Palindromic Companion Forms for Matrix Polynomials of Odd Degree [*]

Fernando De Terán[†]   Froilán M. Dopico[‡]   D. Steven Mackey [§]

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Abstract

The standard way to solve polynomial eigenvalue problems $P(\lambda)x = 0$ is to convert the matrix polynomial $P(\lambda)$ into a matrix pencil that preserves its spectral information– a process known as linearization. When $P(\lambda)$ is palindromic, the eigenvalues, elementary divisors, and minimal indices of $P(\lambda)$ have certain symmetries that can be lost when using the classical first and second Frobenius companion linearizations for numerical computations, since these linearizations do not preserve the palindromic structure. Recently new families of pencils have been introduced with the goal of finding linearizations that retain whatever structure the original $P(\lambda)$ might possess, with particular attention to the preservation of palindromic structure. However, no general construction of palindromic linearizations valid for all palindromic polynomials has as yet been achieved. In this paper we present a family of linearizations for odd degree polynomials $P(\lambda)$ which are palindromic whenever $P(\lambda)$ is, and which are valid for all palindromic polynomials of odd degree. We illustrate our construction with several examples. In addition, we establish a simple way to recover the minimal indices of the polynomial from those of the linearizations in the new family.

Key words. matrix polynomial, matrix pencil, Fiedler pencils, palindromic, companion form, minimal indices, structured linearization

AMS subject classification. 65F15, 15A18, 15A21, 15A22

1   Introduction

Consider an $n \times n$ matrix polynomial with degree $k \geq 2$ over an arbitrary field $\mathbb{F}$, i.e.,

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n \times n}, \quad A_k \neq 0 \,. \tag{1.1}$$

Then $P(\lambda)$ is said to be *T-palindromic* [22] if $A_i^T = A_{k-i}$ for $i = 0, 1, \ldots, k$, or in other words if

$$\operatorname{rev} P(\lambda) = P(\lambda)^T,$$

where $\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^k \lambda^i A_{k-i}$ denotes the *reversal* polynomial of $P(\lambda)$. For polynomials over the particular field $\mathbb{F} = \mathbb{C}$, one can also consider $P(\lambda)$ that are *∗-palindromic* [18, 22], i.e., polynomials that satisfy $A_i^* = A_{k-i}$ for $i = 0, 1, \ldots, k$, or equivalently $\operatorname{rev} P(\lambda) = P(\lambda)^*$, where $*$ denotes conjugate transpose. Since everything that we do in this paper for T-palindromic polynomials works equally well for $*$-palindromic polynomials, from now on we will just refer to "palindromic" polynomials for the sake of simplicity, except in those few situations in this introduction where the distinction is significant. Polynomials $P(\lambda)$ satisfying $\operatorname{rev} P(\lambda) = -P(\lambda)^T$ or $\operatorname{rev} P(\lambda) = -P(\lambda)^*$, sometimes referred to as *anti-palindromic* polynomials [22], are also of some interest, and can be handled in a similar manner.

Palindromic polynomials arise in a number of application areas. For example, the mathematical modelling and numerical simulation of the behavior of periodic surface acoustic wave filters [13, 26], as well as the analysis of rail track vibrations produced by high speed trains [12, 14, 15, 22], each lead to a quadratic T-palindromic polynomial eigenvalue problem. Also, discrete-time optimal control problems can be formulated as $*$-palindromic eigenproblems of degree 2 and higher [4].

The spectral structure of palindromic matrix polynomials enjoys certain symmetries. For example, the elementary divisors of T-palindromic polynomials corresponding to eigenvalues $\lambda_0 \neq \pm 1$ always come in pairs $(\lambda - \lambda_0)^s, (\lambda - 1/\lambda_0)^s$ [20, 22, 24]. For palindromic polynomials $P(\lambda)$ that are *singular*[1], the minimal indices also are paired; if $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_\ell$ and $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_m$ are respectively the left and right minimal indices of $P(\lambda)$, then $\ell = m$ and $\eta_j = \varepsilon_j$ for $j = 1, \ldots, \ell$ [6, Thm. 3.6].

The usual way to numerically solve polynomial eigenproblems for *regular* polynomials $P(\lambda)$ is to first linearize $P(\lambda)$ into a matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk \times nk}$, then compute the eigenvalues and eigenvectors of $L(\lambda)$ using well-known algorithms for general matrix pencils. When $P(\lambda)$ is singular, linearizations can also be used to compute the minimal indices and bases of $P(\lambda)$ [6, 7]. The classical approach uses the *first* or *second Frobenius companion forms* of $P(\lambda)$ as linearizations [7, 11]. However, these companion forms are never palindromic, even when $P(\lambda)$ is. Consequently, the rounding errors inherent in numerical computations may destroy the symmetry of elementary divisors and minimal indices of palindromic polynomials if such unstructured linearizations are employed. A numerical procedure that preserves palindromic structure throughout the computation would thus be more appropriate than employing some standard method designed for use on general polynomials. In order to gain more accuracy and reliability in the numerical solution of palindromic eigenvalue and minimal index problems by linearization, then, two steps should be addressed:

(1) Design linearizations that share the palindromic structure of $P(\lambda)$.

(2) Develop specific numerical methods for computing eigenvalues and minimal indices of palindromic pencils, methods that preserve and exploit the palindromic structure throughout the computation.

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[1] Recall that an $n \times n$ polynomial $P(\lambda)$ is *singular* if $\det P(\lambda) \equiv 0$, i.e., if all the coefficients of the scalar polynomial $\det P(\lambda)$ are zero, and it is *regular* otherwise.

Step (2) has been addressed for the regular case in [23], where a structured Schur-like form for T-palindromic pencils and an algorithm to compute it are presented. Additional structure-preserving algorithms for T-palindromic pencils are developed in [17] and [25]. Step (1) has been addressed in [22], but again only for regular palindromic polynomials $P(\lambda)$; note also that the presence of eigenvalues at $\lambda_0 = \pm 1$ was found to be problematic in the T-palindromic case. In [22], necessary and sufficient conditions are given for the existence of palindromic linearizations within certain special families of matrix pencils associated with $P(\lambda)$ that were introduced in [21]. A procedure to construct these structured linearizations, when they exist, is also given in [22]. However, the problem of finding palindromic linearizations that are valid for *all* palindromic polynomials $P(\lambda)$, regular and singular, with no restrictions on the eigenvalues of $P(\lambda)$, remained open.

In order to probe for intrinsic obstructions to the existence of palindromic linearizations, the Smith forms of palindromic matrix polynomials were analyzed in [24], and necessary conditions on the structure of the elementary divisors of such polynomials were obtained. One striking feature that emerges from this analysis is a clear dichotomy between the behavior of *even* and *odd* degree T-palindromic polynomials; the elementary divisors of all odd degree T-palindromic polynomials satisfy one common set of necessary conditions, while the elementary divisors of all even degree T-palindromic polynomials satisfy a slightly *different* set of necessary conditions. Thus it is possible for an even degree palindromic polynomial $P(\lambda)$ to have an elementary divisor structure that is *incompatible* with that of every palindromic pencil; for such a $P(\lambda)$ it is impossible to have any palindromic linearization at all. For example, it is shown in [24] that no palindromic pencil can have the same elementary divisor structure as that of the quadratic palindromic polynomial

$$ Q(\lambda) = \begin{bmatrix} \lambda^2 + 1 & 2\lambda \\ 2\lambda & \lambda^2 + 1 \end{bmatrix} = \lambda^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad (1.2) $$

and hence this $Q(\lambda)$ has no palindromic linearization.

By contrast, the Smith form results of [24] reveal no elementary divisor incompatibility between palindromic pencils and arbitrary odd degree palindromic polynomials. This suggests not only that every odd degree palindromic polynomial has a palindromic linearization, but even that it might be possible to construct *companion-like* palindromic linearizations for odd degree palindromic polynomials, i.e., linearizations with the following desirable properties.

Definition 1.1 (Companion Forms/Palindromic Companion Forms)**.**
A companion form *for general $n \times n$ matrix polynomials $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ of degree k is an $nk \times nk$ matrix pencil $\mathcal{C}_P(\lambda) = \lambda X + Y$ such that if X and Y are viewed as block $k \times k$ matrices with $n \times n$ blocks, then:*

(a) *each block of X and Y is either 0_n or $\pm I_n$ or $\pm A_i$ for $i = 0, 1, \ldots, k$, and*

(b) *$\mathcal{C}_P(\lambda)$ is a strong linearization for* every *$n \times n$ polynomial $P(\lambda)$ of degree k, regular or singular, over an arbitrary field.*

A palindromic companion form *is a companion form with the additional property that $\mathcal{C}_P(\lambda)$ is a palindromic pencil whenever $P(\lambda)$ is a palindromic polynomial.*

Remark 1.2. Note that the Fiedler pencils studied in [7] are all companion forms in the sense of Definition 1.1, but none of them are palindromic companion forms. Although the fact that Fiedler pencils *are* companion forms seems to be widely known, to the best of our knowledge a formal proof of property (a) has never been presented. We present such a proof in Section 3 as a simple consequence of some other results developed in this paper.

