) \geq \mathcal{L}(\mathcal{M}_2)$, and (iii) $\mathcal{J}_\mu(\mathcal{M}_1) \leq \mathcal{J}_\mu(\mathcal{M}_2)$ for all $\mu \in \mathbb{C} \cup \{\infty\}$ (because $p(\mathcal{M}) = n - \operatorname{rank}(\mathcal{M})$).

We will make use of the following result whose proof is immediate.

LEMMA 3.1. *Let $\mathcal{T} = \operatorname{diag}(\mathcal{G}, \mathcal{H})$ be a block-diagonal matrix pencil. If $\mathcal{G} \in \overline{\mathcal{O}}(\mathcal{M}_1)$ and $\mathcal{H} \in \overline{\mathcal{O}}(\mathcal{M}_2)$, for some matrix pencils $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\overline{\mathcal{O}}(\mathcal{T}) \subseteq \overline{\mathcal{O}}(\operatorname{diag}(\mathcal{M}_1, \mathcal{M}_2))$.*

Our main result characterizes the set of singular pencils with rank at most r through a set of maximal orbits of pencils with rank exactly r.

THEOREM 3.2. *Let r be an integer such that $1 \leq r \leq \min\{m, n\} - 1$. Let us define, in the set of $m \times n$ complex matrix pencils with rank r, the following $r + 1$ KCFs:*

$$\mathcal{K}_a(\lambda) = \operatorname{diag}(\underbrace{L_{\alpha+1}, \dots, L_{\alpha+1}}_{s}, \underbrace{L_\alpha, \dots, L_\alpha}_{n-r-s}, \underbrace{L^T_{\beta+1}, \dots, L^T_{\beta+1}}_{t}, \underbrace{L^T_\beta, \dots, L^T_\beta}_{m-r-t}) \tag{1}$$

for $a = 0, 1, \dots, r$, where $\alpha = \lfloor a/(n-r) \rfloor$, $s = a \bmod (n-r)$, $\beta = \lfloor (r-a)/(m-r) \rfloor$, and $t = (r-a) \bmod (m-r)$. Then,

(i) *For every $m \times n$ pencil $\mathcal{M}(\lambda)$ with rank at most r, there exists an integer a such that $\overline{\mathcal{O}}(\mathcal{K}_a) \supseteq \overline{\mathcal{O}}(\mathcal{M})$.*

(ii) *$\overline{\mathcal{O}}(\mathcal{K}_a) \not\supseteq \overline{\mathcal{O}}(\mathcal{K}_{a'})$ whenever $a \neq a'$.*

(iii) *The set of $m \times n$ complex matrix pencils with rank at most r is a closed set equal to $\bigcup_{0 \leq a \leq r} \overline{\mathcal{O}}(\mathcal{K}_a)$.*

Proof. For each $a = 0, 1, \dots, r$, let $\mathcal{D}_a$ be the set of block-diagonal matrix pencils in the form $\operatorname{diag}(\mathcal{G}, \mathcal{H})$, where $\mathcal{G}$ and $\mathcal{H}$ are, respectively, $a \times (a+n-r)$ and $(m-a) \times (r-a)$ matrix pencils. The generic KCFs of $\mathcal{G}$ and $\mathcal{H}$ are, respectively, $\operatorname{diag}(L_{\alpha+1}, \overset{s)}{\dots}, L_{\alpha+1}, L_\alpha, \overset{n-r-s)}{\dots}, L_\alpha)$ and $\operatorname{diag}(L^T_{\beta+1}, \overset{t)}{\dots}, L^T_{\beta+1}, L^T_\beta, \overset{m-r-t)}{\dots}, L^T_\beta)$ (where $\overset{k)}{\dots}$ means that there is a series of exactly k equal terms), with α, β, s, and t as in the statement (see [3, Corollary 7.1]). Now, to prove the first part of the theorem, it remains to show only that any matrix pencil of rank at most r is strictly equivalent to a block-diagonal pencil in $\mathcal{D}_a$, for some $a = 0, 1, \dots, r$, and apply Lemma 3.1.

