BIT
2002, Vol. 42, No. 1, pp. 084–109

0006-3835/02/4201-0084 $16.00

PERTURBATION THEORY FOR SIMULTANEOUS BASES OF SINGULAR SUBSPACES *

FROILÁN M. DOPICO and JULIO MORO †

Departamento de Matemáticas, Universidad Carlos III, Avda. de la Universidad, 30, 28911 Leganés, Madrid, Spain. emails: dopico@math.uc3m.es, jmoro@math.uc3m.es

Abstract.

New perturbation theorems are proved for simultaneous bases of singular subspaces of real matrices. These results improve the absolute bounds previously obtained in [6] for general (complex) matrices. Unlike previous results, which are valid only for the Frobenius norm, the new bounds, as well as those in [6] for complex matrices, are extended to any unitarily invariant matrix norm. The bounds are complemented with numerical experiments which show their relevance for the algorithms computing the singular value decomposition. Additionally, the differential calculus approach employed allows to easily prove new $\sin\Theta$ perturbation theorems for singular subspaces which deal independently with left and right singular subspaces.

AMS subject classification: 65F15, 15A18, 65F35, 15A42.

Key words: Singular value decomposition, perturbation bounds, derivatives of orthogonal projections.

1 Introduction.

In [6] it was questioned whether the simultaneous bases of singular subspaces of an m by n matrix A had the same sensitivity as the subspaces themselves. The results showed that the two sensitivities coincide when the perturbation is multiplicative (i.e. the perturbed matrix is $\widetilde{A} = D_1^* A D_2$), both depending on a relative gap between singular values (see Li [12]) instead of the usual absolute gap. In the case of arbitrary perturbations ($\widetilde{A} = A + E$), however, it was found that *the simultaneous bases may be much more sensitive than their associated singular subspaces* if the smallest singular values of the matrix A are among those corresponding to the chosen subspaces (see Theorems 1.1 and 1.2 below). Hence, we may conclude that the behavior of simultaneous bases is completely satisfactory under multiplicative perturbations, but not necessarily when the perturbations are additive. This last case is the one we shall address in the present paper.

To briefly summarize the results of [6] for the case of arbitrary additive perturbations, let A and $\widetilde{A}$ be two arbitrary matrices in $\mathbb{C}^{m\times n}$, $m \geq n$ with conformally

*Received July 2000. Revised September 2001. Communicated by Françoise Chaitin-Chatelin.
†This work was supported by research grant BFM2000-0008 of the Spanish Ministry of Science and Technology.

partitioned singular value decompositions (hereafter, SVD)

$$A = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}, \tag{1.1}$$

$$\widetilde{A} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 & \widetilde{U}_3 \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & \widetilde{\Sigma}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{bmatrix} \tag{1.2}$$

where $\Sigma_1 \in \mathbb{C}^{k\times k}$, $\Sigma_2 \in \mathbb{C}^{(n-k)\times(n-k)}$ and * denotes the conjugate transpose. No particular order is assumed on the singular values. Define the residuals

$$\begin{aligned} R &= A\widetilde{V}_1 - \widetilde{U}_1\widetilde{\Sigma}_1 = (A - \widetilde{A})\widetilde{V}_1, \\ S &= A^*\widetilde{U}_1 - \widetilde{V}_1\widetilde{\Sigma}_1 = (A^* - \widetilde{A}^*)\widetilde{U}_1. \end{aligned} \tag{1.3}$$

In this setting, the classical result by Wedin [20] bounds the sines of the canonical angles (see [18]) between the column spaces $\mathcal{R}(U_1)$ of U_1 and $\mathcal{R}(\widetilde{U}_1)$ of $\widetilde{U}_1$, as well as between the column spaces of V_1 and $\widetilde{V}_1$. If we denote by $\Theta(U_1, \widetilde{U}_1)$ the matrix of canonical angles between $\mathcal{R}(U_1)$ and $\mathcal{R}(\widetilde{U}_1)$, and by $\Theta(V_1, \widetilde{V}_1)$ the matrix of canonical angles between $\mathcal{R}(V_1)$, $\mathcal{R}(\widetilde{V}_1)$, Wedin proved in the Frobenius norm the following theorem:

THEOREM 1.1. *Let A and $\widetilde{A}$ be two $m\times n$ ($m \ge n$) complex matrices with SVDs* (1.1) *and* (1.2). *Define*

$$\delta = \min_{\substack{\widetilde{\mu}\in\sigma(\widetilde{\Sigma}_1) \\ \mu\in\sigma_{ext}(\Sigma_2)}} |\widetilde{\mu} - \mu| \tag{1.4}$$

where, for any matrix B, $\sigma(B)$ denotes the set of its singular values and

$$\sigma_{ext}(\Sigma_2) = \begin{cases} \sigma(\Sigma_2) \cup \{0\} & \text{if } m > n, \\ \sigma(\Sigma_2) & \text{if } m = n. \end{cases} \tag{1.5}$$

If $\delta > 0$ then

$$\sqrt{\|\sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + \|\sin\Theta(V_1, \widetilde{V}_1)\|_F^2} \le \frac{\sqrt{\|R\|_F^2 + \|S\|_F^2}}{\delta}. \tag{1.6}$$

Of course, the bound (1.6) implies that if both residuals are small, and the gap δ is not too small, the canonical angles are also small. However, as was observed in [6], this *does not mean* necessarily that the differences $U_1 - \widetilde{U}_1$ and $V_1 - \widetilde{V}_1$ are small, only that there exist orthonormal bases of $\mathcal{R}(U_1)$ and $\mathcal{R}(V_1)$ which are close to the orthonormal bases formed by the columns of $\widetilde{U}_1$ and $\widetilde{V}_1$. This is equivalent to the existence of two $k\times k$ unitary matrices P and Q such that both $\|U_1P - \widetilde{U}_1\|_F$ and $\|V_1Q - \widetilde{V}_1\|_F$ are small (see [18, Theorem I.5.2]). However, as remarked in

[6, Section 1], P and Q *should be the same* if we want to take the columns of the matrices $\widetilde{U}_1$ and $\widetilde{V}_1$ as reliable approximations of a pair of singular vector matrices corresponding to nonzero singular values of the unperturbed matrix A. Hence, we need to bound a different, more specific measure of the distance between simultaneous bases, in order to determine their sensitivity. The approach taken in [6, Theorem 2.1] is to bound the quantity[1]

$$\min_{W \text{unitary}} \sqrt{\|U_1 W - \widetilde{U}_1\|_F^2 + \|V_1 W - \widetilde{V}_1\|_F^2}, \tag{1.7}$$

which is zero whenever $\widetilde{U}_1$, $\widetilde{V}_1$ are a pair of simultaneous bases of singular subspaces of the unperturbed matrix A. The following result was obtained in [6]:

THEOREM 1.2. *Let A and $\widetilde{A}$ be two $m \times n$ ($m \geq n$) complex matrices with SVDs* (1.1) *and* (1.2). *Define*

$$\delta_b = \min \left\{ \delta \ , \ \sigma_{\min}(\widetilde{\Sigma}_1) + \sigma_{\min}(\Sigma_1) \right\} \tag{1.8}$$

where δ is the classical gap given by (1.4)*, and $\sigma_{\min}(\Sigma_1)$, $\sigma_{\min}(\widetilde{\Sigma}_1)$ denote, respectively, the minimum of the singular values of Σ_1 and $\widetilde{\Sigma}_1$. If $\delta_b > 0$ then*

$$\min_{W \text{unitary}} \sqrt{\|U_1 W - \widetilde{U}_1\|_F^2 + \|V_1 W - \widetilde{V}_1\|_F^2} \leq \sqrt{2} \frac{\sqrt{\|R\|_F^2 + \|S\|_F^2}}{\delta_b}. \tag{1.9}$$

Moreover, the left hand side of (1.9) *is minimized for $W = YZ^*$, where YSZ^* is any SVD of $U_1^* \widetilde{U}_1 + V_1^* \widetilde{V}_1$, and the equality can be attained.*

The bound in Theorem 1.2 differs from that in Wedin's Theorem 1.1, modulo numerical factors, in its dependence on the gap δ_b instead of on δ. This new gap, however, coincides with δ if $m > n$, since in that case $\delta \leq (\sigma_{\min}(\widetilde{\Sigma}_1) + \sigma_{\min}(\Sigma_1))$. Hence, only in the square case $m = n$ is there any significant difference between the sensitivity of simultaneous bases and that of the corresponding singular subspaces. Moreover, this different behavior appears only if Σ_1 contains the smallest singular values of A.

In the present paper we focus on the study of arbitrary additive perturbations ($\widetilde{A} = A + E$), significantly improving the bound (1.9) in the special (but frequent) case when the matrices A and $\widetilde{A}$ are real, provided the size of the perturbations is appropriately restricted (see Theorem 3.2 below for the details). If we, additionally, suppose that $k = 1$, i.e. that the singular subspaces $\mathcal{R}(U_1)$, $\mathcal{R}(V_1)$ are one-dimensional (by far the most common situation in numerical practice), the perturbation bounds may be further refined. Theorem 3.2 shows that in this case the variation of simultaneous bases is similar to that of subspaces, both depending on the usual gap δ between singular values. Again, in this case it will be necessary to restrict the size of the admissible perturbations. Both results will be illustrated through numerical experiments which clearly display how the output

[1]This quantity is an extension to pairs of subspaces of the unitarily invariant metric on the set of subspaces introduced by Paige [17] in a different context (see also [18, §II.4.2]).

of the usual algorithms for computing the SVD adequately reflects the influence of the corresponding perturbation bounds.

A remark is in order concerning the different behavior under perturbation of simultaneous bases for real and for complex matrices. This difference is analogous to the one already observed in the perturbation theory of unitary polar factors (see [1, 10, 16, 11, 3]). In fact, this latter theory can be deduced as a particular case of the perturbation theory for simultaneous bases developed both here and in [6]. However, we do not pursue this line of thought to keep the presentation concise.