From the above discussion of (1.2), it is clear that palindromic companion forms cannot exist for degree $k = 2$. Indeed, similar examples can be fashioned to show that palindromic companion forms cannot exist for *any* even degree k. Thus we focus attention in this paper on the odd degree case, showing how to explicitly construct families of palindromic companion forms for each odd degree k. Our construction of these palindromic companion forms is based on the Fiedler pencils, a family of linearizations introduced in [1] for regular polynomials, and extended and further analyzed in [7] and [2] for both regular and singular matrix polynomials. Because of the close connection between the family of linearizations introduced in this work and the Fiedler companion pencils, it is not too surprising that these new linearizations also turn out to be companion forms; what requires considerable work is to prove that they are *palindromic* companion forms.

Another important advantage of the Fiedler linearizations is that they allow the recovery of the minimal indices of a matrix polynomial from those of the linearization by means of very simple formulas [2, 7]. We will see that this property is inherited by the palindromic linearizations constructed in this paper. It is important to stress that minimal indices are intrinsic quantities associated with any singular matrix polynomial, and are relevant in many control problems [8, 16].

The paper is organized as follows. In Section 2 we introduce the basic definitions, background facts, and notations used throughout the paper, including the Fiedler pencils and their basic properties. Then in Section 3 certain block matrices closely related to the Fiedler pencils, but with some factors deleted, are introduced and algorithmically constructed. These block matrices will be the basis of our construction of palindromic companion linearizations. These linearizations are then introduced in Section 4 and their basic properties established. We also prove some useful structural properties of these linearizations and provide a number of concrete examples. Section 5 then shows how any of the palindromic companion forms constructed in Section 4 can be simply modified to become an *anti-palindromic companion form*, i.e., a companion form that produces an anti-palindromic linearization whenever the original polynomial is anti-palindromic. In Section 6 we show how the minimal indices of any singular odd degree matrix polynomial can be recovered in an extremely simple way from the minimal indices of any one of our palindromic linearizations. Finally, some conclusions are discussed in Section 7.

2   Basic definitions and background

In this paper we follow the notation and definitions from [7]. In particular, $\mathbb{F}(\lambda)$ will denote the field of rational functions with coefficients in the field $\mathbb{F}$, and I_ℓ is the $\ell \times \ell$ identity matrix. Since the $n \times n$ identity appears frequently throughout the paper (n being the size of $P(\lambda)$ in (1.1)), for this particular size we drop the subscript and denote it simply by I.

The spectral structure of a regular matrix polynomial $P(\lambda)$ is comprised of its *finite* and *infinite elementary divisors* (see definition in [9]). For singular matrix polynomials $P(\lambda)$, there is an additional structure comprised of the *minimal indices*. Since minimal indices

are considered only in Section 6, the formal definition will be postponed until that section. We just mention here that minimal indices are related to the existence of *right* and *left null vectors* of $P(\lambda)$, that is, nonzero vectors $x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1}$ and $y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times n}$ such that $P(\lambda)x(\lambda) \equiv 0$ and $y(\lambda)^T P(\lambda) \equiv 0$. The existence of such null vectors leads us to introduce the notion of *right* and *left nullspaces* of $P(\lambda)$. These are the following vector subspaces over $\mathbb{F}(\lambda)$:

$$
\begin{aligned}
\mathcal{N}_r(P) &:= \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} \,:\, P(\lambda)x(\lambda) \equiv 0 \right\}, \\
\mathcal{N}_\ell(P) &:= \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times n} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T \right\}.
\end{aligned}
$$

Notice that, since $P(\lambda)$ is square, we have $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_\ell(P)$.

Two matrix pencils $L_1(\lambda)$ and $L_2(\lambda)$ are *strictly equivalent* if there exist two invertible constant matrices E, F such that

$$
E \cdot L_1(\lambda) \cdot F = L_2(\lambda).
$$

The notion of strict equivalence is also applicable to matrix polynomials with degree greater than one, but in this paper we will only need it for matrix pencils. It is well known [9] that two pencils of the same size are strictly equivalent if and only if they have the same elementary divisors and minimal indices.

Now we recall the notion of linearization as introduced in [11], and also the related notion of strong linearization introduced in [10] and named in [19]. Note that a *unimodular matrix* is a square matrix polynomial whose determinant is a nonzero constant in $\mathbb{F}$.

Definition 2.1. *A matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk \times nk}$ is a* linearization *of an $n \times n$ matrix polynomial $P(\lambda)$ of degree k if there exist two unimodular $nk \times nk$ matrices $U(\lambda)$ and $V(\lambda)$ such that*

$$
U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2.1}
$$

or, in other words, if $L(\lambda)$ is unimodularly equivalent to $\mathrm{diag}(I_{(k-1)n}, P(\lambda))$. A linearization $L(\lambda)$ is called a strong linearization *if $\mathrm{rev}\, L(\lambda)$ is also a linearization of $\mathrm{rev}\, P(\lambda)$.*

The relevance of linearizations (resp., strong linearizations) in the study of both regular and singular matrix polynomials lies in the fact that these are the only matrix pencils preserving the dimension of the left and right null spaces and the finite (resp., finite and infinite) elementary divisors of the polynomial [6, Lemma 2.3].

Note that the size of linearizations in Definition 2.1 is assumed to be exactly $nk \times nk$. Linearizations with other sizes have been considered recently in [3], and their minimal possible size has been determined in [5]. In particular, it is shown in [5] that every strong linearization of a regular $n \times n$ matrix polynomial with degree k must have size exactly $nk \times nk$. Since we are interested in finding companion forms, i.e., strong linearizations valid for *all* matrix polynomials of degree k (including regular ones), in this paper we consider only linearizations of size $nk \times nk$.

Our construction of palindromic linearizations is based on the *Fiedler pencils*, introduced in [1] for regular matrix polynomials, and later extended in [7] to the singular case. To construct these pencils for the polynomial $P(\lambda)$ in (1.1) we need the following block-partitioned matrices:

$$
M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \qquad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix}, \tag{2.2}
$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I & \\ & I & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1. \tag{2.3}$$

These $kn \times kn$ matrices are viewed as $k \times k$ block-matrices with blocks all of size $n \times n$, and are the basic factors used to build the *Fiedler pencils* [1, 7] of $P(\lambda)$:

$$\lambda M_k - M_{i_0} M_{i_1} \cdots M_{i_{k-1}}, \tag{2.4}$$

where $(i_0, i_1, \ldots, i_{k-1})$ is any possible permutation of the n-tuple $(0, 1, \ldots, k-1)$. The following fact is fundamental for the development in the rest of the paper.

Theorem 2.2 ([1, 7]). *Let $P(\lambda)$ be an $n \times n$ matrix polynomial (regular or singular). Then any Fiedler pencil of $P(\lambda)$ is a strong linearization for $P(\lambda)$.*

This result was shown to hold for regular $P(\lambda)$ over $\mathbb{F} = \mathbb{C}$ in [1], while a proof valid for arbitrary regular and singular polynomials over an arbitrary field $\mathbb{F}$ was given in [7]. As background for the work in this paper, this fact is crucial in guaranteeing that our construction produces strong linearizations of $P(\lambda)$.

We recall the commutativity relations

$$M_i M_j = M_j M_i \quad \text{for } |i - j| \neq 1, \tag{2.5}$$

that will be used later. Unless otherwise stated, the matrices M_i for $i = 0, \ldots, k$ are built from the coefficients of the matrix polynomial $P(\lambda)$ in (1.1). When necessary, we will explicitly indicate the dependence on a certain matrix polynomial $Q(\lambda)$ with the notation $M_i(Q)$. This convention will also be applied to other matrices appearing in this paper.

In the following example we exhibit a Fiedler pencil for polynomials of degree $k = 5$.

Example 2.3. Let $k = 5$ and $(i_0, i_1, i_2, i_3, i_4) = (3, 4, 0, 1, 2)$. Then the Fiedler pencil associated with this permutation is

$$\lambda M_5 - M_3 M_4 M_0 M_1 M_2 = \begin{bmatrix} \lambda A_5 + A_4 & -I & 0 & 0 & 0 \\ A_3 & \lambda I & A_2 & -I & 0 \\ -I & 0 & \lambda I & 0 & 0 \\ 0 & 0 & A_1 & \lambda I & -I \\ 0 & 0 & A_0 & 0 & \lambda I \end{bmatrix}.$$

Example 2.3 illustrates the general structure of the Fiedler pencils. The zero-degree term contains all the coefficients of $P(\lambda)$ except the leading one, i.e. A_k, together with $k-1$ identity blocks (with minus signs). The remaining blocks of this term are null blocks. The first-degree coefficient contains the leading coefficient of $P(\lambda)$ in the $(1, 1)$ position together with $k-1$ identities in the remaining diagonal positions. Again, all other blocks are zero.

3 Fiedler-like block matrices with deleted factors

For further developments, we construct matrices analogous to the ones in the zero-degree term of (2.4), but with some of the factors missing. For working effectively with this type

of matrix we introduce the following notation: let $s \leq k$ be a positive integer, and let $C_s :=$ $\{j_1, \ldots, j_s\} \subseteq \{0, 1, \ldots, k-1\}$ be a set of s distinct numbers. Also let $\tau : C_s \rightarrow \{1, 2, \ldots, s\}$ be a bijection. Then we consider the matrix

$$\mathbb{M}_\tau := M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(s)} . \tag{3.1}$$

Notice that $\tau(j)$ for $j \in C_s$ describes the position of the matrix M_j in the product defining $\mathbb{M}_\tau$. Observe that $\mathbb{M}_\tau$ can be obtained from the zero-degree term of one of the Fiedler pencils (2.4) by removing $k - s$ of the M_j factors.