Let $\mathcal{M}(\lambda)$ be a matrix pencil with rank $r' \leq r$ and KCF given by

$$\mathcal{K}_{\mathcal{M}}(\lambda) = \operatorname{diag}(L_{\alpha_1}, \dots, L_{\alpha_{n-r'}}, L^T_{\beta_1}, \dots, L^T_{\beta_{m-r'}}, J),$$

where J is the regular structure of the KCF. Then, since $r' \leq r$, we can consider

$$\mathcal{G} = \operatorname{diag}(L_{\alpha_1}, \dots, L_{\alpha_{n-r}})$$

and $\mathcal{H}$ being the block-diagonal matrix pencil containing the remaining blocks in $\mathcal{K}_{\mathcal{M}}(\lambda)$. Notice that $\mathcal{G}$ is of size $a \times (a+n-r)$, with $a = \alpha_1 + \dots + \alpha_{n-r}$. Then $\mathcal{M}(\lambda)$ is equivalent to $\operatorname{diag}(\mathcal{G}, \mathcal{H})$, and this last matrix pencil is in the class $\mathcal{D}_a$.

Now, we show that $\overline{\mathcal{O}}(\mathcal{K}_a) \not\supseteq \overline{\mathcal{O}}(\mathcal{K}_{a'})$ whenever $a \neq a'$. For this, it suffices to check that for distinct $a, a' \in \{0, 1, \dots, r\}$ the simplified versions of the three conditions (i), (ii), and (iii) in Theorem 2.1 do not hold simultaneously. This fact is immediate, because $a > a'$ implies (with the same notation as in the statement)

$$\alpha > \alpha' \quad \text{or} \quad \alpha = \alpha' \text{ and } s > s',$$

which implies $\mathcal{R}(\mathcal{K}_a) > \mathcal{R}(\mathcal{K}_{a'})$, and also

$$\beta < \beta' \quad \text{or} \quad \beta = \beta' \text{ and } t < t',$$

which implies $\mathcal{L}(\mathcal{K}_a) < \mathcal{L}(\mathcal{K}_{a'})$.

Finally, notice that the third item in Theorem 3.2 is a direct consequence of the first item, $\operatorname{rank}(\mathcal{K}_a(\lambda)) = r$ for all a, and the fact that the set of pencils of rank at most r is closed. ∎

Theorem 3.2 is, in essence, a consequence of Corollary 7.1 in [3], though we have stated it using the concepts and terminology from [5]. Notice also that although the rank of the KCFs in (1) is exactly r, the closures of their orbits include the set of all pencils with rank smaller than or equal to r.

Next, we pay attention to the codimension of the orbits $\mathcal{O}(\mathcal{K}_a)$ of the generic KCFs of pencils with rank at most r. We will see that these codimensions are distinct if $m \neq n$. In this case, the codimension (dimension) of the set of matrix pencils with rank at most r is defined, according to [11], as the least (largest) of the codimensions (dimensions) of $\mathcal{O}(\mathcal{K}_a)$, for $a = 0, 1, \ldots, r$.

THEOREM 3.3. *Let r be an integer such that $1 \leq r \leq \min\{m, n\} - 1$, and let $\mathcal{K}_a(\lambda)$, for $a = 0, 1, \ldots, r$, be the $r+1$ KCFs defined in* (1). *Then*

1. *The codimension of $\mathcal{O}(\mathcal{K}_a)$ is $(n-r)(2m-r) + a(m-n)$.*
2. *The codimension of the set of $m \times n$ complex matrix pencils with rank at most r is equal to*
   (i) *$(n-r)(2m-r)$ if $m \geq n$, and*
   (ii) *$(m-r)(2n-r)$ if $m \leq n$.*

Proof. The first item is a direct consequence of [3, Theorem 2.2]. The second item follows from computing $\min_a\{(n-r)(2m-r) + a(m-n)\}$. ∎

