XXI Congreso de Ecuaciones Diferenciales y Aplicaciones
XI Congreso de Matemática Aplicada
Ciudad Real, 21-25 septiembre 2009
(pp. 1–9)

Linearizations of matrix polynomials: Sharp lower bounds for the dimension and structures

Fernando de Terán[1], Froilán M. Dopico[2], D. Steven Mackey[3]

[1] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. E-mail:* `fteran@math.uc3m.es`.
[2] *Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain. E-mail:* `dopico@math.uc3m.es`
[3] *Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA. E-mail:* `steve.mackey@wmich.edu`.

Palabras clave: Singular matrix polynomials, Linearizations, Structured linearizations, Dimension

Resumen

In this note we will survey some recent results on linearizations of singular matrix polynomials. We also present new results regarding structured linearizations of structured matrix polynomials.

1. Introduction

A square *matrix polynomial* of size $n \times n$ and degree k is a polynomial of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n\times n}, \quad A_k \neq 0 \,, \tag{1}$$

where $\mathbb{F}$ is the field of real or complex numbers. Our focus is on singular matrix polynomials. A square matrix polynomial $P(\lambda)$ is said to be *singular* if $\det P(\lambda)$ is identically zero, and it is said to be *regular* otherwise. The *normal rank* of $P(\lambda)$ is the dimension of the largest non-identically zero minor of $P(\lambda)$.

The matrix polynomial (1) naturally arise associated with linear systems of differential equations

$$A_k x^{(k)}(t) + A_{k-1} x^{(k-1)}(t) + \ldots + A_1 x'(t) + A_0 x(t) = f(t) \,, \tag{2}$$

where $x(t)$ is a vector-valued function (unknown) with n coordinates, $x^{(j)}(t)$ denotes the jth derivative of $x(t)$ and $f(t)$ is another vector-valued function with n coordinates. Of particular relevance is the case of linear systems of second order, appearing in many engineering applications. The matrix polynomials of second degree associated with these systems are usually known as *vibrating systems* and they have been widely treated in the literature (see, for instance, the survey [12] and the references therein). The solubility of the system (2) and its solutions, if any, can be analyzed through the *eigenstructure* of the associated matrix polynomial (1) [13, 7, 11]. This structure consists of the *finite* and *infinite elementary divisors* and the *minimal indices* [6, 8].

An useful approach to address the study and computation of the spectral structure of regular matrix polynomials is through the use of *linearizations*. A matrix pencil $L(\lambda) = \lambda X + Y$ is a *linearization* of the matrix polynomial (1) (regular or singular) if there exist two unimodular matrices $E(\lambda)$ and $F(\lambda)$ such that

$$E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_s \end{bmatrix}, \tag{3}$$

or in other words, if $L(\lambda)$ is *equivalent* to $\operatorname{diag}(P(\lambda), I_s)$. If we define $\operatorname{rev}P(\lambda) := \lambda^k P(1/\lambda)$ for the *reversal* polynomial of $P(\lambda)$, then a linearization $L(\lambda)$ of $P(\lambda)$ is called a *strong linearization* if $\operatorname{rev}L(\lambda)$ is also a linearization of $\operatorname{rev}P(\lambda)$. The advantage of strong linearizations is that they allow us to reduce a problem of higher degree to a problem of degree one, without affecting the elementary divisors associated with both the finite and the infinite eigenvalues of the matrix polynomial. The classical approach is to use as linearizations the *first and second companion forms* [7]. Recently, new families of linearizations have been introduced, for regular matrix polynomials, in [2, 9, 1]. On the other hand, the first works containing a systematic study of linearizations for singular matrix polynomials are [4, 5], where it is shown, in particular, that the families introduced in [2, 9, 1] are still strong linearizations for singular matrix polynomials.