Definition 3.1. *Let $\tau : C_s \rightarrow \{1, 2, \ldots, s\}$ be a bijection. For $j \in C_s$ we say that τ has a consecution at j if $j + 1 \in C_s$ and $\tau(j) < \tau(j+1)$, and that τ has an inversion at j if $j + 1 \in C_s$ and $\tau(j) > \tau(j+1)$.*

The following theorem provides an algorithm to construct the matrix $\mathbb{M}_\tau$ without performing multiplications. `Algorithm 1` in Theorem 3.2 will be used to establish certain properties of the matrix $\mathbb{M}_\tau$ that are needed in Section 4. We assume that all matrices appearing in `Algorithm 1` are block-partitioned matrices with $n \times n$ blocks, and that MAT-LAB *notation for submatrices is used on block indices. We will follow this convention in the rest of the paper.*

Theorem 3.2. *Let $P(\lambda)$ be the matrix polynomial in (1.1) with degree $k \geq 2$, let $C_s = \{j_1, j_2, \ldots, j_s\} \subseteq \{0, 1, \ldots, k-1\}$ be a set of s distinct numbers such that $0 \in C_s$, let $\tau : C_s \rightarrow \{1, 2, \ldots, s\}$ be a bijection, and let $\mathbb{M}_\tau$ be the matrix defined in (3.1). Then* `Algorithm 1` *below computes $\mathbb{M}_\tau$.*

`Algorithm 1`: Computes $\mathbb{M}_\tau$ for given $P(\lambda)$, C_s and τ

if τ has a consecution at 0

$$W_0 = \begin{bmatrix} -A_1 & I \\ -A_0 & 0 \end{bmatrix}$$

elseif τ has an inversion at 0

$$W_0 = \begin{bmatrix} -A_1 & -A_0 \\ I & 0 \end{bmatrix}$$

else   % this happens if $1 \notin C_s$

$$W_0 = \begin{bmatrix} I & 0 \\ 0 & -A_0 \end{bmatrix}$$

endif

for $i = 1 : k - 2$

    if τ has a consecution at i

$$W_i = \begin{bmatrix} -A_{i+1} & I & 0 \\ W_{i-1}(:, 1) & 0 & W_{i-1}(:, 2 : i+1) \end{bmatrix}$$

    elseif τ has a inversion at i

$$W_i = \begin{bmatrix} -A_{i+1} & W_{i-1}(1, :) \\ I & 0 \\ 0 & W_{i-1}(2 : i+1, :) \end{bmatrix}$$

    elseif ($i \notin C_s$ and $i + 1 \in C_s$)

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$$W_i = \begin{bmatrix} -A_{i+1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & W_{i-1}(2:i+1, 2:i+1) \end{bmatrix}$$

else   % this happens if $i+1 \notin C_s$

$$W_i = \begin{bmatrix} I & 0 \\ 0 & W_{i-1} \end{bmatrix}$$

endif

endfor

$\mathbb{M}_\tau = W_{k-2}$

Proof. The proof proceeds by induction on the degree k. The result is obvious for $k=2$ because in this case there are only three possibilities for $\mathbb{M}_\tau$, namely: $\mathbb{M}_\tau = M_0 M_1$ if τ has a consecution at 0, $\mathbb{M}_\tau = M_1 M_0$ if τ has an inversion at 0 and $\mathbb{M}_\tau = M_0$ if $1 \notin C_s$. A direct computation shows that these three matrices correspond to the matrices computed by `Algorithm 1` for $k=2$.

Assume now that the result is true for all matrix polynomials of degree $k-1 \geq 2$, and let us prove it for the polynomial $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ of degree k and the bijection $\tau : C_s \to \{1, 2, \ldots, s\}$, where $C_s = \{j_1, j_2, \ldots, j_s\}$ is as specified in the statement of the theorem. Notice first that the matrices $M_i(P)$ defined in (2.2) and (2.3) for $P(\lambda)$ satisfy

$$M_i(P) = \mathrm{diag}(I, M_i(Q)), \qquad \text{for } i = 0, \ldots, k-2, \tag{3.2}$$

where $M_i(Q)$ are the $n(k-1) \times n(k-1)$ matrices corresponding to the polynomial $Q(\lambda) = \sum_{i=0}^{k-1} \lambda^i A_i$. In the proof, we distinguish four cases that correspond to the four possibilities in the "if" statement inside the "for loop" of `Algorithm 1` for $i = k-2$.

Case 1. If τ has a consecution at $k-2$ then the commutativity relations (2.5) imply

$$\mathbb{M}_\tau(P) = M_{i_1}(P) \cdots M_{i_{s-1}}(P) \, M_{k-1}(P),$$

where $(i_1, \ldots, i_{s-1})$ is a permutation of $C_s \setminus \{k-1\}$. Notice that for $i = 0, 1, \ldots, k-2$, $i \in C_s \setminus \{k-1\}$ if and only if $i \in C_s$. Now by using (3.2) we can write

$$\mathbb{M}_\tau(P) = \mathrm{diag}(I, \mathbb{M}_{\widetilde{\tau}}(Q)) \, M_{k-1}(P), \tag{3.3}$$

where $\widetilde{\tau} : C_s \setminus \{k-1\} \to \{1, 2, \ldots, s-1\}$ is a bijection such that for $i = 0, 1, \ldots, k-3$, the bijection $\widetilde{\tau}$ has a consecution (resp., inversion) at i if and only if τ has a consecution (resp., inversion) at i. So `Algorithm 1` applied to $Q(\lambda)$, $C_s \setminus \{k-1\}$ and $\widetilde{\tau}$ produces the same W_{k-3} as `Algorithm 1` applied to $P(\lambda)$, C_s and τ. Therefore, the induction hypothesis guarantees that $\mathbb{M}_{\widetilde{\tau}}(Q) = W_{k-3}$. Finally, we perform the simple block product in (3.3) as follows

$$\mathbb{M}_\tau(P) = \begin{bmatrix} I & 0 & 0 \\ 0 & W_{k-3}(:,1) & W_{k-3}(:, 2:k-1) \end{bmatrix} \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I_{(k-2)n} \end{bmatrix}$$

$$= \begin{bmatrix} -A_{k-1} & I & 0 \\ W_{k-3}(:,1) & 0 & W_{k-3}(:, 2:k-1) \end{bmatrix},$$

which is precisely the matrix obtained for $i = k - 2$ in the "for loop" in `Algorithm 1` when τ has a consecution at $k - 2$.

Case 2. If τ has an inversion at $k - 2$ then the proof is similar to that of *Case 1*, but with $M_{k-1}(P)$ placed on the left, that is,

$$\mathbb{M}_\tau(P) = M_{k-1}(P)\, M_{i_1}(P) \cdots M_{i_{s-1}}(P) = M_{k-1}(P)\, \mathrm{diag}\,(I, \mathbb{M}_{\widetilde{\tau}}(Q))\,.$$

Case 3. If $k - 2 \notin C_s$ and $k - 1 \in C_s$, we can argue as in *Case 1* and write again

$$\mathbb{M}_\tau(P) = M_{i_1}(P) \cdots M_{i_{s-1}}(P) M_{k-1}(P) = \mathrm{diag}\,(I, \mathbb{M}_{\widetilde{\tau}}(Q))\, M_{k-1}(P)\,. \qquad (3.4)$$

Then by the induction hypothesis,

$$\mathbb{M}_{\widetilde{\tau}}(Q) = W_{k-3} = \begin{bmatrix} I & \\ & W_{k-4} \end{bmatrix},$$

where W_{k-3} and W_{k-4} are the matrices obtained for $i = k - 3, k - 4$ in the "for loop" of `Algorithm 1` applied to $Q(\lambda)$, $C_s \setminus \{k-1\}$ and $\widetilde{\tau}$; these are the same as the W_{k-3} and W_{k-4} matrices obtained when `Algorithm 1` is applied to $P(\lambda)$, C_s and τ. Finally, performing the block product in (3.4) we get

$$\mathbb{M}_\tau(P) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & W_{k-3}(2:k-1, 2:k-1) \end{bmatrix} \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I_{(k-2)n} \end{bmatrix}$$
$$= \begin{bmatrix} -A_{k-1} & I & 0 \\ I & 0 & 0 \\ 0 & 0 & W_{k-3}(2:k-1, 2:k-1) \end{bmatrix},$$

which is precisely the matrix obtained for $i = k - 2$ in the "for loop" of `Algorithm 1` when $k - 2 \notin C_s$ and $k - 1 \in C_s$.

Case 4. If $k - 1 \notin C_s$, we have that $C_s \setminus \{k-1\} = C_s$, and so we can simply write

$$\mathbb{M}_\tau(P) = \mathrm{diag}\,(I, \mathbb{M}_\tau(Q)).$$

By the induction hypothesis, $\mathbb{M}_\tau(Q) = W_{k-3}$ with W_{k-3} the matrix obtained for $i = k-3$ in the "for loop" in `Algorithm 1`. Therefore $\mathbb{M}_\tau(P) = W_{k-2}$, and the proof is complete. $\square$

Observe that from Theorem 3.2 it is immediate that the $n \times n$ blocks of $\mathbb{M}_\tau$ are 0_n or I_n or $-A_i$ for $i \in C_s$. In addition, Theorem 3.2 can also be used to construct the zero degree term of any Fiedler pencil (2.4). This fact can be combined with Theorem 2.2 to give a rigorous proof of the following result.