It should be noticed that the techniques used to prove the theorems for real matrices are considerably more sophisticated than those employed in [6] for the general, complex, case. As in [1, 10, 16, 3], the basic tool is matrix differential calculus or, to be more precise in this paper, derivatives of orthogonal projectors onto invariant subspaces of Hermitian matrices. Although comparatively more involved, this approach has the advantage of directly leading in Theorem 5.1 to new bounds on the variation of singular subspaces for general complex matrices, which deal *separately* with the left and the right singular subspaces. This is an important remark, since Wedin's Theorem 1.1 and almost any other $\sin\Theta$-like theorem in the literature for arbitrary additive perturbations are all joint bounds, in the sense that both singular subspaces, left and right, seem to be influenced by the existence of small singular values in Σ_1. However, it is well known that only the left subspace $\mathcal{R}(U_1)$ should be affected by the presence of singular values close to zero in Σ_1 [18, Section V.4.1]. Finally, we point out that these separate bounds in Theorem 5.1 are not obtained following the obvious route of dealing with the matrices A^*A and AA^*, since this would lead to gaps between squares of singular values, not between the singular values themselves.

The paper is organized as follows: Section 2 presents the basic matrix differential calculus results required throughout the article. We stress that taking a point of view strictly confined to matrix analysis simplifies to a large extent the discussion, allowing for a compact, self-contained presentation of differentiability results of spectral projectors of (Hermitian) matrices, a topic where analytic function theory is usually the tool of choice (see, for instance, [9, Section II.6] or [4, Section 2.7]). Furthermore, simple, explicit formulas are provided for all the relevant derivatives. This said, one should still bear in mind that Section 2 is rather technical and, although it deserves a careful reading, the reader not interested in the technical details may skip the proofs and concentrate on the statements of Corollaries 2.2 and 2.3. Besides differentiability results, Section 2 contains several notations and block decompositions frequently used throughout the rest of the paper.

Section 3 contains perturbation bounds for quantity (1.7) in the Frobenius norm for real matrices, illustrated in Section 4 with several numerical experiments. Section 5 contains the announced new subspace perturbation bounds, separately dealing with left and right singular subspaces. Finally, we present in Section 6 perturbation bounds for simultaneous bases valid for any unitarily invariant matrix norm.

The following notation will be used throughout the paper: the conjugate transpose of a matrix A is denoted by A^*, its spectral norm (the largest singular value of A, also called 2-norm) by $\|A\|_2$ and its Frobenius norm by $\|A\|_F$. The transpose of the *real* matrix A is denoted by A^T. We write $\sigma(A)$ for the set of the singular values of A. If A is square, $\mathcal{L}(A)$ is the set of its eigenvalues. If B is an m by n matrix, $\mathcal{R}(B)$ stands for the subspace of $\mathbb{C}^m$ spanned by the columns of B, and to shorten the notation, we denote by $\sin\Theta(X_1, \widetilde{X}_1)$ the matrix of sines of the canonical angles between the subspaces $\mathcal{R}(X_1)$ and $\mathcal{R}(\widetilde{X}_1)$ (see [18] for the definition of canonical angles). Finally, $|||\cdot|||$ denotes any family of normalized unitarily invariant matrix norms.

2 Derivatives of orthogonal projectors.

Consider an arbitrary n by n complex Hermitian matrix H with unitary spectral decomposition

$$H = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} Q_1^* \\ Q_2^* \end{bmatrix}, \tag{2.1}$$

where $Q_1 \in \mathbb{C}^{n\times k}$,

$$\Lambda_1 = \operatorname{diag}\{\lambda_1, \dots, \lambda_k\} \in \mathbb{R}^{k\times k},$$

$$\Lambda_2 = \operatorname{diag}\{\lambda_{k+1}, \dots, \lambda_n\} \in \mathbb{R}^{(n-k)\times(n-k)},$$

and

$$\lambda_i \neq \lambda_j, \quad \text{whenever} \quad i \in \{1, \dots, k\},\ j \in \{k+1, \dots, n\}. \tag{2.2}$$

Hence, *no particular order is assumed on the eigenvalues* in Λ_1, Λ_2, only that they are disjoint. In this setting, the orthogonal projector onto the invariant subspace $\mathcal{R}(Q_1)$ spanned by the columns of Q_1 is

$$\Pi_1 = Q_1 Q_1^*.$$

Given any Hermitian perturbation matrix $\Delta H \in \mathbb{C}^{n\times n}$, we are interested in the uniparametric matrix family

$$H(t) = H + t\,\Delta H, \qquad 0 \le t \le 1,$$

with unitary diagonalizations

$$H(t) = \begin{bmatrix} Q_1(t) & Q_2(t) \end{bmatrix} \begin{bmatrix} \Lambda_1(t) & 0 \\ 0 & \Lambda_2(t) \end{bmatrix} \begin{bmatrix} Q_1(t)^* \\ Q_2(t)^* \end{bmatrix}, \tag{2.3}$$

where $Q_1(t) \in \mathbb{C}^{n\times k}$,

$$\Lambda_1(t) = \operatorname{diag}\{\lambda_1(t), \dots, \lambda_k(t)\} \in \mathbb{R}^{k\times k},$$

$$\Lambda_2(t) = \operatorname{diag}\{\lambda_{k+1}(t), \dots, \lambda_n(t)\} \in \mathbb{R}^{(n-k)\times(n-k)},$$

with $\lim_{t\to 0} \lambda_l(t) = \lambda_l$, $l = 1, \dots, n$. Thus, the orthogonal projector onto $\mathcal{R}(Q_1(t))$ is

$$\Pi_1(t) = Q_1(t)\, Q_1(t)^*.$$

Now we are in the position to prove the main result in this section.

THEOREM 2.1. *Let $H \in \mathbb{C}^{n\times n}$ be Hermitian with unitary diagonalization* (2.1) *satisfying* (2.2)*, and define*

$$\delta = \min_{\substack{\lambda \in \mathcal{L}(\Lambda_1) \\ \mu \in \mathcal{L}(\Lambda_2)}} |\lambda - \mu|. \tag{2.4}$$

For any given n by n Hermitian perturbation matrix ΔH, consider the matrix family $H(t) = H + t\,\Delta H$, with unitary diagonalizations (2.3) *and let $\Pi_1(t)$ be the orthogonal projector onto the subspace spanned by the columns of $Q_1(t)$.*

If

$$\|\Delta H\|_2 < \frac{\delta}{2}, \tag{2.5}$$

then the projection $\Pi_1(t)$ is continuously differentiable for $t \in [0,1]$ and

$$\frac{d\Pi_1}{dt}(t) = Q(t) \left[\begin{array}{c|c} 0 & \Gamma(t) \circ \Phi(t) \\ \hline \Gamma(t)^* \circ \Phi(t)^* & 0 \end{array}\right] Q(t)^*, \tag{2.6}$$

where $Q(t) = [Q_1(t)\ Q_2(t)]$, the symbol $\circ$ stands for the Hadamard product,

$$\Phi(t) = Q_1(t)^*\, \Delta H\, Q_2(t)$$

and $\Gamma(t)$ stands for the k by $n-k$ gap matrix

$$\Gamma(t) = \left[\frac{1}{\lambda_i(t) - \lambda_j(t)}\right]_{\substack{i=1,\ldots,k \\ j=k+1,\ldots,n}}. \tag{2.7}$$

PROOF. Although in principle we are only interested in $H(t)$ for $t \in [0,1]$, we begin by extending, for technical reasons, the range of t to a larger interval $(-t_+, t_+)$ for any $t_+ > 1$ satisfying

$$t_+ < \frac{\delta}{2\,\|\Delta H\|_2}. \tag{2.8}$$

Then, conditions (2.5) and (2.2) guarantee, using Weyl's eigenvalue perturbation Theorem, that

$$\lambda_i(t) \neq \lambda_j(t), \quad \text{for} \quad i \in \{1,\ldots,k\},\ j \in \{k+1,\ldots,n\}$$

for every $t \in (-t_+, t_+)$. Furthermore, if $\lambda_{\min}$ and $\lambda_{\max}$ denote, respectively, the smallest and largest eigenvalue of H, then

$$\lambda_i(t) \in (\alpha, \omega) \qquad \text{for every} \ \ t \in (-t_+, t_+),$$

where $\alpha = \lambda_{\min} - \delta/2$, $\omega = \lambda_{\max} + \delta/2$.

Now, we define a function $f : (\alpha,\omega) \longrightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{i=1}^{k} [\lambda_i - \tau, \lambda_i + \tau], \\ 0 & \text{if } x \in \bigcup_{j=k+1}^{n} [\lambda_j - \tau, \lambda_j + \tau], \end{cases}$$

where $\tau = t_+\|\Delta H\|_2$. Outside the τ-neighborhoods of the λ's, the function f is defined in such a way that it is continuously differentiable on the whole interval (α, ω).

Note first that condition (2.8) guarantees that the intervals where $f = 0$ have no intersection with those where $f = 1$. Moreover, again by Weyl's Theorem, for every $t \in (-t_+, t_+)$ each $\lambda_i(t)$ is in $[\lambda_i - \tau, \lambda_i + \tau]$ for $i = 1, \ldots, n$, so

$$f(H(t)) = Q(t) \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} Q(t)^* = Q_1(t)\, Q_1(t)^* = \Pi_1(t).$$

Now, we apply [8, Theorem 6.6.30] to the matrix function $f(H(t)) = \Pi_1(t)$ on the open interval $(-t_+, t_+)$ (this is the reason why we extended $H(t)$ beyond $[0, 1]$), obtaining that $\Pi_1(t)$ is continuously differentiable in the whole interval (in particular, in $[0, 1]$). Furthermore, its derivative is

$$\frac{d\Pi_1}{dt}(t) = Q(t)\, [\mathcal{D}_f(t) \circ \Psi(t)]\; Q(t)^*, \qquad t \in (-t_+, t_+),$$

where $\circ$ stands for the Hadamard product, $\Psi(t) = Q(t)^*\, \Delta H\, Q(t)$ and the n by n matrix $\mathcal{D}_f(t)$ is given by

$$\mathcal{D}_f(t) = [\Delta f(\lambda_r(t), \lambda_s(t))]_{r,s=1}^{n}\,,$$

with the divided differences $\Delta f(\cdot, \cdot)$ defined as

$$\Delta f(x, y) = \begin{cases} \dfrac{f(x) - f(y)}{x - y} & \text{if } x \neq y, \\ f'(x) & \text{if } x = y. \end{cases}$$

Given our choice of the function f, it is clear that

$$\Delta f(\lambda_r(t), \lambda_s(t)) = 0$$

if either both $r, s \in \{1, \ldots, k\}$ or both $r, s \in \{k + 1, \ldots, n\}$. On the other hand,

$$\Delta f(\lambda_r(t), \lambda_s(t)) = \Delta f(\lambda_s(t), \lambda_r(t)) = \frac{1}{\lambda_r(t) - \lambda_s(t)}$$

whenever $r \in \{1, \ldots, k\},\; s \in \{k + 1, \ldots, n\}$. Hence,

$$\mathcal{D}_f(t) = \begin{bmatrix} 0 & \Gamma(t) \\ \Gamma(t)^* & 0 \end{bmatrix}, \quad t \in (-t_+, t_+),$$

with $\Gamma(t)$ defined as in (2.7). The proof is completed by writing the corresponding block partition of $\Psi(t)$. □

Once we have a result for the spectral Hermitian problem, one can easily obtain useful results for the SVD of general rectangular matrices $A(t) = A + tE$, A, $E \in \mathbb{C}^{m\times n}$, $m \geq n$, via the $(m + n)$ by $(m + n)$ Hermitian Jordan–Wielandt matrices [18, Section I.4.1]

$$H = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}, \qquad \Delta H = \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} \tag{2.9}$$

associated with A and E.