Unlike what it happens with the elementary divisors, linearizations of singular matrix polynomials do not necessarily preserve the minimal indices. For instance, the first and the second companion forms do not have the same minimal indices as the matrix polynomial [4]. In Section 2 we will show how to recover the minimal indices of the matrix polynomial from the minimal indices of the linearizations in each of the families mentioned in the previous paragraph.

The main disadvantage of linearizations is that they increase the dimension of the problem. In the classical definition of linearization the dimension of the identity block in the right hand side of (3) is $s = (k-1)n$, but it is natural to ask whether or not linearizations with lower dimension exist and, if they exist, to ask for the minimum dimension of any linearization of a given matrix polynomial $P(\lambda)$. This question has been solved in [3]. In Section 3 we show the main result included in that paper.

In many applied problems the coefficient matrices of the associated matrix polynomial (1) have a particular structure (e.g. symmetric, skew-symmetric, palindromic) which reflects the properties of the underlying physical model [10]. In the context of the numerical calculation of eigenvalues through linearizations, a desirable property of the linearization is the preservation of the structure of the polynomial. This property is not satisfied, for instance, by the companion forms. Therefore the rounding errors inherent to numerical computations may destroy qualitative aspects of the spectrum when using companion forms. On the other hand, it is shown in [10] that for most structured regular polynomials $P(\lambda)$ (symmetric, palindromic, even or odd) there are structured linearizations in the sets of pencils introduced in [9]. This fact turns these linearizations into suitable ones for the calculation of eigenvalues of structured matrix polynomials. Nonetheless, the problem of finding an easily constructible (from polynomial data) structured linearization, valid for any structured polynomial, remained open. In Section 4 we will show a procedure to construct, from the coefficients of the polynomial, T-palindromic linearizations of a given T-palindromic matrix polynomial which is valid for any matrix polynomial with odd degree. On the contrary, we will show that, if the polynomial is of even degree, some conditions on its eigenstructure must hold in order to exist a T-palindromic linearization. In particular, we will show that most singular T-palindromic polynomials of even degree k do not have any T-palindromic linearization of size $nk \times nk$.

2. Linearizations of singular matrix polynomials and the recovery of minimal indices

2.1. The Fiedler pencils

We begin by recalling the families of matrix pencils defined in [2]. They are built up through the matrices

$$M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix} \tag{4}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1, \tag{5}$$

which are the building factors needed to define the *Fiedler pencils.*

Definition 1 (Fiedler Pencils) [5] *Let* $P(\lambda)$ *be the matrix polynomial in* (1) *and* M_i, $i = 0, \ldots, k$, *be the matrices defined in* (4) *and* (5). *Given any bijection* $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$, *the* Fiedler pencil *of* $P(\lambda)$ *associated with* σ *is the* $nk \times nk$ *matrix pencil*

$$F_\sigma(\lambda) := \lambda M_k - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}. \tag{6}$$

Note that $\sigma(i)$ *is the position of the factor* M_i *in the product* $M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}$ *giving the zero-degree term in* (6).

The first main result in this section is the following.

Theorem 1 [2, 5] *Let* $P(\lambda)$ *be an* $n \times n$ *matrix polynomial of degree* k. *Then, for any bijection* $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$, *the Fiedler pencil* $F_\sigma(\lambda)$ *is a strong linearization of* $P(\lambda)$.

Theorem 1 was first proved in [2] for regular matrix polynomials using methods that cannot be extended to singular matrix polynomials. With different techniques the authors have extended it to singular matrix polynomials in [5], where the name *Fiedler pencils* is introduced.

As mentioned in Section 1, the eigenstructure of a singular matrix polynomial consists of the elementary divisors and the minimal indices. Minimal indices comprise the "singular part" of the eigenstructure of the matrix polynomial. To define this minimal indices we first introduce Definitions 2 and 3. The vector space $\mathbb{F}(\lambda)^n$ is the space of n-tuples with entries in the field of rational functions $\mathbb{F}(\lambda)$. The *degree* of a vector polynomial is the greatest degree of its components, and the *order* of a polynomial basis of a vector subspace of $\mathbb{F}(\lambda)^n$ is defined as the sum of the degrees of its vectors [6, p. 494].