Corollary 3.3. *Any Fiedler pencil is a companion form for general square matrix polynomials of degree k.*

We finish this section by establishing a further simple corollary of Theorem 3.2 that will be used in Section 4.

Corollary 3.4. *Using the same notation and assumptions as in Theorem 3.2 (except that here $k \geq 3$), let $W_0, W_1, \ldots, W_{k-2} = \mathbb{M}_\tau$ be the sequence of matrices computed by `Algorithm 1`. Also recall that the MATLAB notation used here refers to block indices. Then for $i = 0, \ldots, k - 3$ we have*

(a) $\mathbb{M}_\tau(k-i:k, k-i:k) = W_i(2:i+2, 2:i+2)$, *and*

(b) *The block row $\mathbb{M}_\tau(j,:)$ is obtained from the block row $W_i(j+2-k+i,:)$, for $j = k-i,\ldots,k$, by adding $k-i-2$ zero blocks of size $n \times n$ in certain positions. Similarly, the block column $\mathbb{M}_\tau(:,j)$ is obtained from $W_i(:, j+2-k+i)$, for $j = k-i,\ldots,k$, by adding $k-i-2$ zero blocks of size $n \times n$ in certain positions.*

Proof. Part (a): The result for $i = k-3$ follows directly from the way W_{k-2} is obtained from W_{k-3} in `Algorithm 1`, which implies that

$$\mathbb{M}_\tau(3:k, 3:k) = W_{k-2}(3:k, 3:k) = W_{k-3}(2:k-1, 2:k-1).$$

Now proceed by (downwards) induction: we assume that the result is true for an index $i+1$ such that $1 \le i+1 \le k-3$, then prove that it is true for index i. By the induction assumption $\mathbb{M}_\tau(k-i-1:k, k-i-1:k) = W_{i+1}(2:i+3, 2:i+3)$. On the other hand, by the way W_{i+1} is obtained from W_i in `Algorithm 1`, it is clear that $W_{i+1}(3:i+3, 3:i+3) = W_i(2:i+2, 2:i+2)$. Combining the two identities above we get $\mathbb{M}_\tau(k-i:k, k-i:k) = W_i(2:i+2, 2:i+2)$, which is the desired result for part (a).

Part (b): We prove the result only for block rows; the argument for block columns is completely analogous. The result for $i = k-3$ follows directly from `Algorithm 1`. Again we proceed by (downwards) induction: we assume that the result is true for an index $i+1$ such that $1 \le i+1 \le k-3$, then show that it holds for index i. This induction assumption implies that

$$\mathbb{M}_\tau(j,:), \quad \text{for } j = k-i-1,\ldots,k, \text{ is obtained from} \quad W_{i+1}(j+3-k+i,:) \qquad (3.5)$$

by adding zero blocks. On the other hand, by the way W_{i+1} is obtained from W_i in `Algorithm 1`, it is clear that

$$W_{i+1}(j,:), \quad \text{for } j = 3,\ldots,i+3, \text{ is obtained from} \quad W_i(j-1,:) \qquad (3.6)$$

by adding one zero block. Combining (3.5) and (3.6) gives the desired result. $\qquad\square$

4 Palindromic companion forms for odd degree

The technical results presented in Section 3 allow us in this section to achieve the main goal of this paper: the construction of palindromic companion forms for any odd degree, i.e., strong linearizations $\lambda X + Y$ for any odd degree matrix polynomial $P(\lambda)$ with coefficients as in (1.1), such that $\lambda X + Y$ is palindromic whenever $P(\lambda)$ is. For this purpose, we will construct pencils that are *strictly equivalent to certain Fiedler pencils*, and that satisfy

$$X^T = Y \qquad \text{whenever} \qquad A_{k-i} = A_i^T \quad \text{for all } i = 0, 1, \ldots, k. \qquad (4.1)$$

Observe that Theorem 2.2 guarantees immediately that the pencils we construct are strong linearizations for any odd degree $P(\lambda)$.

The initial step in our strategy can be viewed as multiplying a selected Fiedler pencil by the inverses of some of the M_i matrices in (2.3); these M_i are always invertible for $i = 1, \ldots, k-1$, and have inverses given by

$$M_i^{-1} = \begin{bmatrix} I_{(k-i-1)n} & & & \\ & 0 & I & \\ & I & A_i & \\ & & & I_{(i-1)n} \end{bmatrix}. \qquad (4.2)$$

So, starting with (2.4), we obtain a pencil $\lambda X' + Y'$ where X' is a product of M_k times some inverses M_i^{-1} and $-Y'$ is a product of those M_j matrices that have not been inverted (this product always includes M_0). An analogous strategy was introduced in [1, Corollary 2.4, Theorem 3.1] to build self-adjoint linearizations of self-adjoint regular matrix polynomials. However, for preserving palindromic structure two additional steps are needed: first we reverse the order of the block rows of $\lambda X' + Y'$, then we change the signs of a selected subset of block rows. Both steps can be performed via strict equivalences on $\lambda X' + Y'$.

One important choice in this construction strategy is to select which inverses M_i^{-1} are to be used in the formation of X', in order to achieve property (4.1). Note that a simple necessary condition follows easily from (4.1): if for some $j = 1, \ldots, k-1$ the factor M_j is part of Y, then X *must* contain the factor M_i^{-1} with "complementary" index $i = k - j$. This key fact forces the degree k to be odd, and plays a central role in our construction of palindromic linearizations based on Fiedler pencils. Before addressing the general construction leading to our main Theorem 4.8, we illustrate this initial discussion with Example 4.1.

Example 4.1. Let $k = 5$ and set

$$\mathbb{M}_0 = M_0 M_1 M_2, \qquad \mathbb{M}_1 = M_3^{-1} M_4^{-1} M_5 \,.$$

Then

$$\lambda \mathbb{M}_1 - \mathbb{M}_0 = \begin{bmatrix} -I & \lambda I & 0 & 0 & 0 \\ 0 & -I & \lambda I & 0 & 0 \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & 0 & A_1 & \lambda I & -I \\ 0 & 0 & A_0 & 0 & \lambda I \end{bmatrix}. \tag{4.3}$$

Note that (4.3) is strictly equivalent to the Fiedler pencil $\lambda M_5 - M_4 M_3 M_0 M_1 M_2$. Reversing the order of the block rows in (4.3), we get the strictly equivalent pencil

$$\begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & -I & \lambda I & 0 & 0 \\ -I & \lambda I & 0 & 0 & 0 \end{bmatrix}.$$

Finally, if we change the sign of the fourth and fifth block rows, then we obtain the pencil

$$\lambda X + Y = \begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3 + A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix},$$

which satisfies (4.1). Observe that the block rows whose signs have been changed have only $\pm I$, $\pm \lambda I$ and 0 blocks.

Example 4.1 and the paragraphs preceding it sketch a procedure to construct palindromic linearizations for odd degree matrix polynomials from Fiedler pencils, comprised of the following three main steps:

(S1) Build up a pencil $\lambda \mathbb{M}_1 - \mathbb{M}_0$ that is strictly equivalent to a Fiedler pencil, where $\mathbb{M}_0$ is a product of M_0 and *half* of the M_i matrices for $i = 1, \ldots, k-1$, and $\mathbb{M}_1$ is a product of M_k and the matrices M_{k-i}^{-1} with "complementary indices" to those in $\mathbb{M}_0$.

(S2) Reverse the order of the block rows of $\lambda \mathbb{M}_1 - \mathbb{M}_0$.

(S3) Change the sign of appropriate block rows in the pencil obtained in **(S2)**.

In subsequent developments we adopt the following notation for simplicity. For the matrices M_i introduced in (2.2) and (2.3), we define for $j = 0, 1, \ldots, k-1$

$$\widetilde{M}_{k-j} := \begin{cases} M_k & \text{if } j = 0, \\ M_{k-j}^{-1} & \text{otherwise}. \end{cases} \qquad (4.4)$$

The ordering and the selection of the factors in the pencil $\lambda \mathbb{M}_1 - \mathbb{M}_0$ in step **(S1)** above will be crucial in our construction. This is established in Definition 4.2.

Definition 4.2 (Admissible index set and associated pencils).
Let $P(\lambda)$ be the matrix polynomial (1.1), let the degree k be odd, and $h := (k+1)/2$. Then a subset $C \subset \{0, 1, \ldots, k-1\}$ is said to be an admissible index set *if*

- $0 \in C$,

- $C = \{j_1, \ldots, j_h\}$ *has cardinality h, and*

- $C \cap \{k - j_1, \ldots, k - j_h\} = \emptyset$.