To be more precise, consider an arbitrary matrix $A \in \mathbb{C}^{m\times n}$, $m \geq n$, with singular value decomposition given by (1.1) where

$$\Sigma_1 = \text{diag}\{\sigma_1, \ldots, \sigma_k\} \in \mathbb{R}^{k\times k},$$

$$\Sigma_2 = \text{diag}\{\sigma_{k+1}, \ldots, \sigma_n\} \in \mathbb{R}^{(n-k)\times(n-k)}$$

and

$$\sigma_i \neq \sigma_j, \quad \text{whenever} \quad i \in \{1, \ldots, k\}, \quad j \in \{k+1, \ldots, n\}. \tag{2.10}$$

Again, no particular order is assumed on the singular values. For any given matrix $E \in \mathbb{C}^{m\times n}$ we consider the uniparametric family of perturbed matrices $A(t) = A + t\,E$, with SVD

$$A(t) = \begin{bmatrix} U_1(t) & U_2(t) & U_3(t) \end{bmatrix} \begin{bmatrix} \Sigma_1(t) & 0 \\ 0 & \Sigma_2(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1(t)^* \\ V_2(t)^* \end{bmatrix}, \tag{2.11}$$

where

$$\Sigma_1(t) = \text{diag}\{\sigma_1(t), \ldots, \sigma_k(t)\},$$

$$\Sigma_2(t) = \text{diag}\{\sigma_{k+1}(t), \ldots, \sigma_n(t)\},$$

with $\lim_{t\to 0} \sigma_i(t) = \sigma_i, \ i = 1, \ldots, n$.

COROLLARY 2.2. *Let A be an arbitrary complex $m \times n$ $(m \geq n)$ matrix with SVD given by* (1.1) *and satisfying* (2.10). *Define*

$$\rho \ = \min \left\{ \min_{\substack{\lambda\in\sigma(\Sigma_1) \\ \mu\in\sigma_{ext}(\Sigma_2)}} |\lambda - \mu|, \ 2\sigma_{\min}(\Sigma_1) \right\}, \tag{2.12}$$

where $\sigma_{ext}(\cdot)$ is given by (1.5). *For any m by n perturbation matrix E, consider the matrix family $A(t) = A + t\,E$, $0 \leq t \leq 1$, with SVDs* (2.11) *and let $P_1(t)$ be the orthogonal projector onto the subspace of $\mathbb{C}^{m+n}$ spanned by the columns of the $(m+n)$ by k matrix*

$$X_1(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1(t) \\ V_1(t) \end{bmatrix}.$$

If

$$\|E\|_2 < \frac{\rho}{2}, \tag{2.13}$$

then the projection $P_1(t)$ is continuously differentiable for $t \in [0,1]$ and

$$\frac{dP_1}{dt}(t) = \begin{bmatrix} X_1(t) & X_2(t) \end{bmatrix} \left[\begin{array}{c|c} 0 & G(t) \circ F(t) \\ \hline G(t)^* \circ F(t)^* & 0 \end{array} \right] \begin{bmatrix} X_1(t)^* \\ X_2(t)^* \end{bmatrix}, \tag{2.14}$$

where

$$X_2(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1(t) & U_2(t) & U_2(t) & \sqrt{2}U_3(t) \\ -V_1(t) & V_2(t) & -V_2(t) & 0 \end{bmatrix},$$

the symbol $\circ$ *stands for the Hadamard product,*

$$F(t) = X_1(t)^* \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} X_2(t)$$

and $G(t)$ *stands for the* $k \times (m+n-k)$ gap matrix

$$G(t) = \left[\; G_1(t) \; \middle| \; G_2^+(t) \; \middle| \; G_2^-(t) \; \middle| \; G_0(t) \; \right]$$

whose blocks are given, respectively, by

$$\begin{aligned} G_1(t) &= \left[\frac{1}{\sigma_i(t) + \sigma_j(t)} \right]_{\substack{i=1,\ldots,k \\ j=1,\ldots,k}} \in \mathbb{C}^{k \times k}, \\ G_2^+(t) &= \left[\frac{1}{\sigma_i(t) - \sigma_j(t)} \right]_{\substack{i=1,\ldots,k \\ j=k+1,\ldots,n}} \in \mathbb{C}^{k \times (n-k)}, \\ G_2^-(t) &= \left[\frac{1}{\sigma_i(t) + \sigma_j(t)} \right]_{\substack{i=1,\ldots,k \\ j=k+1,\ldots,n}} \in \mathbb{C}^{k \times (n-k)}, \\ G_0(t) &= \left[\frac{1}{\sigma_i(t)} \right]_{\substack{i=1,\ldots,k \\ j=1,\ldots,m-n}} \in \mathbb{C}^{k \times (m-n)}. \end{aligned}$$

PROOF. Apply Theorem 2.1 to the Jordan–Wielandt matrices (2.9), with $\Lambda_1 = \mathcal{L}(\Sigma_1)$ and $\Lambda_2 = \mathcal{L}(-\Sigma_1) \cup \mathcal{L}(\Sigma_2) \cup \mathcal{L}(-\Sigma_2) \cup \{0\}$. □

REMARK 2.1. In the previous Corollary it is implicitly assumed that Σ_1 is nonsingular, otherwise $\rho = 0$ and restriction (2.13) does not hold. The same happens in several other Theorems of this paper.

REMARK 2.2. In the nonsquare ($m > n$) case condition (2.13) can be relaxed to

$$\|E\|_2 < \min \left\{ \frac{1}{2} \min_{\substack{\lambda \in \sigma(\Sigma_1) \\ \mu \in \sigma(\Sigma_2)}} |\lambda - \mu| \; , \; \sigma_{\min}(\Sigma_1) \right\},$$

if we take into account the existence of $m - n$ "ghost" zero singular values which do not change for $0 \le t \le 1$. The same is possible in the next corollary.

We conclude this section with a second corollary, similar to Corollary 2.2, which will be used in Section 5.

COROLLARY 2.3. *Let* A *be an arbitrary complex* $m \times n$ $(m \ge n)$ *matrix with SVD given by* (1.1) *and satisfying* (2.10). *Define*

$$g_{ext} \; = \min_{\substack{\lambda \in \sigma(\Sigma_1) \\ \mu \in \sigma_{ext}(\Sigma_2)}} |\lambda - \mu|, \tag{2.15}$$

where $\sigma_{ext}(\cdot)$ *is given by* (1.5). *For any* m *by* n *perturbation matrix* E, *consider the matrix family* $A(t) = A + t\,E$, $0 \le t \le 1$, *with SVDs* (2.11) *and let* $P_1^{\pm}(t)$ *be the orthogonal projector onto the subspace of* $\mathbb{C}^{m+n}$ *spanned by the columns of the* $(m+n)$ *by* $2k$ *matrix*

$$Y_1(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1(t) & U_1(t) \\ V_1(t) & -V_1(t) \end{bmatrix}.$$

If

$$\|E\|_2 < \frac{g_{ext}}{2},$$

then the projection $P_1^{\pm}(t)$ *is continuously differentiable for* $t \in [0,1]$ *and*

$$\frac{dP_1^{\pm}}{dt}(t) = \begin{bmatrix} Y_1(t) & Y_2(t) \end{bmatrix} \left[\begin{array}{c|c} 0 & \mathcal{G}(t) \circ \mathcal{F}(t) \\ \hline \mathcal{G}(t)^* \circ \mathcal{F}(t)^* & 0 \end{array}\right] \begin{bmatrix} Y_1(t)^* \\ Y_2(t)^* \end{bmatrix}, \tag{2.16}$$

where

$$Y_2(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} U_2(t) & U_2(t) & \sqrt{2}U_3(t) \\ V_2(t) & -V_2(t) & 0 \end{bmatrix},$$

the symbol $\circ$ *stands for the Hadamard product,*

$$\mathcal{F}(t) = Y_1(t)^* \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} Y_2(t)$$

and $\mathcal{G}(t)$ *stands for the* $2k \times (m+n-2k)$ gap matrix

$$\mathcal{G}(t) = \left[\begin{array}{c|c|c} G_2^+(t) & G_2^-(t) & G_0(t) \\ \hline -G_2^-(t) & -G_2^+(t) & -G_0(t) \end{array}\right]$$

with $G_0(t)$, $G_2^+(t)$ *and* $G_2^-(t)$ *defined as in Corollary* 2.2.

Proof. Apply Theorem 2.1 to the Jordan–Wielandt matrices (2.9), with $\Lambda_1 = \mathcal{L}(\Sigma_1) \cup \mathcal{L}(-\Sigma_1)$ and $\Lambda_2 = \mathcal{L}(\Sigma_2) \cup \mathcal{L}(-\Sigma_2) \cup \{0\}$. □

3 Theorem for bases of real matrices in Frobenius norm.

The following preliminary lemma is a straightforward consequence of Corollary 2.2. This lemma allows us to prove Theorem 3.2, one of the main results in this paper. Since we are going to deal with real matrices, all the matrices appearing in Corollary 2.2 have to be understood as real matrices, in particular conjugate transpose are just transpose matrices.