Definition 2 *The* right *and* left nullspaces *of the $n \times n$ matrix polynomial $P(\lambda)$, denoted by $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$ respectively, are the following subspaces of $\mathbb{F}(\lambda)^n$:*

$$\begin{aligned} \mathcal{N}_r(P) &:= \left\{x(\lambda) \in \mathbb{F}(\lambda)^n \,:\, P(\lambda)x(\lambda) \equiv 0\right\} , \\ \mathcal{N}_\ell(P) &:= \left\{y(\lambda) \in \mathbb{F}(\lambda)^n \,:\, y^T(\lambda)P(\lambda) \equiv 0\right\} . \end{aligned}$$

Definition 3 [6] *Let $\mathcal{V}$ be a subspace of $\mathbb{F}(\lambda)^n$. A* minimal basis *of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$.*

It can be shown [6] that for any given subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. Specializing $\mathcal{V}$ to be the left and right nullspaces of a singular matrix polynomial gives Definition 4; here $\deg(p(\lambda))$ denotes the degree of the vector polynomial $p(\lambda)$.

Definition 4 *Let $P(\lambda)$ be a square singular matrix polynomial, and let the sets $\{y_1(\lambda), \ldots, y_p(\lambda)\}$ and $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ be minimal bases of, respectively, the left and right nullspaces of $P(\lambda)$, ordered such that $\deg(y_1) \le \deg(y_2) \le \cdots \le \deg(y_p)$ and $\deg(x_1) \le \deg(x_2) \le \cdots \le \deg(x_p)$. Let $\eta_i = \deg(y_i)$ and $\varepsilon_i = \deg(x_i)$ for $i = 1, \ldots, p$. Then $\eta_1 \le \eta_2 \le \cdots \le \eta_p$ and $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$ are, respectively, the* left *and* right minimal indices *of $P(\lambda)$.*

As mentioned above, strong linearizations do not necessarily preserve the minimal indices of the polynomial, and the natural question is to ask whether or not there is a simple relationship between these quantities. In [5] the authors give the answer to this question for the Fiedler pencils $F_\sigma(\lambda)$. To state it we introduce the following definition.

Definition 5 *Let $s \le k$ be some positive integer and $C_s := \{j_1, \ldots, j_s\}$ be a set of s distinct nonnegative integers satisfying $0 \le j_i \le k-1$, for $i = 1, \ldots, s$. Let $\tau : C_s \to \{1, \ldots, s\}$ be a bijection. For $j \in \{j_1, j_2, \ldots, j_s\}$ we say that σ has a* consecution *at j if $j+1 \in \{j_1, j_2, \ldots, j_s\}$ and $\tau(j) < \tau(j+1)$, and that τ has an* inversion *at j if $j+1 \in \{j_1, j_2, \ldots, j_s\}$ and $\tau(j) > \tau(j+1)$*

We denote by $c(\tau)$ (resp. $i(\tau)$) the number of consecutions (resp. inversions) of τ.

The previous definition has been stated for a set C_s with cardinality $s \leq k$. In the present section we consider only the case $s = k$ and bijections $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$. Sets with cardinality $s \leq k$ will appear in Section 4.

Theorem 2 gives the relationship between the minimal indices of $P(\lambda)$ and $F_\sigma(\lambda)$.

Theorem 2 [5] *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda) \in \mathbb{F}(\lambda)^{nk \times nk}$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $c(\sigma), i(\sigma)$ be, respectively, the number of consecutions and inversions of σ. If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$\varepsilon_1 + i(\sigma) \leq \varepsilon_2 + i(\sigma) \leq \cdots \leq \varepsilon_p + i(\sigma)\,,$$

are the right minimal indices of $F_\sigma(\lambda)$.