In addition, given any bijection $\tau : C \to \{1, 2, \ldots, h\}$, the pencil of $P(\lambda)$ associated with C and τ is the $nk \times nk$ matrix pencil

$$L_\tau(\lambda) := \lambda \widetilde{M}_{k-\tau^{-1}(h)} \cdots \widetilde{M}_{k-\tau^{-1}(2)} \widetilde{M}_{k-\tau^{-1}(1)} - M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(h)}. \qquad (4.5)$$

For brevity, we denote the coefficients of this pencil by

$$\mathbb{M}_0 := M_{\tau^{-1}(1)} M_{\tau^{-1}(2)} \cdots M_{\tau^{-1}(h)}, \quad \mathbb{M}_1 := \widetilde{M}_{k-\tau^{-1}(h)} \cdots \widetilde{M}_{k-\tau^{-1}(2)} \widetilde{M}_{k-\tau^{-1}(1)}. \qquad (4.6)$$

The construction of admissible index sets is simple; partitioning $\{1, 2, \ldots, k-1\}$ as a union $\cup_{j=1}^{(k-1)/2} \{j, k-j\}$ of complementary pairs, any admissible index set can be formed by taking exactly one element from $\{j, k-j\}$ for each $j = 1, 2, \ldots, (k-1)/2$, and then adding 0. Given an admissible index set C there are many possible bijections τ, so for $P(\lambda)$ there may be several distinct pencils $L_\tau(\lambda)$ associated with the index set C. Nevertheless, every pencil $L_\tau(\lambda)$ can be obtained by multiplying some Fiedler pencil of $P(\lambda)$ on the left and/or on the right by the inverses of the matrices $M_{k-j_1}, \ldots, M_{k-j_h}$ with $j_\ell \neq 0$. Therefore every pencil $L_\tau(\lambda)$ is strictly equivalent to a Fiedler pencil, and hence is always a strong linearization of $P(\lambda)$ by Theorem 2.2. Finally, observe that any admissible index set C is a particular case of the index sets C_s considered in Section 3, with $s = h$. Thus the matrix $\mathbb{M}_0$ in (4.6) is a special case of the matrix $\mathbb{M}_\tau$ in (3.1), and all the results of Section 3 apply to $\mathbb{M}_0$. Note also that for the arguments in the next section 4.1, it will be helpful to bear in mind that $\tau(j)$ for $j \in C$ specifies the position of the factor M_j in the product defining $\mathbb{M}_0$.

The above discussion together with `Algorithm 1` makes it clear that $\mathbb{M}_0$ satisfies property (a) in Definition 1.1. The fact that $\mathbb{M}_1$ also satisfies this property follows from

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Lemma 4.3 below. In this lemma and its proof, $M_i(P)$ denotes any of the matrices defined in (2.2)-(2.3) for $P(\lambda)$, while $M_i(-\mathrm{rev}\,P)$ denotes the corresponding matrices for the matrix polynomial $-\mathrm{rev}\,P(\lambda)$. An analogous notation is used for the matrices $\widetilde{M}_{k-j}$ defined in (4.4). For completeness, we consider in Lemma 4.3 an arbitrary number of $\widetilde{M}$ factors, not just products with exactly $h = (k+1)/2$ factors. From now on, $R \in \mathbb{F}^{nk \times nk}$ denotes the $k \times k$ block reverse identity matrix with $n \times n$ blocks, that is

$$R := \begin{bmatrix} & & I_n \\ & \cdot^{\cdot^{\cdot}} & \\ I_n & & \end{bmatrix},$$

with the property that $R^2 = I_{nk}$.

Lemma 4.3. *Let $P(\lambda)$ be the matrix polynomial in (1.1) with degree $k \geq 2$, let $C_s = \{j_1, j_2, \ldots, j_s\} \subseteq \{0, 1, \ldots, k-1\}$ be any set of s distinct numbers such that $0 \in C_s$ and $1 \leq s \leq k$, and let $\tau : C_s \to \{1, 2, \ldots, s\}$ be a bijection. Then*

$$\widetilde{M}_{k-\tau^{-1}(s)}(P) \cdots \widetilde{M}_{k-\tau^{-1}(1)}(P) = R\left(M_{\tau^{-1}(s)}(-\mathrm{rev}\,P) \cdots M_{\tau^{-1}(1)}(-\mathrm{rev}\,P)\right)R. \quad (4.7)$$

Furthermore, the right-hand side of (4.7) may be constructed by first using Algorithm 1 *to construct $\mathbb{M}_{\mathrm{rev}\,\tau}$ for the matrix polynomial $-\mathrm{rev}\,P$, where $\mathrm{rev}\,\tau : C_s \to \{1, 2, \ldots, s\}$ is the bijection defined by $\mathrm{rev}\,\tau(j) := s + 1 - \tau(j)$, and then reversing the order of the block rows and block columns of $\mathbb{M}_{\mathrm{rev}\,\tau}(-\mathrm{rev}\,P)$.*

Proof. Use $R^2 = I_{nk}$ to write

$$\widetilde{M}_{k-\tau^{-1}(s)}(P) \cdots \widetilde{M}_{k-\tau^{-1}(1)}(P) = R\left(R\,\widetilde{M}_{k-\tau^{-1}(s)}(P)\,R\right) \cdots \left(R\,\widetilde{M}_{k-\tau^{-1}(1)}(P)\,R\right)R.$$

Next use (4.2), (4.4), (2.2)-(2.3), and the fact that the ith degree coefficient of $\mathrm{rev}\,P(\lambda)$ is A_{k-i} to see that

$$R\,\widetilde{M}_{k-j}(P)\,R = M_j(-\mathrm{rev}\,P), \quad \text{for } j = 0, \ldots, k-1,$$

and equation (4.7) follows. For the construction of the right-hand side of (4.7), simply note that the order of the M_i matrices in $\mathbb{M}_{\mathrm{rev}\,\tau}$ is reversed with respect to their order in $\mathbb{M}_\tau$ in (3.1). $\qquad\square$

4.1 Technical lemmas

We gather in this subsection four technical lemmas that are used in the proof of the main result of the paper, i.e., Theorem 4.8. These lemmas investigate the block structure of the matrix $\mathbb{M}_0 \in \mathbb{F}^{nk \times nk}$ introduced in Definition 4.2, viewed as a $k \times k$ block matrix with $n \times n$ blocks.

Lemma 4.4. *Let $\mathbb{M}_0$ be as in Definition 4.2. Then the following statements hold.*

(a) *If τ has a consecution at i for some $0 \leq i \leq k-2$, then the $(k-i)$th block-column of $\mathbb{M}_0$ contains exactly one identity block, and all of its remaining blocks are zero.*

(b) *If τ has an inversion at i for some $0 \leq i \leq k-2$, then the $(k-i)$th block-row of $\mathbb{M}_0$ contains exactly one identity block, and all of its remaining blocks are zero.*

Proof. The result is an immediate consequence of Corollary 3.4 and `Algorithm 1` in Theorem 3.2. We only prove (a); the proof of (b) is analogous. For $i = k - 2$, the result follows from `Algorithm 1` and the fact that $\mathbb{M}_0 = W_{k-2}$. For other i recall that, from Corollary 3.4 (b), we know that $\mathbb{M}_0(:, k - i)$ is obtained from $W_i(:, 2)$ by adding zero blocks. But if τ has a consecution at i, then $W_i(:, 2) = [I\ \ 0]^T$ by `Algorithm 1`. $\square$

Lemma 4.5. *Let $\mathbb{M}_0$ be as in Definition* 4.2, *and* $0 \le i \le k-1$. *If $i \notin C$, then the $(k-i)$th block-row of $\mathbb{M}_0$ contains exactly one identity block, and all of its remaining blocks are zero. The same is true for the $(k-i)$th block-column of $\mathbb{M}_0$.*

Proof. Recall that $k \ge 3$ and that $i > 0$ since $i \notin C$. We prove the result for block-rows; the argument for block-columns is analogous. If $i = k - 1$, then `Algorithm 1` gives $\mathbb{M}_0 = W_{k-2} = \text{diag}(I, W_{k-3})$ and the result is proven. If $0 < i \le k - 2$, then $\mathbb{M}_0(k - i, :)$ has the same nonzero blocks as $W_i(2, :)$. This follows from Corollary 3.4(b) for $i < k - 2$, and from $\mathbb{M}_0 = W_{k-2}$ for $i = k - 2$. Therefore in the rest of the proof we focus on proving that $W_i(2, :)$ has only one nonzero block equal to I. `Algorithm 1` provides two possibilities for W_i when $i \notin C$:

$$
W_i = \begin{bmatrix} -A_{i+1} & I & \\ I & 0 & \\ & & W_{i-1}(2 : i + 1, 2 : i + 1) \end{bmatrix} \quad \text{if } i + 1 \in C,
$$

or $W_i = \text{diag}(I, W_{i-1})$ if $i+1 \notin C$. But $i \notin C$ in `Algorithm 1` implies that $W_{i-1}(1, :) = [I\ \ 0]$. So in any case $W_i(2, :)$ contains exactly one identity block and its remaining blocks are zero. $\square$

Lemma 4.6. *Let $\mathbb{M}_0$ be as in Definition* 4.2. *Then the following statements hold.*

(a) *The matrix $\mathbb{M}_0$ contains exactly $k - 1$ identity blocks.*

(b) *If the (i, j) block-entry of $\mathbb{M}_0$, with $i \ne j$, is equal to I, then a block $-A_d$, for some $0 \le d \le k - 1$, is in the ith block-row or in the jth block-column of $\mathbb{M}_0$.*

Proof. Part (a) follows from `Algorithm 1`, that constructs $\mathbb{M}_0$ in $k-1$ steps. Observe that in each step exactly one identity block is added. This is evident in all cases except when $i \notin C$ and $i + 1 \in C$, for $i \ge 1$. In this case W_i is obtained by adding as nonzero blocks $-A_{i+1}$ and two I blocks, while at the same time removing the first block-row and the first block-column of W_{i-1}. But $i \notin C$ implies $W_{i-1}(1, :) = [I\ \ 0]$ and $W_{i-1}(:, 1) = [I\ \ 0]^T$, so the net result is that exactly one I is added.