Lemma 3.1. *Let* A *and* E *to be two* $m \times n$ $(m \geq n)$ *real matrices and consider the matrix family* $A(t) = A + tE$ *with* $t \in [0,1]$ *under the assumptions of Corollary* 2.2. *Define the usual order* $\sigma_1(t) \geq \cdots \geq \sigma_k(t)$ *on the singular values of* $\Sigma_1(t)$*, and also*

$$\rho_R(t) = \min\left\{ \min_{\substack{\lambda \in \sigma(\Sigma_1(t)) \\ \mu \in \sigma_{ext}(\Sigma_2(t))}} |\lambda - \mu| \ , \ \sigma_{k-1}(t) + \sigma_k(t) \right\}, \tag{3.1}$$

1. *If* $k > 1$ *then*

$$\left\| \frac{dP_1}{dt}(t) \right\|_F \leq \frac{\sqrt{2}\,\|E\|_F}{\rho_R(t)}. \tag{3.2}$$

2. *If* $k = 1$ *then* $\Sigma_1(t)$ *is a number, and*

$$\left\| \frac{dP_1}{dt}(t) \right\|_F \leq \frac{\sqrt{2}\,\|E\|_2}{\min_{\mu \in \sigma_{ext}(\Sigma_2(t))} |\Sigma_1(t) - \mu|}. \tag{3.3}$$

Proof. From equation (2.14) we get

$$\left\| \frac{dP_1}{dt}(t) \right\|_F = \left\| \left[\begin{array}{c|c} 0 & G(t) \circ F(t) \\ \hline G(t)^T \circ F(t)^T & 0 \end{array} \right] \right\|_F .$$

The key point in the proof is that the diagonal elements of $F(t)$, those corresponding to the diagonal elements of matrix $G_1(t)$ defined in Corollary 2.2, are equal to zero. The proof of this is trivial, although it should be stressed that this fact is true only for real matrices, not for complex ones. Thus if we bound from below the remaining elements of $G(t)$ with $\rho_R(t)$, we arrive at the inequality

$$\left\| \frac{dP_1}{dt}(t) \right\|_F \leq \frac{1}{\rho_R(t)} \left\| \begin{bmatrix} 0 & F(t) \\ F(t)^T & 0 \end{bmatrix} \right\|_F . \tag{3.4}$$

Finally, denoting $X(t) = [X_1(t)\ X_2(t)]$, the properties of the Frobenius norm and the definition of $F(t)$ lead us to

$$\left\| \frac{dP_1}{dt}(t) \right\|_F \leq \frac{1}{\rho_R(t)} \left\| X(t)^T \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} X(t) \right\|_F = \frac{1}{\rho_R(t)} \left\| \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right\|_F ,$$

from which the first part of the lemma follows easily.

In the one-dimensional case ($k = 1$), one can take advantage of $F(t)$ being a row vector and $G_1(t)$ a number (which is canceled out by the $(1, 1)$ zero element of $F(t)$) to get

$$\left\| \frac{dP_1}{dt}(t) \right\|_F \leq \frac{1}{\min\limits_{\mu \in \sigma_{ext}(\Sigma_2(t))} |\Sigma_1(t) - \mu|} \left\| \begin{bmatrix} 0 & F(t) \\ F(t)^T & 0 \end{bmatrix} \right\|_F ,$$

instead of (3.4). The following trivial fact:

$$\left\| \begin{bmatrix} 0 & F(t) \\ F(t)^T & 0 \end{bmatrix} \right\|_F = \sqrt{2} \|F(t)\|_F = \sqrt{2} \|F(t)\|_2 \leq \sqrt{2} \|E\|_2$$

completes the proof. □

It is worth noting that in the previous lemma, the bound (3.3) for the derivative of the orthogonal projection in the one-dimensional case depends only on the usual gap for singular subspaces. This does not happen for complex matrices, so it is something particular of real matrices. More on this question will be commented in the section devoted to numerical experiments.

Some remarks on Lemma 3.1 are in order:

Remark 3.1. It is also possible to obtain an analogous result for complex matrices, but in this case $\rho_R(t)$ has to be replaced by $\rho(t)$, defined like (2.12) but with all the involved quantities depending on t. This approach leads us to a result similar to Theorem 3.2 below, but with ρ instead of ρ_R in the bound (3.8). The theorem so obtained is weaker than Theorem 1.2.

Remark 3.2. Notice that the "new" term $\sigma_{k-1}(t) + \sigma_k(t)$ in $\rho_R(t)$ is only relevant in the square case. Thus in the rest of this section we restrict ourselves to the case $m = n$.

REMARK 3.3. In the second part of the lemma, for $k = 1$, it is possible to bound the spectral norm of the derivative of $P_1(t)$ just suppressing the square root of 2. However this does not lead to any improvement in Theorem 3.2.

We prove our first perturbation theorem combining the bounds in the previous lemma with the technique developed in [1]. As usual, A and $\widetilde{A}$ denote, from now on, the unperturbed and perturbed matrices, respectively. The relationship with the family $A(t) = A + tE$ is the obvious one: $A = A(0)$ and $\widetilde{A} = A(1)$. Unless otherwise stated, all quantities denoted with a tilde correspond to $\widetilde{A}$.

For the sake of clarity, we state the theorem with all the necessary assumptions.

THEOREM 3.2. *Let A and $\widetilde{A} = A+E$ be two $n\times n$ real matrices with conformally partitioned real SVDs*

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{bmatrix} \begin{bmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & \widetilde{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \widetilde{V}_1^T \\ \widetilde{V}_2^T \end{bmatrix}.$$

Define

$$\rho = \min\left\{ \min_{\substack{\lambda\in\sigma(\Sigma_1)\\ \mu\in\sigma(\Sigma_2)}} |\lambda-\mu|\,,\ 2\,\sigma_{\min}(\Sigma_1) \right\}, \tag{3.5}$$

$\sigma(\Sigma_1) = \{\sigma_1 \geq \ldots \geq \sigma_k\}$ *and*

$$\rho_R = \min\left\{ \min_{\substack{\lambda\in\sigma(\Sigma_1)\\ \mu\in\sigma(\Sigma_2)}} |\lambda-\mu|\,,\ \sigma_{k-1}+\sigma_k \right\}. \tag{3.6}$$

If

$$\|E\|_2 < \frac{\rho}{2}, \tag{3.7}$$

then

$$\min_{W \text{orthogonal}} \sqrt{\|U_1W-\widetilde{U}_1\|_F^2 + \|V_1W-\widetilde{V}_1\|_F^2} \leq -\frac{\|E\|_F}{\|E\|_2}\ln\left(1-\frac{2\|E\|_2}{\rho_R}\right). \tag{3.8}$$

Moreover, if Σ_1 is 1×1 then

$$\min_{w\in\{-1,1\}} \sqrt{\|U_1w-\widetilde{U}_1\|_2^2 + \|V_1w-\widetilde{V}_1\|_2^2} \leq -\ln\left(1-\frac{2\|E\|_2}{\min_{\mu\in\sigma(\Sigma_2)}|\Sigma_1-\mu|}\right). \tag{3.9}$$

The left hand sides of the previous inequalities are minimized for $W = YZ^T$, where YSZ^T is any SVD of $U_1^T\widetilde{U}_1 + V_1^T\widetilde{V}_1$.

REMARK 3.4. The first order bound obtained from (3.8) by expanding the logarithm differs from the one following from (1.9) replacing the residuals by $\|E\|_F$ only in the presence of ρ_R instead of δ_b. If we suppose the perturbed singular values approximately equal to the unperturbed ones then it may happen that $\rho_R \gg \delta_b$. Thus, simultaneous bases of singular subspaces can be much more sensitive for complex than for real matrices. More on the relevance of this point will be commented at the end of this section and in the next one.

REMARK 3.5. In Section 4 we will see that the restriction (3.7) on the size of the perturbation is necessary and not an artifact of the differential proof.

PROOF OF THEOREM 3.2 We only prove (3.8). The one-dimensional case follows similarly. As in Corollary 2.2, we consider the matrix family $A(t) = A + tE$, for $t \in [0, 1]$. In the first place, notice that Weyl's perturbation theorem for singular values and (3.7) implies

$$\rho_R(t) \geq \rho_R - 2t\|E\|_2 > 0.$$

Thus, using Lemma 3.1 on the projectors $P_1 = P_1(0)$ and $\widetilde{P}_1 = P_1(1)$, we obtain

$$\begin{aligned} \|\widetilde{P}_1 - P_1\|_F &\leq \int_0^1 \left\| \frac{dP_1}{dt}(t) \right\|_F dt \leq \sqrt{2}\,\|E\|_F \int_0^1 \frac{dt}{\rho_R(t)} \\ &\leq \sqrt{2}\,\|E\|_F \int_0^1 \frac{dt}{\rho_R - 2t\|E\|_2} = -\frac{\|E\|_F}{\sqrt{2}\,\|E\|_2} \ln\left(1 - \frac{2\|E\|_2}{\rho_R}\right). \end{aligned} \tag{3.10}$$

Now, consider the $(m + n) \times k$ matrices with orthonormal columns:

$$X_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} \quad \text{and} \quad \widetilde{X}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix}, \tag{3.11}$$

and notice that

$$\begin{aligned} \min_{W \text{orthogonal}} \sqrt{\|U_1 W - \widetilde{U}_1\|_F^2 + \|V_1 W - \widetilde{V}_1\|_F^2} &= \sqrt{2} \min_{W \text{orthogonal}} \|X_1 W - \widetilde{X}_1\|_F \\ &= \sqrt{2}\sqrt{\|I - \cos\Theta(X_1, \widetilde{X}_1)\|_F^2 + \|\sin\Theta(X_1, \widetilde{X}_1)\|_F^2} \\ &\leq 2\|\sin\Theta(X_1, \widetilde{X}_1)\|_F = \sqrt{2}\|\widetilde{P}_1 - P_1\|_F, \end{aligned}$$

where we have used [18, Theorem II.4.11] (or the original reference [17]) for the second equality and [18, Theorem I.5.5] for the last one. The final bound follows from combining the previous bound with the bound (3.10) on the difference of projectors. The solution of the orthogonal Procrustes problem for X_1 and $\widetilde{X}_1$ (see [7, Section 12.4.1]) implies that the orthogonal matrix $W = YZ^T$ minimizes the corresponding left-hand sides of (3.8) and (3.9). □