If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then

$$\eta_1 + c(\sigma) \leq \eta_2 + c(\sigma) \leq \cdots \leq \eta_p + c(\sigma)\,,$$

are the left minimal indices of $F_\sigma(\lambda)$.

2.2. The vector spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$

Now we recall the families of matrix pencils defined in [9]. In this section we will follow the notation $\Lambda := \left[\lambda^{k-1}, \lambda^{k-2}, \ldots, \lambda, 1\right]^T$ for the vector of decreasing powers of λ and $\otimes$ for the Kronecker product.

Let us consider the following sets of matrix pencils introduced in [9]:

$$\begin{aligned}
\mathbb{L}_1(P) &:= \left\{L(\lambda) \,:\, L(\lambda)(\Lambda \otimes I_n) = v \otimes P(\lambda),\, v \in \mathbb{F}^k\right\}, \\
\mathbb{L}_2(P) &:= \left\{L(\lambda) \,:\, (\Lambda^T \otimes I_n)L(\lambda) = w^T \otimes P(\lambda),\, w \in \mathbb{F}^k\right\}
\end{aligned} \tag{7}$$

The vectors v and w in (7) are referred to, respectively, as the "right ansatz" and "left ansatz" vectors of $L(\lambda)$. It is proved in [9] that, for any square matrix polynomial $P(\lambda)$, regular or singular, the sets in (7) are vector subspaces of the vector space of all $nk \times nk$ matrix pencils over $\mathbb{F}$. In this note we will focus on the set $\mathbb{L}_1(P)$, but an analogous treatment can be carried out with the set $\mathbb{L}_2(P)$.

The next result gives us a simple way to detect strong linearizations in $\mathbb{L}_1(P)$.

Theorem 3 [9, 4] *Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ with $A_k \neq 0$ be a (regular or singular) $n \times n$ matrix polynomial and $L(\lambda) = \lambda X + Y \in \mathbb{L}_1(P)$ with right ansatz vector $v \neq 0$. Suppose $M \in \mathbb{F}^{k \times k}$ is any nonsingular matrix such that $Mv = \alpha e_1$ for some number $\alpha \neq 0$. Then the pencil $(M \otimes I_n)L(\lambda)$ can be written as*

$$(M \otimes I_n)L(\lambda) = \lambda \left[\begin{array}{c|c} \alpha A_k & X_{12} \\ \hline 0 & -Z \end{array}\right] + \left[\begin{array}{c|c} Y_{11} & \alpha A_0 \\ \hline Z & 0 \end{array}\right], \tag{8}$$

with $Z \in \mathbb{F}^{(k-1)n \times (k-1)n}$. If Z is nonsingular then $L(\lambda)$ is a strong linearization of $P(\lambda)$.

Now, we can state the main theorem of this section. This result shows the relationship between minimal bases of $P(\lambda)$ and minimal bases of linearizations in $\mathbb{L}_1(P)$ and, as a consequence, the relationship between the minimal indices.

Theorem 4 [4] *Let $P(\lambda)$ be an $n\times n$ matrix polynomial of degree k. Suppose $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector $v \neq 0$ is a linearization of $P(\lambda)$. Then the following holds.*

(a) *If $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$(k-1)+\varepsilon_1 \;\leq\; (k-1)+\varepsilon_2 \;\leq\; \cdots \;\leq\; (k-1)+\varepsilon_p$$

are the right minimal indices of $L(\lambda)$.