Part (b): We will prove by induction that the result is true for every matrix $W_0, W_1, \ldots,$ $W_{k-2} = \mathbb{M}_0$ computed by `Algorithm 1`. The result is obviously true for W_0. Assume that it is true for W_{i-1} with $i - 1 \ge 0$, and let us prove it for W_i. Getting W_i from W_{i-1} according to `Algorithm 1`, a simple inspection shows that those off-diagonal identity blocks of W_i that are not in W_{i-1} satisfy the condition of the statement. For those off-diagonal identity blocks of W_i that are in W_{i-1}, note that:

(1) off-diagonal blocks of W_{i-1} remain as off-diagonal blocks of W_i,

(2) the block-rows and block-columns of W_{i-1} corresponding to off-diagonal identity blocks are contained in W_i.

The result of part (b) then follows from (1) and (2). □

Lemma 4.7. *Let $\mathbb{M}_0$ be as in Definition 4.2, $0 \leq i \leq k-1$, and recall that $h = (k+1)/2$. Then the $(k-i, k-i)$ block-entry of $\mathbb{M}_0$ is equal to I if and only if $i \notin C$ and $i+1 \notin C$. In particular, the (h, h) block-entry of $\mathbb{M}_0$ is never equal to I.*

Proof. Recall that $k \geq 3$. Consider first the case $i = k-1$, i.e., $k-i = 1$. Then according to `Algorithm 1`, $\mathbb{M}_0(1,1) = W_{k-2}(1,1) = I$ if and only if $k-1 \notin C$. This proves the result for $i = k-1$ because $k \notin C$ by definition.

Now we consider $i = k-2$, i.e., $k-i = 2$. Then according to `Algorithm 1`, $\mathbb{M}_0(2,2) = W_{k-2}(2,2) = I$ if and only if $k-1 \notin C$ and $W_{k-3}(1,1) = I$. Use again `Algorithm 1` to see that $W_{k-3}(1,1) = I$ if and only if $k-2 \notin C$. This proves the result for $i = k-2$.

Finally consider $i \leq k-3$, and use Corollary 3.4(a) to establish that $\mathbb{M}_0(k-i, k-i) = I$ if and only if $W_i(2,2) = I$. This never happens for $i = 0 \in C$ because $W_0(2,2) \neq I$. For $i \geq 1$, `Algorithm 1` says that $W_i(2,2) = I$ if and only if $i+1 \notin C$ and $W_{i-1}(1,1) = I$, which is equivalent to $i+1 \notin C$ and $i \notin C$.

Observe that $h-1$ and h are "complementary indices", since $h = k - (h-1)$. Thus $\mathbb{M}_0(h,h) \neq I$, since Definition 4.2 for admissible index sets C does not allow $h-1 \notin C$ and $h \notin C$. Note that $2 \leq h \leq k-1$. □

4.2 Main result, consequences and examples

Now we can state and prove the most important result in this work, Theorem 4.8, which presents a simple procedure to construct a family of palindromic companion forms for odd degree matrix polynomials.

Theorem 4.8 (Palindromic companion forms for odd degree polynomials).
Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, with $A_i \in \mathbb{F}^{n \times n}$ and $A_k \neq 0$, be a (regular or singular) matrix polynomial of odd degree $k \geq 3$, let $h = (k+1)/2$, and let $C \subset \{0, 1, \ldots, k-1\}$ be an admissible index set. Let $\tau : C \rightarrow \{1, 2, \ldots, h\}$ be a bijection, and let $L_\tau(\lambda)$ be the pencil of $P(\lambda)$ associated with C and τ, as defined in (4.5). Define $S_\tau \in \mathbb{F}^{nk \times nk}$ as the $k \times k$ block-diagonal matrix whose $n \times n$ diagonal block $S_\tau(i,i)$ is given for $i = 1, \ldots, k$ by

$$S_\tau(i,i) := \begin{cases} -I & \text{if} \quad \begin{cases} \tau \text{ has an inversion at } i-1, \text{ or} \\ \tau \text{ has a consecution at } k-i, \text{ or} \\ i \in C \text{ and } i-1 \notin C \end{cases} \\ I & \text{otherwise} \end{cases} . \quad (4.8)$$

Then the pencil $S_\tau \cdot R \cdot L_\tau(\lambda)$ is a palindromic companion form for all square matrix polynomials of odd degree k.

Remark 4.9. The reader is invited to check that the number of $-I$ blocks in S_τ is always $(k-1)/2$. For this purpose, prove first that the three conditions "τ has an inversion at $i-1$", "τ has a consecution at $k-i$", and "$i \in C$ and $i-1 \notin C$" are mutually exclusive, that is, if any of them holds, then the other two do not hold. After this, note that the number of inversions of τ plus the number of consecutions of τ plus the number of indices i such that "$i \in C$ and $i-1 \notin C$" is exactly $h-1 = (k-1)/2$.

Proof of Theorem 4.8. Since $S_\tau \cdot R \cdot L_\tau(\lambda)$ is strictly equivalent to a Fiedler pencil for $P(\lambda)$, we know from Theorem 2.2 that it satisfies property (b) in Definition 1.1. Moreover, Algorithm 1, Lemma 4.3, together with the block structure of S_τ and R guarantee that $S_\tau \cdot R \cdot L_\tau(\lambda)$ satisfies property (a) in Definition 1.1. Thus the only remaining task is to prove that $S_\tau \cdot R \cdot L_\tau(\lambda)$ is palindromic whenever $P(\lambda)$ is. For this, we will use the following notation: $\mathbb{M}_0(P)$ and $\mathbb{M}_1(P)$ are the matrices defined in (4.6) for $P(\lambda)$, while $\mathbb{M}_0(-P)$ and $\mathbb{M}_1(-P)$ are the corresponding matrices for $-P(\lambda)$. The proof will be carried out in two steps:

Step 1. We will prove that, if $P(\lambda)$ is palindromic, then

$$\left(R \cdot \mathbb{M}_1(P) \right)^T = R \cdot \mathbb{M}_0(-P). \qquad (4.9)$$

Step 2. We will prove that

$$R \cdot \mathbb{M}_0(-P) \cdot S_\tau = -S_\tau \cdot R \cdot \mathbb{M}_0(P). \qquad (4.10)$$

Observe that (4.9) and (4.10) easily imply that $S_\tau \cdot R \cdot L_\tau(\lambda)$ is palindromic whenever $P(\lambda)$ is. From (4.5) we have $L_\tau(\lambda) := \lambda \mathbb{M}_1(P) - \mathbb{M}_0(P)$, so that $S_\tau \cdot R \cdot L_\tau(\lambda) = \lambda X + Y$, where $X = S_\tau R \, \mathbb{M}_1(P)$ and $Y = -S_\tau R \, \mathbb{M}_0(P)$. But

$$X^T = (R \cdot \mathbb{M}_1(P))^T \cdot S_\tau^T = R \cdot \mathbb{M}_0(-P) \cdot S_\tau = -S_\tau \cdot R \cdot \mathbb{M}_0(P) = Y \,,$$

which means that $S_\tau \cdot R \cdot L_\tau(\lambda)$ is palindromic. Note that in (4.9) we are assuming that P is palindromic, whereas (4.10) is true for an arbitrary $n \times n$ polynomial P of odd degree k.

Step 1. From Lemma 4.3 and $R^2 = I_{nk}$ we get that

$$R \, \mathbb{M}_1(P) \ = \ M_{\tau^{-1}(h)}(-\operatorname{rev} P) \cdots M_{\tau^{-1}(1)}(-\operatorname{rev} P) \cdot R \,.$$

Therefore, if P is palindromic, i.e., $\operatorname{rev} P(\lambda) = P(\lambda)^T$, then

$$R \, \mathbb{M}_1(P) \ = \ M_{\tau^{-1}(h)}(-P^T) \cdots M_{\tau^{-1}(1)}(-P^T) \cdot R \,. \qquad (4.11)$$

Finally, from (4.11) and the fact that $(M_i(-P^T))^T = M_i(-P)$ for $i = 0, \ldots, k$, we obtain (4.9) by transposition:

$$\begin{aligned} \left(R \, \mathbb{M}_1(P) \right)^T &= R \cdot M_{\tau^{-1}(1)}(-P) \cdots M_{\tau^{-1}(h)}(-P) \\ &= R \cdot \mathbb{M}_0(-P). \end{aligned}$$

Step 2. Now we address the proof of (4.10). We will use the matrix $\widetilde{S}_\tau := RS_\tau R \in \mathbb{F}^{nk \times nk}$. Viewed as a $k \times k$ block matrix with $n \times n$ blocks, $\widetilde{S}_\tau$ is block diagonal with diagonal blocks

$$\widetilde{S}_\tau(i,i) = S_\tau(k+1-i, k+1-i) \quad \text{for } i = 1, \ldots, k.$$

Observe that if $S_\tau(i,i) = -I$ and $H \in \mathbb{F}^{nk \times nk}$ is an arbitrary matrix viewed as a $k \times k$ block matrix with $n \times n$ blocks, then the ith block-column of HS_τ is minus the ith block-column of H, whereas the ith block-row of $S_\tau H$ is minus the ith block-row of H.