When comparing Theorem 3.2 for real matrices with Theorem 1.2 for general complex matrices some precautions have to be taken because the quantity δ_b defined in Theorem 1.2 involves both perturbed and unperturbed singular values, while ρ_R in Theorem 3.2 only involves the unperturbed ones. This is due to the generality of Theorem 1.2, which allows perturbations of any size. As observed in Remark 3.1, it is possible to obtain for complex matrices a result weaker than Theorem 1.2, but analogous to Theorem 3.2, just by replacing ρ_R by ρ. Thus, it is clear that the bound for real matrices, involving in ρ_R the sum of the two smallest singular values of Σ_1, can be much smaller than the bound for complex matrices, which involves in ρ (or in δ_b) twice the smallest singular value of Σ_1. The situation is much better in the one-dimensional real case, since the sensitivity

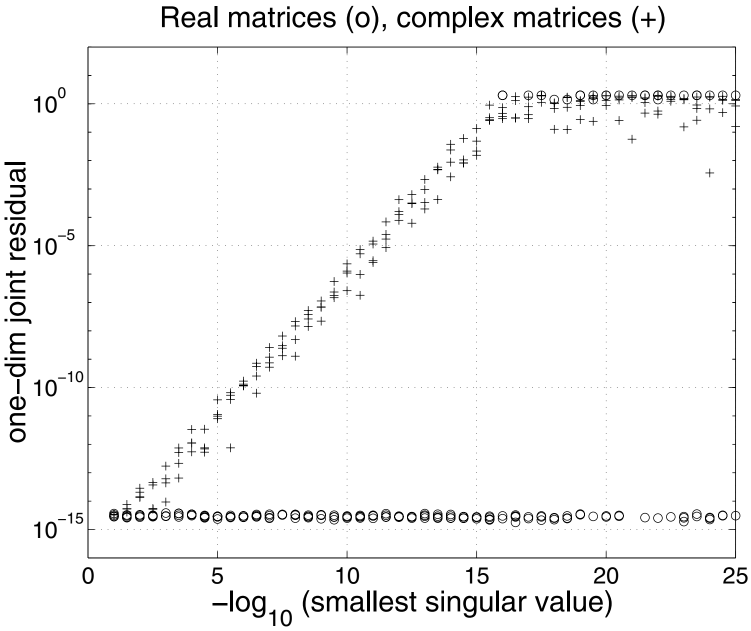

Figure 4.1: Joint residual given by equation (1.7) vs. $-\log_{10}$ (smallest singular value) in the one-dimensional case described in Section 4.

of simultaneous bases is the same as that of singular subspaces, both depending on the *usual gap for singular subspace variations*. In this case the bound (3.9) for simultaneous bases[2] does not show any explicit dependence on the size of Σ_1. However, it should be noticed that this dependence appears implicitly in Theorem 3.2 in the restriction (3.7) on the size of the perturbation. More on the relevance of all these questions will be discussed in the next section.

4 Numerical experiments.

We present in this section numerical experiments done in MATLAB 5.3, showing that the different sensitivity of simultaneous bases and singular subspaces predicted for square matrices in both [6] and Section 3 above is observed in the behavior of usual SVD algorithms. Also, it will be shown that the cases of real and of complex matrices exhibit in practice different behavior only in the one-dimensional case $k = 1$. Although (3.8) in Theorem 3.2 implies that, in theory, the simultaneous bases can be much less sensitive for $k > 1$ in the real than in the complex case, this difference is extremely unlikely to be observed in numerical

[2]The fundamental idea explaining the sensitivity of simultaneous bases in the one-dimensional case is easily understood using on Jordan–Wielandt matrices the well-known results in [7, Section 7.2.4, 2nd ed.] on eigenvector sensitivity. However, this simple approach demands all the singular values to be distinct and imposes severe, and unnecessary, restrictions on the size of the perturbation.

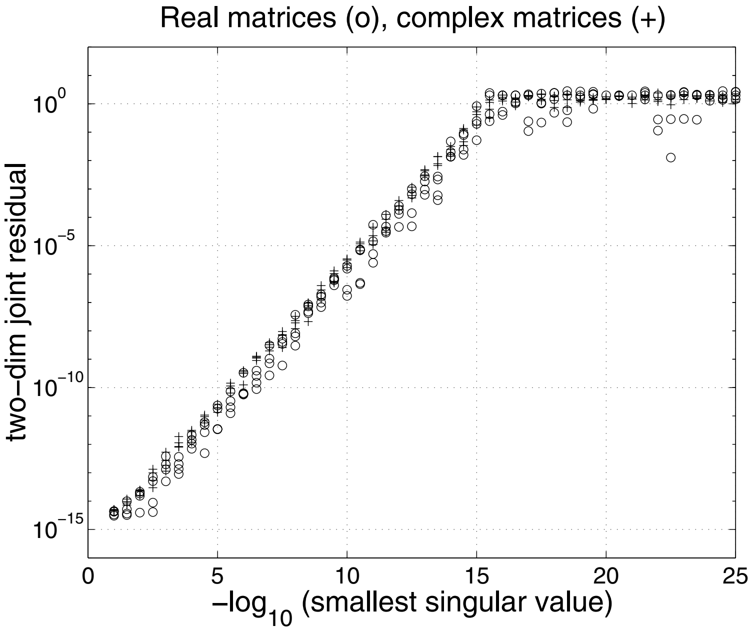

Figure 4.2: Joint residual given by equation (1.7) vs. $-\log_{10}$ (smallest singular value) in the two-dimensional case described in Section 4.

practice. The reason is that Σ_1 will only be taken to have dimension larger than 1 if the singular values $\sigma_1, \ldots, \sigma_k$ of Σ_1 form a tight cluster. Hence, $\sigma_{k-1} + \sigma_k$ is practically equal to $2\,\sigma_k$, and $\rho_R \approx \rho$.

We have first generated 50×50 random real and complex matrices of the form $A = U\,S\,V^*$, where S is a diagonal matrix with all but the last entry being random numbers uniformly distributed between 1 and 4, and a last diagonal entry 10^{-j}. This ensures that the one-dimensional singular subspaces associated with the smallest singular value 10^{-j} are well-determined (the corresponding gap is large). The matrices U and V are random finite precision matrices, orthogonal for real matrices and unitary for complex ones. We will compute the SVD of these matrices A using the standard MATLAB command. The output will be compared with U, S, V. We remind the reader that the MATLAB algorithm essentially produces the SVD of a perturbed matrix $A + E$, with $\|E\|_2 = O(\epsilon_M)\|A\|_2$, where ϵ_M is the unit roundoff [7, p. 261].

The exponent j varies from 1 to 25 in steps of size 0.5. Four real and four complex matrices are generated for each value of j. In Figure 4.1 we plot in logarithmic scale the joint residual (1.7) for the singular vectors corresponding to the smallest singular value 10^{-j} versus the exponent j. The numerical results in Figure 4.1 are those predicted by Theorems 1.2 and 3.2 for $k = 1$, i.e., in the complex case the joint residual increases with the decrease of the smallest singular value, and bases are much more sensitive than subspaces (in this case, the sines between the unperturbed and perturbed singular subspaces are of order ϵ_M). In the real case, however, the joint residual remains at order ϵ_M until the smallest singular value becomes smaller than, approximately, the unit roundoff which is, more or less, the size of the perturbation $\|E\|_2$ introduced by the algorithm. At that point, the condition (3.7) no longer holds and Theorem 3.2 does not apply. From this point on, the joint residual takes only two different values, one still of order ϵ_M and the other value approximately 2 (the maximum possible value for the one-dimensional joint residual). The reason why there are no values in between is that in the one-dimensional real case, if the singular subspaces are well-determined, the only way the computed bases can fail is to change a sign. When this happens the error is very large. This is also the case in Example 1.1 in [6]. These numerical results show that the restriction on the size of the perturbation (3.7) in Theorem 3.2 is necessary and is not an artifact of the proof.

The second experiment is essentially equal to the first one, with the only difference that the diagonal matrix S has two last diagonal entries 10^{-j}. Thus, $k = 2$ with well-determined two-dimensional singular subspaces associated with the two smallest singular values. The results of this experiment are plotted in Figure 4.2. As predicted, the joint residual behaves in the same way in both the real and the complex case, growing with j, since $\sigma_{k-1} + \sigma_k \approx 2\,\sigma_k$.

5 Separate absolute sin Θ bounds

As noted in [18, Section V.4.1], a fundamental defect of Wedin's Theorem 1.1, appearing in the $m > n$ case, is that, although the right singular subspace is insensitive to the size of $\sigma_{min}(\Sigma_1)$, this cannot be seen in a joint bound for left and right singular subspaces like (1.6). Although there are indirect ways to circumvent this problem, using the joint bound on the orthogonal complements [18, Section V.4.1], we take here a direct approach which solves this difficulty.

One possibility which immediately comes to mind is applying Davis and Kahan's $\sin\Theta$ Theorem [5] for Hermitian matrices on either AA^* or A^*A, but this leads to gaps between the squares of the singular values, which may be much smaller than the usual gaps.

There are separate bounds in the literature for left and right singular subspaces in the relative setting. Additive perturbations are treated in [14] using relative perturbation results for invariant subspaces of the Hermitian matrix A^*A. The corresponding bounds depend on gaps between squares of singular values, but in this case this is not an important disadvantage, since the gaps are relative and are roughly equivalent to the usual relative gaps. Multiplicative perturbations are addressed in [13], distinguishing the different influence of left and right perturbations on left and right singular subspaces. It should be noted that in this relative setting the size of $\sigma_{min}(\Sigma_1)$ is not relevant for the sensitivity of singular subspaces, as pointed out in [6, footnote in p. 400]. Thus, the above two references address interesting problems which are different from ours.

As far as we know, the following Theorem is a new contribution in the absolute setting, valid for additive perturbations. Its most relevant feature is that, even in the nonsquare case, the bound on the sines of the canonical angles between right singular subspaces does not depend on $\sigma_{\min}(\Sigma_1)$, while this dependence is present in the corresponding bound for left singular subspaces.