(b) *Every minimal basis of $\mathcal{N}_r(L)$ is of the form $\{\Lambda \otimes x_1(\lambda), \ldots, \Lambda \otimes x_p(\lambda)\}$, where $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ is a minimal basis of $\mathcal{N}_r(P)$.*

Moreover, if the matrix Z in (8) *is nonsingular, then*

(a) *the left minimal indices of $L(\lambda)$ and $P(\lambda)$ are the same,*

(b) *every minimal basis of $\mathcal{N}_\ell(P)$ is of the form $\{(v^T \otimes I_n)y_1(\lambda), \ldots, (v^T \otimes I_n)y_p(\lambda)\}$, where $\{y_1(\lambda), \ldots, y_p(\lambda)\}$ is a minimal basis of $\mathcal{N}_\ell(L)$.*

Note that Theorem 4 is true under certain conditions, namely, that $L(\lambda) \in \mathbb{L}_1(P)$ is a linearization of $P(\lambda)$, and, for the second part of the statement, that the matrix Z in (8) is nonsingular. These are not strong restrictions, as the following result shows.

Theorem 5 (Linearizations are generic in $\mathbb{L}_1(P)$) [4]
Let $P(\lambda)$ be an $n \times n$ matrix polynomial (regular or singular). Then, for almost every pencil in $\mathbb{L}_1(P)$ the matrix Z in (8) *is nonsingular. As a consequence, almost every pencil in $\mathbb{L}_1(P)$ is a strong linearization for $P(\lambda)$.*

3. Dimension of linearizations

In this section, we consider matrix polynomials of size $m \times n$. This includes both rectangular and square matrix polynomials. The classical definition of linearization fixes the size s of the identity matrix in (3) to be $s = (k-1)\,\text{mín}\{m, n\}$. This is the case, for instance, of the linearizations studied in Section 2. We will see that if $P(\lambda)$ is singular then strong linearizations with lower dimension may exist and we will provide a sharp lower bound on this dimension. On the contrary, we will see that in the regular case all strong linearizations have dimension $nk \times nk$.

Definition 6 *Let $P(\lambda)$ be a matrix polynomial of degree k with normal rank equal to r. The* finite degree *of $P(\lambda)$ is the degree of the greatest common divisor of all the $r \times r$ minors of $P(\lambda)$. The* infinite degree *of $P(\lambda)$ is the multiplicity of the zero root in the greatest common divisor of all the $r \times r$ minors of* $\text{rev}P(\lambda)$.

Now we can state the main result of this section.

Theorem 6 [3] *Let $P(\lambda)$ be an $m \times n$ matrix polynomial with normal rank r, finite degree α and infinite degree β. Then the following statements hold.*

1. *There exists a linearization of* $P(\lambda)$ *with dimension*

$$(\text{máx}\{\alpha, r\} + m - r) \times (\text{máx}\{\alpha, r\} + n - r)\,, \tag{9}$$

and there are no linearizations of $P(\lambda)$ *with dimension smaller than* (9).

2. *There exists a strong linearization of* $P(\lambda)$ *with dimension*

$$(\text{máx}\{\alpha + \beta, r\} + m - r) \times (\text{máx}\{\alpha + \beta, r\} + n - r)\,, \tag{10}$$

and there are no strong linearizations of $P(\lambda)$ *with dimension smaller than* (10).

If $P(\lambda)$ is a regular matrix polynomial then $m = n = r$ and $\alpha + \beta = nk$, so, by Theorem 6, the minimal size for a strong linearization of $P(\lambda)$ is $nk \times nk$. This in, in fact, the only size for linearizations of regular matrix polynomials [3, Th. 3.2].

4. Structured linearizations

Structured matrix polynomials arise in many applied problems. To preserve the spectral properties associated to these structures in the numerical solution of the polynomial eigenvalue problem through linearizations it is important to construct linearizations preserving the structure in order to apply an appropriate numerical method to the corresponding linear eigenvalue problem (see the references in [10] for some already know structured methods for matrix pencils). The following definition comprises the structures we are dealing with. For conciseness, the symbol $\star$ is used as an abbreviation for transpose T in the real case and for either T or conjugate transpose $*$ in the complex case.