The identities $S_\tau R = R\widetilde{S}_\tau$ and $S_\tau^2 = I$ allow us to show that (4.10) is equivalent to

$$\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau = -\mathbb{M}_0(P) \,. \qquad (4.12)$$

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Therefore we focus on proving (4.12) in the remainder of the argument, which relies on Lemmas 4.4, 4.5, 4.6 and 4.7, and is somewhat messy, although elementary. In what follows all matrices are viewed as $k \times k$ block matrices with $n \times n$ blocks, and we often use MATLAB notation on block indices. For brevity we use expressions like $H(i, :) = [0 \cdots 0 \; I \; 0 \cdots 0]$ to indicate that the ith block-row of H has only one nonzero block equal to I that can be in any block-entry, including the first and the last ones.

From $\mathtt{Algorithm\ 1}$ for constructing $\mathbb{M}_0(P)$ and $\mathbb{M}_0(-P)$ and Lemma 4.6, it is easy to see that: (1) $\mathbb{M}_0(P)$ has $k - 1$ blocks equal to I, h blocks $-A_{j_1}, \ldots, -A_{j_h}$, where $C = \{j_1, \ldots, j_h\}$, and the remaining blocks are zero; and, (2) if the blocks $-A_{j_1}, \ldots, -A_{j_h}$ in $\mathbb{M}_0(P)$ are replaced by $A_{j_1}, \ldots, A_{j_h}$, then $\mathbb{M}_0(-P)$ is obtained. Therefore, as S_τ and $\widetilde{S}_\tau$ are block diagonal with diagonal blocks $\pm I$, proving (4.12) is equivalent to proving that the *only* effect of $\widetilde{S}_\tau$ and S_τ in the product $\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau$ is transforming all $k - 1$ identity blocks of $\mathbb{M}_0(-P)$ into minus identities or, equivalently in terms of block-entries, that

$$\mathbb{M}_0(-P)\,(i, j) = -\left(\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau\right)(i, j), \quad \text{whenever } \mathbb{M}_0(-P)(i, j) = I, \qquad (4.13)$$

$$\mathbb{M}_0(-P)\,(i, j) = \left(\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau\right)(i, j), \qquad \text{otherwise}, \qquad (4.14)$$

for $1 \le i, j \le k$. We will prove (4.13)-(4.14) through the following three steps:

(a) We will prove that if $S_\tau(j, j) = -I$, $1 \le j \le k$, then $\mathbb{M}_0(-P)\,(:, j) = [0 \cdots 0 \; I \; 0 \cdots 0]^T$.

(b) We will prove that if $\widetilde{S}_\tau(i, i) = -I$, $1 \le i \le k$, then $\mathbb{M}_0(-P)\,(i, :) = [0 \cdots 0 \; I \; 0 \cdots 0]$.

(c) We will prove that if $\mathbb{M}_0(-P)\,(i, j) = I$, then $\widetilde{S}_\tau(i, i) \ne -I$ or $S_\tau(j, j) \ne -I$.

Observe that **(a)**, **(b)** and **(c)** imply that each $-I$ block in S_τ and $\widetilde{S}_\tau$ has only the effect of transforming one identity block of $\mathbb{M}_0(-P)$ into a minus identity block of $\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau$. But this means that all identity blocks of $\mathbb{M}_0(-P)$ are transformed into minus identity blocks of $\widetilde{S}_\tau \cdot \mathbb{M}_0(-P) \cdot S_\tau$, because the total number of $-I$ blocks in S_τ and $\widetilde{S}_\tau$ is $k - 1$.

Proof of **(a)**: $S_\tau(j, j) = -I$ implies that "τ has an inversion at $j - 1$", or "τ has a consecution at $k - j$", or "$j \in C$ and $j - 1 \notin C$". Let us analyze separately these three possibilities. If "τ has an inversion at $j - 1$", then $j \in C$, which is equivalent to $k - j \notin C$, and Lemma 4.5 implies the result. If "τ has a consecution at $k - j$", then Lemma 4.4(a) implies the result. Finally, if "$j \in C$ and $j - 1 \notin C$", then $k - j \notin C$ and Lemma 4.5 implies the result.

Proof of **(b)**: $\widetilde{S}_\tau(i, i) = S_\tau(k + 1 - i, k + 1 - i) = -I$ implies that "τ has an inversion at $k - i$", or "τ has a consecution at $i - 1$", or "$k + 1 - i \in C$ and $k - i \notin C$". Let us analyze separately these three possibilities. If "τ has an inversion at $k - i$", then Lemma 4.4(b) implies the result. If "τ has a consecution at $i - 1$", then $i \in C$, which is equivalent to $k - i \notin C$, and Lemma 4.5 implies the result. Finally, if "$k + 1 - i \in C$ and $k - i \notin C$", then Lemma 4.5 again implies the result.

Proof of **(c)**: For $i \ne j$ proceed by contradiction: assume $\widetilde{S}_\tau(i, i) = -I$ and $S_\tau(j, j) = -I$. Therefore, from **(b)** and **(a)**, $\mathbb{M}_0(-P)\,(i, :) = [0 \cdots 0 \; I \; 0 \cdots 0]$ and $\mathbb{M}_0(-P)\,(:, j) = [0 \cdots 0 \; I \; 0 \cdots 0]^T$. This implies $\mathbb{M}_0(-P)\,(i, j) \ne I$ by Lemma 4.6(b).

For $i = j$, we give a direct argument. $\mathbb{M}_0(-P)\,(i, i) = I$ implies $k - i \notin C$ and $k - i + 1 \notin C$ by Lemma 4.7. This is equivalent to $i \in C$ and $i - 1 \in C$, by Definition 4.2. So in this situation the definition of S_τ implies that $S_\tau(i, i) = -I$ holds only if "τ has an inversion at

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$i-1$", and that $\widetilde{S}_\tau(i,i) = S_\tau(k+1-i, k+1-i) = -I$ holds only if "τ has a consecution at $i-1$". Therefore $S_\tau(i,i) \neq -I$ or $\widetilde{S}_\tau(i,i) \neq -I$. $\qquad\Box$

Theorem 4.8 provides *many* strong linearizations $S_\tau \cdot R \cdot L_\tau(\lambda)$ for $P(\lambda)$ that are palindromic whenever $P(\lambda)$ is. Note first of all that there are $2^{(k-1)/2}$ different[2] admissible index sets C, and that for each of these sets C there exist many different bijections $\tau : C \to \{1, 2, \ldots, h\}$. In this context, it is important to note that different bijections of the same C may produce the same linearization due to the commutativity relations (2.5); as a consequence, we see that different index sets C may produce quite different *numbers* of distinct linearizations. This can be readily observed in Table 4.1. However, if $C_1 \neq C_2$ are distinct admissible index sets, then a linearization associated with C_1 is never equal to any linearization associated with C_2, because the set of coefficients of $P(\lambda)$ appearing in the zero-degree terms of these two linearizations must be different.

We present next some concrete examples of the various palindromic companion forms provided by Theorem 4.8, both to emphasize the ease of construction of these palindromic linearizations from the coefficients of the polynomial, as well as to highlight how certain selections of the index set C and the bijection τ can produce some particularly simple patterns.

Example 4.10. Let $k \geq 3$ be an odd integer. Consider the admissible index set

$$C = \{2j \ : \ j = 0, 1, \ldots, (k-1)/2\} = \{0, 2, 4, \ldots, k-1\}\,,$$

and the bijection $\tau : C \to \{1, 2, \ldots, h\}$ defined by $\tau(2j) = j+1$, for $j = 0, 1, \ldots, (k-1)/2$. Then the pencil (4.5) associated with C and τ is

$$L_\tau(\lambda) \ = \ \lambda M_1^{-1} M_3^{-1} \cdots M_{k-2}^{-1} M_k \ - \ M_0 M_2 \cdots M_{k-3} M_{k-1}\,, \qquad (4.15)$$

and the matrix S_τ in (4.8) satisfies $S_\tau(i,i) = I$ for odd i, and $S_\tau(i,i) = -I$ for even i. For $L_\tau(\lambda)$ in (4.15), denote by $L_k(\lambda) := S_\tau \cdot R \cdot L_\tau(\lambda)$ the pencil in the statement of Theorem 4.8 associated with $P(\lambda)$; then we have

$$L_3(\lambda) = \lambda \begin{bmatrix} & I & A_1 \\ & 0 & -I \\ A_3 & & \end{bmatrix} + \begin{bmatrix} & & A_0 \\ I & 0 & \\ A_2 & -I & \end{bmatrix},$$

$$L_5(\lambda) = \lambda \begin{bmatrix} & & I & A_1 \\ & & 0 & -I \\ & I & A_3 & \\ & 0 & -I & \\ A_5 & & & \end{bmatrix} + \begin{bmatrix} & & & A_0 \\ & & I & 0 \\ & & A_2 & -I \\ I & 0 & & \\ A_4 & -I & & \end{bmatrix},$$

---

[2] Recall that $j \in C$, $1 \leq j \leq k-1$, if and only if $k - j \notin C$ and that $\{1, \ldots, k-1\} = \bigcup_{j=1}^{(k-1)/2} \{j, k-j\}$, so there are $2^{(k-1)/2}$ ways of selecting the elements of C.

and a direct inductive argument gives

$$
L_k(\lambda) = \begin{bmatrix}
 & & & & & & \lambda I & \lambda A_1 + A_0 \\
 & & & & & I & 0 & -\lambda I \\
 & & & & \lambda I & \lambda A_3 + A_2 & -I \\
 & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} \\
 & & I & 0 & -\lambda I \\
 & \lambda I & \lambda A_{k-2} + A_{k-3} & -I \\
 I & 0 & -\lambda I \\
 \lambda A_k + A_{k-1} & -I
\end{bmatrix},
$$

which has a reverse block-tridiagonal pattern. Note that by the commutativity relations (2.5) and the analogous relations for the inverses of the matrices M_i's, every bijection $\tau : \{0, 2, 4, \ldots, k-1\} \to \{1, 2, \ldots, h\}$ yields the same pencil $L_k(\lambda)$.