THEOREM 5.1. *Let A and $\widetilde{A} = A + E$ be two matrices in $\mathbb{C}^{m\times n}$, $m \geq n$ with SVDs* (1.1) *and* (1.2). *Define*

$$g = \min_{\substack{\lambda\in\sigma(\Sigma_1)\\ \mu\in\sigma(\Sigma_2)}} |\lambda - \mu| \tag{5.1}$$

and recall the gap

$$g_{ext} = \min_{\substack{\lambda\in\sigma(\Sigma_1)\\ \mu\in\sigma_{ext}(\Sigma_2)}} |\lambda - \mu|,$$

defined in (2.15). *If*

$$\|E\|_2 < \frac{g_{ext}}{2}, \tag{5.2}$$

then

$$\big\| \sin\, \Theta(U_1, \widetilde{U}_1)\big\|_F \leq -\frac{\|E\|_F}{2\,\|E\|_2} \ln\!\left(1 - \frac{2\|E\|_2}{g_{ext}}\right) \tag{5.3}$$

and

$$\big\| \sin\, \Theta(V_1, \widetilde{V}_1)\big\|_F \leq -\frac{\|E\|_F}{2\,\|E\|_2} \ln\!\left(1 - \frac{2\|E\|_2}{g}\right). \tag{5.4}$$

REMARK 5.1. The previous bound (5.4) for right singular subspaces remains valid under the assumption

$$\|E\|_2 < \frac{g}{2},$$

which allows perturbations larger than (5.2). This can be seen reversing the roles of Σ_1 and Σ_2 in Theorem 5.1: the gap g does not change, while g_{ext} is replaced by

$$g'_{ext} = \min_{\substack{\lambda\in\sigma(\Sigma_2)\\ \mu\in\sigma_{ext}(\Sigma_1)}} |\lambda - \mu|.$$

Notice that either $g_{ext} = g$ or $g'_{ext} = g$. The result follows since $\|\sin\,\Theta(V_2, \widetilde{V}_2)\|_F = \|\sin\,\Theta(V_1, \widetilde{V}_1)\|_F$.

REMARK 5.2. It is an important observation that the first order bounds obtained from Theorem 5.1 by expanding the logarithm are, ignoring the difference in gaps, a factor $\sqrt{2}$ smaller than the one which follows directly from Wedin's Theorem 1.1 replacing the residuals with $\|E\|_F$ and assuming the perturbed and unperturbed singular values to be approximately equal. This is to be expected, since Wedin's bound is of an intrinsically joint nature, unlike (5.3) and (5.4), which reflect separately the sensitivities of left and right singular subspaces.

PROOF. As in Corollary 2.3, consider the matrix family $A(t) = A + tE$ for $t \in [0, 1]$, and the orthogonal projection $P_1^{\pm}(t) = Y_1(t)\,Y_1(t)^*$ on the subspace spanned by the columns of

$$Y_1(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1(t) & U_1(t) \\ V_1(t) & -V_1(t) \end{bmatrix}.$$

By construction, the $m \times m$ orthogonal projection $\pi_u(t)$ on the column space of $U_1(t)$ is the upper left $m{\times}m$ block of the $(m{+}n){\times}(m{+}n)$ projection matrix $P_1^{\pm}(t)$. Hence, using Corollary 2.3, we have that $\pi_u(t)$ is continuously differentiable for t in $[0,1]$ and, dropping the t's for convenience,

$$\frac{d\pi_u}{dt} = W_1 \begin{bmatrix} 0 & \mathcal{G} \circ \mathcal{F} \\ \mathcal{G}^* \circ \mathcal{F}^* & 0 \end{bmatrix} W_1^*$$

for

$$W_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} U_1 & U_1 & U_2 & U_2 & \sqrt{2}\,U_3 \end{bmatrix}.$$

Since W_1 is a piece of a unitary matrix and $\|\cdot\|_F$ is unitarily invariant, we obtain

$$\left\| \frac{d\pi_u}{dt} \right\|_F \le \left\| \begin{bmatrix} 0 & \mathcal{G} \circ \mathcal{F} \\ \mathcal{G}^* \circ \mathcal{F}^* & 0 \end{bmatrix} \right\|_F . \tag{5.5}$$

If we define $g_{ext}(t)$ for $A(t)$ as in (2.15) but with all the involved quantities depending on t, then $g_{ext}(t)$ is a lower bound on all denominators in $\mathcal{G}(t)$, and

$$\left\| \frac{d\pi_u}{dt} \right\|_F \le \frac{1}{g_{ext}} \left\| \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{F}^* & 0 \end{bmatrix} \right\|_F . \tag{5.6}$$

Furthermore, the same argument used right after (3.4) shows that

$$\left\| \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{F}^* & 0 \end{bmatrix} \right\|_F \le \left\| \begin{bmatrix} 0 & E \\ E^* & 0 \end{bmatrix} \right\|_F = \sqrt{2}\,\|E\|_F.$$

Summarizing, we finally arrive at

$$\left\| \frac{d\pi_u}{dt}(t) \right\|_F \le \frac{\sqrt{2}\,\|E\|_F}{g_{ext}(t)}, \quad \text{for all} \quad t \in [0,1]. \tag{5.7}$$

As before, Weyl's Theorem implies that

$$g_{ext}(t) \ge g_{ext} - 2t\,\|E\|_2 > 0. \tag{5.8}$$

We now proceed as in the proof of Theorem 3.2, using the fact that [18, Theorem I.5.5] implies

$$\|\sin\Theta(U_1, \widetilde{U}_1)\|_F = \frac{1}{\sqrt{2}}\,\|\pi_u(1) - \pi_u(0)\|_F \le \frac{1}{\sqrt{2}} \int_0^1 \left\| \frac{d\pi_u}{dt}(t) \right\|_F dt.$$

Using the bounds (5.7) and (5.8), the same chain of inequalities in (3.10), with ρ_R replaced by g_{ext}, leads to the bound (5.3) on $\|\sin\Theta(U_1, \widetilde{U}_1)\|_F$.

As to the bound (5.4), notice that the $n \times n$ orthogonal projection $\pi_v(t)$ on the column space of $V_1(t)$ is the lower right $n \times n$ block of $P_1^{\pm}(t)$. Hence, also $\pi_v(t)$ is continuously differentiable for t in $[0,1]$ and

$$\frac{d\pi_v}{dt}(t) = W_2(t) \begin{bmatrix} 0 & \mathcal{G}(t) \circ \mathcal{F}(t) \\ \mathcal{G}(t)^* \circ \mathcal{F}(t)^* & 0 \end{bmatrix} W_2(t)^*$$

for

$$W_2(t) = \frac{1}{\sqrt{2}} \left[\begin{array}{ccccc} V_1(t) & -V_1(t) & V_2(t) & -V_2(t) & 0 \end{array} \right].$$

The null $n \times (m-n)$ rightmost block of $W_2(t)$ makes a big difference in this case, namely that the matrix product above is in fact

$$\frac{d\pi_v}{dt}(t) = \widehat{W}_2(t) \left[\begin{array}{cc} 0 & \widehat{\mathcal{G}}(t) \circ \widehat{\mathcal{F}}(t) \\ \widehat{\mathcal{G}}(t)^* \circ \widehat{\mathcal{F}}(t)^* & 0 \end{array} \right] \widehat{W}_2(t)^*,$$

where, using the notation of Corollary 2.3,

$$\widehat{W}_2(t) = \frac{1}{\sqrt{2}} \left[\begin{array}{cccc} V_1(t) & -V_1(t) & V_2(t) & -V_2(t) \end{array} \right],$$

$$\widehat{\mathcal{G}}(t) = \left[\begin{array}{c|c} G_2^+(t) & G_2^-(t) \\ \hline -G_2^-(t) & -G_2^+(t) \end{array} \right],$$

and the partitions

$$\begin{aligned} \mathcal{F}(t) &= \left[\widehat{\mathcal{F}}(t) \mid * \right], \\ \mathcal{G}(t) &= \left[\widehat{\mathcal{G}}(t) \mid * \right] \end{aligned} \tag{5.9}$$

are conformal. Hence, neither U_3 nor the gap matrix $G_0(t)$ play any role whatsoever in the bounds on the derivative of $\pi_v(t)$. This is the key to obtain different bounds for left and right singular subspaces.

Once we have this information, exactly the same procedure used above to bound $d\pi_u/dt$ leads directly to the bound (5.4), simply replacing $g_{ext}(t)$ with $g(t)$, defined as (5.1) with all the involved quantities depending on t. □

REMARK 5.3. As in Remark 2.2, the condition (5.2) on the range of validity of (5.3) may be relaxed when $m > n$ to

$$\|E\|_2 < \min\{\frac{g}{2}, \sigma_{min}(\Sigma_1)\}$$

at the cost of replacing the bound (5.3) by

$$\|\sin\Theta(U_1, \widetilde{U}_1)\|_F \leq -\frac{\|E\|_F}{2\,\|E\|_2} \left[\ln\left(1 - \frac{2\|E\|_2}{g}\right) + 2\,\ln\left(1 - \frac{\|E\|_2}{\sigma_{min}(\Sigma_1)}\right) \right].$$

The crucial point to do this is using the partitions (5.9) to separate the roles of the gap matrices G_2^+ and G_2^- from that of G_0 in (5.5).

REMARK 5.4. Versions for general unitarily invariant norms of Theorem 5.1 are also possible assuming further restrictions on the distribution of the singular values. We do not develop this idea to keep the paper relatively concise.

6 Theorems in unitarily invariant norms.

To obtain bounds in all unitarily invariant norms further restrictions on the singular values have to be imposed. While in Theorem 1.2 (respectively, in Theorem 3.2) only the disjointness between $\sigma(\widetilde{\Sigma}_1)$ and $\sigma_{ext}(\Sigma_2)$ (respectively, between $\sigma(\Sigma_1)$ and $\sigma(\Sigma_2)$) is necessary, to obtain Theorems in unitarily invariant norms we have to impose $\sigma(\widetilde{\Sigma}_1)$ and $\sigma_{ext}(\Sigma_2)$ (respectively $\sigma(\Sigma_1)$ and $\sigma(\Sigma_2)$ in the case of square real matrices) to be separated by two intervals. These requirements are similar to those used in $\sin\Theta$ theorems for arbitrary unitarily invariant norms [20].