Definition 7 *Let* $P(\lambda)$ *be the* $n \times n$ *matrix polynomial as in* (1)*, and define the associated polynomial* $P^\star(\lambda) := \sum_{i=0}^{k} \lambda^i A_i^\star$. *Then* $P(\lambda)$ *is said to be* $\star$-symmetric *if* $P^\star(\lambda) = P(\lambda)$, *and it is said to be* $\star$-palindromic *if* $\text{rev}P^\star(\lambda) = P(\lambda)$.

The following theorem shows how to construct T-palindromic linearizations of T-palindromic polynomials with odd degree. From now on, $R := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix}$ denotes the block $k \times k$ reverse identity.

Theorem 7 *Let* k *be an odd number,* $P(\lambda) = \sum_{i=1}^{k} \lambda^i A_i$, *with* $A_i \in \mathbb{C}^{n\times n}$ *and* $A_k \neq 0$, *be a square matrix polynomial (regular or singular) of degree* k, *and* M_i, *for* $i = 0, 1, \ldots, k$, *be the matrices defined in* (4-5). *Set*

$$\widetilde{M}_{k-j} := \begin{cases} M_k & \text{if } j = 0 \\ M_{k-j}^{-1} & \text{if } j \neq 0 \end{cases}. \tag{11}$$

Set $\overline{k} = (k+1)/2$ *and let* $C = \{j_1, j_2, \ldots, j_{\overline{k}}\} \subseteq \{0, 1, 2, \ldots, k\}$ *be a set of* $\overline{k}$ *distinct numbers satisfying*

- $0 \in C$
- $C \bigcap \{k - j_1, \ldots, k - j_{\overline{k}}\} = \emptyset$.

Let $\tau : C \to \{1, 2, \ldots, \overline{k}\}$ *be a bijection,* $L_\tau(\lambda)$ *the pencil*

$$L_\tau(\lambda) := \lambda \widetilde{M}_{k-\tau^{-1}(\overline{k})} \cdots \widetilde{M}_{k-\tau^{-1}(2)} \widetilde{M}_{k-\tau^{-1}(1)} - M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(\overline{k})}$$

and S *the* $nk \times nk$ *block-diagonal matrix whose diagonal block* $S(i,i)$*, for* $i = 1, \ldots, k$*, is defined by*

$$S(i,i) := \begin{cases} -I & \text{if} \quad \begin{cases} \tau \text{ has an inversion at } i-1, \text{ or} \\ \tau \text{ has a consecution at } k-i, \text{ or} \\ i \in C \text{ but } i-1 \notin C \end{cases} \\ I & \text{otherwise} \end{cases} . \tag{12}$$

Then the pencil $S \cdot R \cdot L_\tau(\lambda)$ *is a* T*-palindromic strong linearization of* $P(\lambda)$ *whenever* $P(\lambda)$ *is* T*-palindromic.*

As a consequence, we have

Corollary 1 *Let* k *be an odd number and* $P(\lambda)$ *be an* $n \times n$ T*-palindromic matrix polynomial of degree* k*. Then, there exists an* $nk \times nk$ T*-palindromic linearization of* $P(\lambda)$*.*

The situation in the case of singular matrix polynomials with even degree is completely different, as the following result shows.

Theorem 8 *Let* k *be an even number. Then the set of* $n \times n$ *singular* $\star$*-palindromic (resp.* $\star$*-symmetric) matrix polynomials of degree* k *having an* $nk \times nk$ $\star$*-palindromic (resp.* $\star$*-symmetric) strong linearization is non-generic in the set of* $n \times n$ *singular* $\star$*-palindromic (resp.* $\star$*-symmetric) matrix polynomials of degree* k*.*

Finally, if we allow linearizations with dimension less than $nk \times nk$ then we have the following characterization of the set of T-palindromic polynomials having a T-palindromic linearization. The multiplicity of an elementary divisor of $P(\lambda)$, mentioned in the statement, is the number of times that this elementary divisor appears in the list of elementary divisors of $P(\lambda)$.