3.1. Irreducibility in Zariski topology. All of the topological ideas used so far refer to the usual topology in $\mathbb{C}^{2mn}$. The Zariski topology was used by Waterhouse to prove that the set of $n \times n$ singular matrix pencils with entries in an arbitrary infinite field has exactly n irreducible components, each of codimension $n+1$ [11, Theorem 1]. In this subsection, we will prove that the closures $\overline{\mathcal{O}}(\mathcal{K}_a(\lambda))$ of the orbits of the KCFs appearing in (1) are irreducible in the Zariski topology. A clear and concise summary of Zariski topology appears in the introduction of [11]. Here we recall only the following ideas: (i) a subset of $\mathbb{C}^q$ is closed in the Zariski topology if it is the set of common zeros of some polynomials, (ii) Zariski-closed sets are closed in the usual sense but the opposite is not true, (iii) a subset of $\mathbb{C}^q$ is irreducible if it is not the union of two relatively closed proper subsets in the Zariski topology, and (iv) every Zariski-closed set is the finite union of maximal irreducible subsets called its irreducible components.

An important result in this context is that the closures of an orbit of strictly equivalent pencils are the same in both the Zariski and the usual topology of $\mathbb{C}^{2mn}$ [7]. Therefore, there is no ambiguity in using the symbol $\overline{\mathcal{O}}(\mathcal{K}_a)$ in this subsection because it refers to exactly the same set as in the rest of this paper. The main result in this section states that orbits of pencils are irreducible.

LEMMA 3.4. *The closure $\overline{\mathcal{O}}(\mathcal{M})$ of the orbit $\mathcal{O}(\mathcal{M})$ of an $m \times n$ complex matrix pencil $\mathcal{M}(\lambda)$ is an irreducible manifold in the Zariski topology.*

Proof. Let us identify the set of matrix pencils of size $m \times n$ with $\mathbb{C}^{2mn}$, where the pencil $\mathcal{M}(\lambda) = A - \lambda B$ is identified with the pair (A, B). Let U be the set of pairs (P, Q) with $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ nonsingular. This is a dense open set of $\mathbb{C}^{m^2+n^2}$ (that is, $\overline{U} = \mathbb{C}^{m^2+n^2}$). Given a matrix pencil (A, B), we can consider the continuous (polynomial) mapping $\varphi_{\mathcal{M}}$ from $\mathbb{C}^{m^2+n^2}$ to $\mathbb{C}^{2mn}$ defined by sending (P,Q) to (PAQ, PBQ). We have $\mathcal{O}(\mathcal{M}) = \varphi_{\mathcal{M}}(U)$. By [11, section 1] we know that $\overline{\varphi_{\mathcal{M}}(\mathbb{C}^{m^2+n^2})}$ is an irreducible set. On the other hand, for every continuous mapping φ, we have $\varphi(\overline{W}) \subset \overline{\varphi(W)}$, where W is an arbitrary set, and this implies $\overline{\varphi(\overline{W})} = \overline{\varphi(W)}$. In the present case, we have

$$\overline{\varphi_{\mathcal{M}}(U)} = \overline{\varphi_{\mathcal{M}}(\mathbb{C}^{m^2+n^2})},$$

and this equals $\overline{\mathcal{O}}(\mathcal{M})$, which concludes the proof. ◻

With this lemma, Theorem 3.2 can be complemented as follows.

THEOREM 3.5. *Let r be an integer number such that $1 \leq r \leq \min\{m,n\} - 1$, and let $\mathcal{K}_a(\lambda)$, for $a = 0, 1, \ldots, r$, be the $r+1$ KCFs defined in* (1)*. Then, the set of $m \times n$ complex matrix pencils with rank at most r is a closed set that has exactly $r+1$ irreducible components in the Zariski topology. These irreducible components are $\overline{\mathcal{O}}(\mathcal{K}_a)$ for $a = 0, 1, \ldots, r$.*

This theorem includes [11, Theorem 1], mentioned at the beginning of this section, as a particular case.

Acknowledgments. The authors thank two anonymous referees for suggesting the use of block-diagonal pencils to prove Theorem 3.2. This idea has enhanced the presentation very much. The authors also thank the editor, Prof. Bo Kågström, for pointing out reference [8].

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