6.1 General complex matrices.

The following simple result will be often used in this section.

LEMMA 6.1. *For any partitioned matrix it holds that*

$$\left\| \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right\|_2 \leq \sqrt{\|A_1\|_2^2 + \|A_2\|_2^2}.$$

PROOF. Notice that

$$[A_1^*, A_2^*] \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = A_1^* A_1 + A_2^* A_2,$$

and apply to this equation the triangle inequality for the spectral norm. The result follows taking into account that $\|B^*B\|_2 = \|B\|_2^2$ for any matrix B. □

In the next lemma we relate the direct generalization for unitarily invariant norms of the joint residual (1.7) to the sines of the canonical angles between the column spaces of X_1 and $\widetilde{X}_1$ defined in (3.11). The relationship with the difference of the corresponding orthogonal projectors P_1 and $\widetilde{P}_1$ is also stated.

LEMMA 6.2. *Let* $|||\cdot|||$ *be any unitarily invariant norm. Then*

$$\min_{W \text{unitary}} \left|\left|\left| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right|\right|\right| \leq 2\sqrt{2}\, |||\sin\Theta(X_1, \widetilde{X}_1)|||,$$

$$\min_{W \text{unitary}} \left|\left|\left| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right|\right|\right| \leq 2\sqrt{2}\, |||P_1 - \widetilde{P}_1|||,$$

$$\min_{W \text{unitary}} \left\| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right\|_2 \leq 2\, \|\sin\Theta(X_1, \widetilde{X}_1)\|_2,$$

$$\min_{W \text{unitary}} \left\| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right\|_2 \leq 2\, \|P_1 - \widetilde{P}_1\|_2.$$

REMARK 6.1. There is no simple way of finding the matrix W which minimizes the left-hand sides of the inequalities of the previous lemma, although the existence of this matrix is easily proved through an elementary compactness argument. Numerical algorithms to compute W in the real case can be found in [19]. In this case all the matrices appearing in Lemma 6.2 are real matrices, so the unitary matrix W has to be a real orthogonal matrix.

PROOF. Applying [18, Theorem I.5.2] and taking into account that $2k \leq (m+n)$, we can prove that there exist two unitary $k \times k$ matrices Z and $\widetilde{Z}$ such that

$$\begin{aligned}
&\min_{W \text{unitary}} \left|\left|\left| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right|\right|\right| \\
&\qquad = \sqrt{2} \min_{W \text{unitary}} \left|\left|\left| \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} Z W \widetilde{Z} - \begin{bmatrix} \cos\Theta(X_1, \widetilde{X}_1) \\ \sin\Theta(X_1, \widetilde{X}_1) \\ 0 \end{bmatrix} \right|\right|\right| \\
&\qquad \leq \sqrt{2} \left|\left|\left| \begin{bmatrix} I - \cos\Theta(X_1, \widetilde{X}_1) \\ -\sin\Theta(X_1, \widetilde{X}_1) \\ 0 \end{bmatrix} \right|\right|\right| \\
&\qquad \leq \sqrt{2} \left(|||I - \cos\Theta(X_1, \widetilde{X}_1)||| + |||\sin\Theta(X_1, \widetilde{X}_1)||| \right) \\
&\qquad \leq 2\sqrt{2}\, |||\sin\Theta(X_1, \widetilde{X}_1)|||.
\end{aligned}$$

The last step follows from the inequality $0 \leq 1 - \cos\theta \leq \sin\theta$ for acute angles using [18, Theorem II.3.7]. This proves part 1 of the lemma. Part 3 follows in a similar way using Lemma 6.1 instead of the triangle inequality in the second inequality of the expression above. Parts 2 and 4 are obtained from parts 1 and 3, respectively, using [18, Theorem I.5.5] and again [18, Theorem II.3.7]. □

Now we are able to prove the following perturbation theorem:

THEOREM 6.3. *Let A and $\widetilde{A}$ be two $m \times n$ $(m \geq n)$ complex matrices with SVDs* (1.1) *and* (1.2). *Assume that*

$$\sigma(\widetilde{\Sigma}_1) \subset [\sigma_{\min}(\widetilde{\Sigma}_1), \beta]$$

and that

$$\sigma(\Sigma_2) \subset [0, \sigma_{\min}(\widetilde{\Sigma}_1) - \delta] \cup [\beta + \delta, +\infty)$$

where β and δ are real positive numbers such that $0 < \sigma_{\min}(\widetilde{\Sigma}_1) < \beta$, $\delta > 0$ and $\sigma_{\min}(\widetilde{\Sigma}_1) - \delta \geq 0$. Define

$$\eta_b = \begin{cases} \delta & \text{if } \sigma(\Sigma_2) \cap [0, \sigma_{\min}(\widetilde{\Sigma}_1) - \delta] \neq \emptyset, \\ \min\{\delta, \sigma_{\min}(\Sigma_1) + \sigma_{\min}(\widetilde{\Sigma}_1)\} & \text{if } \sigma(\Sigma_2) \subset [\beta + \delta, +\infty) \text{ and } m = n, \\ \min\{\delta, \sigma_{\min}(\widetilde{\Sigma}_1)\} & \text{if } \sigma(\Sigma_2) \subset [\beta + \delta, +\infty) \text{ and } m > n. \end{cases}$$

If $\eta_b > 0$ then

1. $$\min_{W \text{unitary}} \left|\left|\left| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right|\right|\right| \leq \frac{2}{\eta_b} \left|\left|\left| \begin{bmatrix} R \\ S \end{bmatrix} \right|\right|\right| \leq 2 \frac{|||R||| + |||S|||}{\eta_b},$$
2. $$\min_{W \text{unitary}} \left\| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right\|_2 \leq \frac{\sqrt{2}}{\eta_b} \left\| \begin{bmatrix} R \\ S \end{bmatrix} \right\|_2 \leq \sqrt{2} \frac{\sqrt{\|R\|_2^2 + \|S\|_2^2}}{\eta_b}.$$

Remark 6.2. The bound for the spectral norm is similar to the bound for the Frobenius norm (1.9), and tighter than the bound for general invariant norms. We have been unable to improve this latter bound.

Proof. This theorem follows from parts 1 and 3 of Lemma 6.2, bounding the norms of the matrix of the sines of the canonical angles between the column spaces of X_1 and $\widetilde{X}_1$. This is accomplished applying the Davis–Kahan $\sin\Theta$ Theorem for unitarily invariant norms [5] to the unperturbed and perturbed Jordan–Wielandt matrices (2.9), and bearing in mind that

$$T_1 = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \widetilde{X}_1 - \widetilde{X}_1 \widetilde{\Sigma}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} R \\ S \end{bmatrix},$$

where R and S are the residuals defined by (1.3). Finally, the triangle inequality is used in part 1, while Lemma 6.1 in part 2. □

6.2 Real matrices.

For the sake of simplicity, we will only consider the case when the sensitivity of simultaneous bases is essentially different from the sensitivity of singular subspaces, that is, singular subspaces associated with the *smallest singular values* of *square* matrices. We can deal with other cases in a similar way.

As in Section 3, we begin with a lemma which bounds the derivatives of the corresponding orthogonal projectors on any unitarily invariant norm. From these bounds, the main result of this subsection, Theorem 6.5, follows easily using again the integral techniques in [1]. The proof of Lemma 6.4, although easy, is rather technical and may be skipped without affecting the rest of the paper.

Lemma 6.4. *Under the same assumptions and with the same notation of Corollary 2.2 and Lemma 3.1, assume that A and E are real square $n \times n$ matrices and that for all $t \in [0,1]$ there exist two numbers $\alpha_t > 0$ and $\delta_t > 0$ such that*

$$\sigma(\Sigma_1(t)) \subset (0, \alpha_t] \quad \text{and} \quad \sigma(\Sigma_2(t)) \subset [\alpha_t + \delta_t, +\infty).$$

1. *Then for any unitarily invariant norm $|||\,.\,|||$*

$$\left|\left|\left| \frac{dP_1}{dt}(t) \right|\right|\right| \leq \left(\frac{1}{\sigma_{k-1}(t) + \sigma_k(t)} + \frac{1}{\delta_t} \right) \left|\left|\left| \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right|\right|\right|,$$

2. *and for the spectral norm $\|\,.\,\|_2$*

$$\left\| \frac{dP_1}{dt}(t) \right\|_2 \leq \frac{\sqrt{2}\|E\|_2}{\min\{\delta_t, \sigma_{k-1}(t) + \sigma_k(t)\}}.$$

Proof. The proof relies heavily on Corollary 2.2, and we are going often to use magnitudes appearing in that Corollary. Moreover, it is convenient to bear in mind that we are dealing with real square matrices, so the matrices $U_3(t)$ and

$G_0(t)$ are not present in the following arguments and conjugate transposes are just transpose matrices. From equation (2.14)

$$\left|\left|\left|\frac{dP_1}{dt}(t)\right|\right|\right| = \left|\left|\left|\left[\begin{array}{c|c} 0 & G(t) \circ F(t) \\ \hline G(t)^T \circ F(t)^T & 0 \end{array}\right]\right|\right|\right|. \tag{6.1}$$

Define

$$X_1^-(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} U_1(t) \\ -V_1(t) \end{bmatrix} \quad \text{and} \quad Y_2(t) = \frac{1}{\sqrt{2}}\begin{bmatrix} U_2(t) & U_2(t) \\ V_2(t) & -V_2(t) \end{bmatrix}.$$

Thus, $F(t)$ can be partitioned in two blocks as $F(t) \equiv [F_1(t) \;\; F_2(t)]$, where

$$F_1(t) = X_1(t)^T \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} X_1^-(t), \qquad F_2(t) = X_1(t)^T \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} Y_2(t).$$

We partition $G(t)$ conformally as

$$G(t) \equiv [G_1(t) \;\; G_2(t)]$$

where, according to the definitions in Corollary 2.2, $G_2(t) \equiv [G_2^+(t) \; G_2^-(t)]$.