Theorem 9 *Let* $P(\lambda)$ *be a* T*-palindromic* $n \times n$ *matrix polynomial of degree* k*. Then, there is a* T*-palindromic strong linearization of* $P(\lambda)$ *with size at most* $nk \times nk$ *if and only if the following two conditions hold:*

(a) *Each elementary divisor with odd degree of* $P(\lambda)$ *associated with* $\lambda_0 = 1$ *occurs with even multiplicity, and*

(b) *each elementary divisor with even degree of* $P(\lambda)$ *associated with* $\lambda_0 = -1$ *occurs with even multiplicity.*

Corollary 1 and Theorem 9 together imply that if $P(\lambda)$ is a T-palindromic matrix polynomial with odd degree then each elementary divisor of $P(\lambda)$ with odd degree associated with $\lambda_0 = 1$ occurs with even multiplicity, and each elementary divisor of $P(\lambda)$ with even degree associated with $\lambda_0 = -1$ occurs with even multiplicity. This is not true for T-palindromic polynomials with even degree. Take, for instance, the scalar polynomial $p(\lambda) = \lambda^2 + 2\lambda + 1$, which is T-palindromic with only one elementary divisor associated with the eigenvalue -1, namely $(\lambda + 1)^2$, having multiplicity 1.

Agradecimientos

F. De Terán was partially supported by the Ministerio de Educación y Ciencia of Spain through grant MTM-2006-05361. F. M. Dopico was partially supported by the Ministerio de Educación y Ciencia of Spain through grant MTM-2006-06671 and the PRICIT program of Comunidad de Madrid through grant SIMUMAT (S-0505/ESP/0158). D. S. Mackey was partially supported by National Science Foundation grant DMS-0713799.

Referencias

[1] A. Amiraslani, R. M. Corless, P. Lancaster, *Linearization of matrix polynomials expressed in polynomial bases*, IMA J. Num. Anal., 29 (2009), 141–157.

[2] E. N. Antoniou, S. Vologiannidis, *A new family of companion forms of polynomial matrices*, Electron. J. Linear Algebra, 11 (2004), 78–87.

[3] F. De Terán, F. M. Dopico, *Sharp lower bounds for the dimension of linearizations of matrix polynomials*, Electron. J. Linear Algebra, 17 (2008), 518–531.

[4] F. De Terán, F. M. Dopico, D. S. Mackey, *Linearizations of singular matrix polynomials and the recovery of minimal indices*, to appear in Electron. J. Linear Algebra.

[5] F. De Terán, F. M. Dopico, D. S. Mackey, *Fiedler companion linearizations and the recovery of minimal indices*, in preparation.

[6] G. D. Forney, *Minimal bases of rational vector spaces, with applications to multivariable linear systems*, SIAM J. Control, 13 (1975), 493–520.

[7] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[8] T. Kailath, *Linear Systems*, Prentice Hall, Englewood Cliffs, New Jersey, 1980.

[9] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, *Vector spaces of linearizations for matrix polynomials*, SIAM J. Matrix Anal. Appl., 28 (2006), 971–1004.

[10] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, *Structured polynomial eigenvalue problems: Good vibrations from good linearizations*, SIAM J. Matrix Anal. Appl., 28 (2006), 1029–1051.

[11] V. Mehrmann, C. Shi, *Transformation of high order linear differential-algebraic systems to first order*, Numer. Algorithms, 42, no. 3-4 (2006), 281–307.

[12] F. Tisseur, K. Meerbergen, *A survey of the quadratic eigenvalue problem*, SIAM Rev., 43 (2001), 234–286.

[13] J. H. Wilkinson, *Linear Differential Equations and Kronecker's Canonical Form*, in *Recent Advances in Numerical Analysis*, de Boor and Golub, eds., Academic Press, New York, 1978.