In the rest of the proof we are going to deal with the problem of bounding

$$G(t) \circ F(t) = [G_1(t) \circ F_1(t) \quad G_2(t) \circ F_2(t)] \tag{6.2}$$

for all unitarily invariant norms. This is equivalent to bounding the derivative of the orthogonal projector $P_1(t)$, as can be seen taking into account equation (6.1) and the fact that, by Fan's dominance theorem [2, Theorem IV.2.2], given two matrices C and D,

$$\left|\left|\left|\begin{bmatrix} 0 & C \\ C^* & 0 \end{bmatrix}\right|\right|\right| \leq \left|\left|\left|\begin{bmatrix} 0 & D \\ D^* & 0 \end{bmatrix}\right|\right|\right| \quad \text{for all unitarily invariant norms,} \tag{6.3}$$

if and only if $|||C||| \leq |||D|||$ for all unitarily invariant norms. However, some care has to be taken because these ideas cannot be applied in a straightforward way.

The first step is to split the problem into two simpler problems using the triangle inequality on

$$[G_1(t) \circ F_1(t) \;\; G_2(t) \circ F_2(t)] = [G_1(t) \circ F_1(t) \;\; 0] + [0 \;\; G_2(t) \circ F_2(t)] \tag{6.4}$$

to obtain

$$|||G(t) \circ F(t)||| \leq |||G_1(t) \circ F_1(t)||| + |||G_2(t) \circ F_2(t)|||.$$

To bound $|||G_1(t) \circ F_1(t)|||$, notice that $F_1(t)$ is a $k \times k$ real skew symmetric matrix. Thus, we can use Corollary 3.3 in [15] to obtain

$$|||G_1(t) \circ F_1(t)||| \leq \frac{1}{\sigma_{k-1}(t) + \sigma_k(t)} |||F_1(t)|||. \tag{6.5}$$

To bound $|||G_2(t)\circ F_2(t)|||$, notice that $G_2(t)\circ F_2(t)$ is the solution of the Sylvester equation $\Sigma_1(t)Z - Z\mathrm{diag}(\Sigma_2(t), -\Sigma_2(t)) = F_2(t)$. Applying the classical result of Davis and Kahan [5] (see also [2, Section VII.2]), we obtain

$$|||G_2(t)\circ F_2(t)||| \le \frac{1}{\delta_t}|||F_2(t)|||. \tag{6.6}$$

Combining equations (6.1) and (6.4), and taking into account that zero rows or columns can be suppressed without changing any unitarily invariant norm, we get

$$\left|\left|\left|\frac{dP_1}{dt}(t)\right|\right|\right| \le \left|\left|\left|\left[\begin{array}{c|c} 0 & G_1(t)\circ F_1(t) \\ \hline G_1(t)^T\circ F_1(t)^T & 0 \end{array}\right]\right|\right|\right| + \left|\left|\left|\left[\begin{array}{c|c} 0 & G_2(t)\circ F_2(t) \\ \hline G_2(t)^T\circ F_2(t)^T & 0 \end{array}\right]\right|\right|\right|,$$

and using the bounds (6.5) and (6.6), and result (6.3) the following bound is obtained

$$\left|\left|\left|\frac{dP_1}{dt}(t)\right|\right|\right| \le \frac{1}{\sigma_{k-1}(t)+\sigma_k(t)}\left|\left|\left|\left[\begin{array}{c|c} 0 & F_1(t) \\ \hline F_1(t)^T & 0 \end{array}\right]\right|\right|\right| + \frac{1}{\delta_t}\left|\left|\left|\left[\begin{array}{c|c} 0 & F_2(t) \\ \hline F_2(t)^T & 0 \end{array}\right]\right|\right|\right|.$$

The first part of the lemma follows from the fact that unitarily invariant norms decrease by pinchings (see [18, Exercise 1, p. 88] or [2, (IV.52), p. 97]), and an almost similar argument to that right after (3.4).

For the second part, notice that from equation (2.14) we get

$$\left\|\frac{dP_1}{dt}(t)\right\|_2 = \|G(t)\circ F(t)\|_2 \le \sqrt{\|G_1(t)\circ F_1(t)\|_2^2 + \|G_2(t)\circ F_2(t)]\|_2^2}$$

applying Lemma 6.1 to equation (6.2). The result follows again from (6.5) and (6.6) after some straightforward manipulations. □

Lemma 6.4 leads to the following theorem:

THEOREM 6.5. *Let A and $\widetilde{A} = A+E$ be two $n\times n$ real matrices with conformally partitioned real singular value decompositions*

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}\begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}, \qquad \widetilde{A} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{bmatrix}\begin{bmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & \widetilde{\Sigma}_2 \end{bmatrix}\begin{bmatrix} \widetilde{V}_1^T \\ \widetilde{V}_2^T \end{bmatrix}.$$

Consider the ordered singular values of Σ_1 to be $\sigma(\Sigma_1) = \{\sigma_1 \ge \cdots \ge \sigma_k\}$ and assume there exist numbers $\alpha > 0$ and $\delta > 0$ such that

$$\sigma(\Sigma_1) \subset (0,\alpha] \quad \text{and} \quad \sigma(\Sigma_2) \subset [\alpha+\delta, +\infty).$$

If

$$\|E\|_2 < \frac{1}{2}\min\{\delta\,,\,2\,\sigma_k\} \tag{6.7}$$

then

$$\min_{W \text{orthogonal}} \left|\left|\left| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right|\right|\right|$$
$$\leq -2\sqrt{2}\, \frac{\left|\left|\left| \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right|\right|\right|}{\|E\|_2} \ln\left(1 - \frac{2\|E\|_2}{\min\{\delta, \sigma_{k-1} + \sigma_k\}}\right).$$

$$\min_{W \text{orthogonal}} \left\| \begin{bmatrix} U_1 \\ V_1 \end{bmatrix} W - \begin{bmatrix} \widetilde{U}_1 \\ \widetilde{V}_1 \end{bmatrix} \right\|_2 \leq -\sqrt{2}\, \ln\left(1 - \frac{2\|E\|_2}{\min\{\delta, \sigma_{k-1} + \sigma_k\}}\right).$$

Proof. Once again, consider the matrix family $A(t) = A + tE$ introduced just before Corollary 2.2. The notation of this corollary will be used again. Using Weyl's perturbation theorem and the restriction (6.7) we get that for any $t \in [0, 1]$

$$\sigma(\Sigma_1(t)) \subset (0, \alpha + t\|E\|_2] \quad \text{and} \quad \sigma(\Sigma_2(t)) \subset [\alpha + \delta - t\|E\|_2, +\infty)$$

with $\alpha + \delta - t\|E\|_2 - (\alpha + t\|E\|_2) = \delta - 2t\|E\|_2 > 0$. Thus, we can apply Lemma 6.4 to get

$$\begin{aligned}
|||\widetilde{P}_1 - P_1||| &\leq \int_0^1 \left|\left|\left| \frac{dP_1}{dt}(t) \right|\right|\right| dt \\
&\leq 2 \left|\left|\left| \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right|\right|\right| \int_0^1 \frac{dt}{\min\{\sigma_{k-1} + \sigma_k, \delta\} - 2t\|E\|_2} \\
&= -\frac{\left|\left|\left| \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \right|\right|\right|}{\|E\|_2} \ln\left(1 - \frac{2\|E\|_2}{\min\{\sigma_{k-1} + \sigma_k, \delta\}}\right).
\end{aligned}$$

The first part of the theorem follows from combining the previous bound with the second part of Lemma 6.2. The result for the spectral norm is proved in a similar way using the corresponding parts of Lemmas 6.2 and 6.4. □

Acknowledgments.

The authors thank Prof. Antonio García for helpful discussions and constructive comments.

REFERENCES

1. A. Barrlund, *Perturbation bounds on the polar decomposition*, BIT, 30 (1989), pp. 101–113.
2. R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.
3. F. Chaitin-Chatelin and S. Gratton, *On the condition number associated with the polar factorization of a matrix*, Numer. Linear Algebra Appl., 7 (2000), pp. 337–354.
4. F. Chatelin, *Eigenvalues of Matrices*, Wiley, New York, 1993.
5. C. Davis and W. Kahan, *The rotation of eigenvectors by a perturbation. III*, SIAM J. Numer. Anal., 7 (1970), pp. 1–46.

6. F. M. Dopico, *A note on* $\sin\Theta$ *theorems for singular subspace variations*, BIT, 40 (2000), pp. 395–403.
7. G. H. Golub and C. F. Van Loan, *Matrix Computations,* 3rd. ed., Johns Hopkins University Press, Baltimore, MD, 1996.
8. R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
9. T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995.
10. C. Kenney and A. J. Laub, *Polar decomposition and matrix sign function condition estimates*, SIAM J. Sci. Statist. Comput., 12 (1991), pp. 488–504.
11. R.-C. Li, *New perturbation bounds for the unitary polar factor*, SIAM J. Matrix Anal. Appl., 16 (1995), pp. 327–332.
12. R.-C. Li, *Relative Perturbation Theory: II. Eigenspace and Singular Subspace Variations*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 471–492.
13. R.-C. Li and G. W. Stewart, *A new relative perturbation theorem for singular subspaces*, Linear Algebra Appl., 313 (2000), pp. 41–51.
14. T. Londré and N. H. Reeh, *A note on relative perturbation bounds*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 357–361.
15. R. Mathias, *The singular values of the Hadamard product of a positive semidefinite and a skew-symmetric matrix*, Linear and Multilinear Algebra, 31 (1992), pp. 57–70.
16. R. Mathias, *Perturbation bounds for the polar decomposition*, SIAM J. Matrix Anal. Appl., 14 (1993), pp. 588–597.
17. C. C. Paige, *A note on a result of Sun Ji-Guang: Sensitivity of the CS and GSV decompositions*, SIAM J. Numer. Anal., 21 (1984), pp. 186–191.
18. G. W. Stewart and J.-G Sun, *Matrix Perturbation Theory*, Academic Press, Boston, MA, 1990.
19. G. A. Watson, *The solution of orthogonal Procrustes problems for a family of orthogonally invariant norms*, Adv. Comput. Math., 2 (1994), pp. 393–405.
20. P.-Å. Wedin, *Perturbation bounds in connection with singular value decomposition*, BIT, 12 (1972), pp. 99–111.