Example 4.11. In this example we show (for polynomials of degree $k = 5$) several of the palindromic linearizations from Theorem 4.8 having special patterns. The reader can easily generalize these patterns to arbitrary odd degrees. First, we illustrate that there exist other "reverse" block-tridiagonal patterns in addition to the one in Example 4.10. For this purpose, choose the admissible index set $C_1 = \{0, 1, 3\}$ and the bijection $\tau_1 : C_1 \to \{1, 2, 3\}$ defined by $\tau_1(0) = 1$, $\tau_1(1) = 2$, $\tau_1(3) = 3$. Then

$$
L_{\tau_1}(\lambda) = \lambda M_2^{-1} M_4^{-1} M_5 - M_0 M_1 M_3 .
$$

The blocks of S_{τ_1} in (4.8) are $S_{\tau_1}(i, i) = I$ for $i = 1, 2, 4$ and $S_{\tau_1}(i, i) = -I$ for $i = 3, 5$. Then

$$
S_{\tau_1} \cdot R \cdot L_{\tau_1}(\lambda) = \lambda \begin{bmatrix}
 & & & & I \\
 & & & I & A_2 \\
 & & & 0 & -I \\
 A_5 & A_4 \\
 0 & -I
\end{bmatrix} + \begin{bmatrix}
 & & & A_0 & 0 \\
 & & & A_1 & -I \\
 & I & 0 \\
 & A_3 & -I \\
 I
\end{bmatrix}
$$

is a palindromic companion form for degree 5 matrix polynomials.

Next, for the admissible index set $C_2 = \{0, 1, 2\}$, we present two palindromic linearizations with maximum block-bandwidth about the anti-diagonal. Let $\tau_2 : C_2 \to \{1, 2, 3\}$ be defined by $\tau_2(0) = 3$, $\tau_2(1) = 2$, $\tau_2(2) = 1$, then

$$
L_{\tau_2}(\lambda) = \lambda M_5 M_4^{-1} M_3^{-1} - M_2 M_1 M_0 .
$$

In this case $S_{\tau_2}(i, i) = I$ for $i = 3, 4, 5$ and $S_{\tau_2}(i, i) = -I$ for $i = 1, 2$, so

$$
S_{\tau_2} \cdot R \cdot L_{\tau_2}(\lambda) = \lambda \begin{bmatrix}
 & & & & -I \\
 & & & -I \\
 0 & I & A_3 \\
 I & 0 & A_4 \\
 0 & 0 & A_5
\end{bmatrix} + \begin{bmatrix}
 & & 0 & I & 0 \\
 & & I & 0 & 0 \\
 & & A_2 & A_1 & A_0 \\
 & -I \\
 -I
\end{bmatrix}
$$

is another palindromic companion form. Observe that the zero and the first degree terms each contain three factors with *consecutive* indices, which causes the structure of the 3×3

block matrix in the upper-right (resp., lower-left) corner in the zero (resp., first) degree term. Other 3×3 structures of this type can be produced by taking different orders of the M_0, M_1 and M_2 factors in the zero degree term. For instance, with $\tau_3 : C_2 \to \{1, 2, 3\}$ defined by $\tau_3(0) = 1 \, , \tau_3(1) = 3 \, , \tau_3(2) = 2$, then

$$L_{\tau_3}(\lambda) = \lambda M_4^{-1} M_3^{-1} M_5 - M_0 M_2 M_1 \, .$$

Now $S_{\tau_3}(i, i) = I$ for $i = 1, 3, 4$ and $S_{\tau_3}(i, i) = -I$ for $i = 2, 5$, yielding the palindromic companion form

$$S_{\tau_3} \cdot R \cdot L_{\tau_3}(\lambda) = \lambda \begin{bmatrix} & & & & I \\ & & & -I & \\ 0 & I & A_3 & & \\ A_5 & 0 & A_4 & & \\ 0 & 0 & -I & & \end{bmatrix} + \begin{bmatrix} 0 & A_0 & 0 \\ I & 0 & 0 \\ A_2 & A_1 & -I \\ & -I & \\ I & & \end{bmatrix} \, .$$

Table 4.1 displays all the distinct pencils that may be constructed for degree $k = 5$ using the procedure of Theorem 4.8, including the examples above.

If we look carefully at the patterns of blocks in the pencils $S_\tau \cdot R \cdot L_\tau(\lambda)$ in Table 4.1, we find that, up to the signs of the identity blocks, these pencils are paired up by block symmetry through the main block anti-diagonal. In particular, the first one is paired with the third one, the second one with the fourth one, the fifth one with the sixth one and the seventh one with the eighth one. The ninth one is self-paired, because, up to signs, it is block symmetric through the main block anti-diagonal. Lemma 4.12 below shows that this is not just a coincidence. Before stating this lemma, let us first recall the concept of *reversal bijection*, used previously in Lemma 4.3. If C is an admissible index set and $\tau : C \to \{1, 2, \ldots, h\}$ is a bijection, then the *reversal bijection* of τ is $\operatorname{rev} \tau : C \to \{1, 2, \ldots, h\}$, defined by $\operatorname{rev} \tau(j) := h + 1 - \tau(j)$. In plain words, the M_j factors of $\mathbb{M}_{\operatorname{rev}\tau}$ are the same as the factors of $\mathbb{M}_\tau$ in (3.1), but placed in reverse order. Then Lemma 4.12 shows how each pencil $S_\tau \cdot R \cdot L_\tau(\lambda)$ constructed in Theorem 4.8 can be naturally paired up with the pencil $S_{\operatorname{rev}\tau} \cdot R \cdot L_{\operatorname{rev}\tau}(\lambda)$. We will also need the block-transpose operation: Let $A = (A_{ij})$ be a block $r \times s$ matrix with $m \times n$ blocks A_{ij}. The *block transpose* of A is the block $s \times r$ matrix $A^{\mathcal{B}}$ with $m \times n$ blocks defined by $\left(A^{\mathcal{B}}\right)_{ij} = A_{ji}$.

Lemma 4.12. *Let τ and $L_\tau(\lambda)$ be as in the statement of Theorem* 4.8, *and let* $\operatorname{rev} \tau$ *be the reversal bijection of τ. Then*

$$R \left(R \cdot L_\tau(\lambda) \right)^{\mathcal{B}} R = R \cdot L_{\operatorname{rev}\tau}(\lambda) \, .$$

Proof. We first recall [20, Chapter 3] that if A and C are block partitioned matrices with $n \times n$ blocks A_{ij} and C_{ij} such that $A_{ij} C_{jp} = C_{jp} A_{ij}$, for all i, j, p, then $(AC)^{\mathcal{B}} = C^{\mathcal{B}} A^{\mathcal{B}}$. This property implies that

$$R \left(R \cdot L_\tau(\lambda) \right)^{\mathcal{B}} R = R \cdot \left(L_\tau(\lambda) \right)^{\mathcal{B}} \cdot R^{\mathcal{B}} R = R \cdot \left(L_\tau(\lambda) \right)^{\mathcal{B}} .$$

Next, it can be proved that $\left(L_\tau(\lambda) \right)^{\mathcal{B}} = L_{\operatorname{rev}\tau}(\lambda)$ with some care. We only sketch the proof. For the zero degree terms $\left(\mathbb{M}_\tau(P) \right)^{\mathcal{B}}$ and $\mathbb{M}_{\operatorname{rev}\tau}(P)$, first note that $\operatorname{rev} \tau$ has a

C	τ	S_τ	$S_\tau \cdot R \cdot L_\tau(\lambda)$
$\{0,1,2\}$	$(1,2,3)$	$\operatorname{diag}(I,I,I,-I,-I)$	$\begin{bmatrix} 0 & 0 & A_0 & 0 & \lambda I \\ 0 & 0 & A_1 & \lambda I & -I \\ \lambda A_5 & \lambda A_4 & \lambda A_3+A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(3,1,2)$	$\operatorname{diag}(-I,I,I,-I,I)$	$\begin{bmatrix} 0 & 0 & I & 0 & -\lambda I \\ 0 & 0 & A_1 & \lambda I & A_0 \\ \lambda I & \lambda A_4 & \lambda A_3+A_2 & -I & 0 \\ 0 & I & -\lambda I & 0 & 0 \\ -I & \lambda A_5 & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(3,2,1)$	$\operatorname{diag}(-I,-I,I,I,I)$	$\begin{bmatrix} 0 & 0 & 0 & I & -\lambda I \\ 0 & 0 & I & -\lambda I & 0 \\ 0 & \lambda I & \lambda A_3+A_2 & A_1 & A_0 \\ \lambda I & -I & \lambda A_4 & 0 & 0 \\ -I & 0 & \lambda A_5 & 0 & 0 \end{bmatrix}$
$\{0,1,2\}$	$(1,3,2)$	$\operatorname{diag}(I,-I,I,I,-I)$	$\begin{bmatrix} 0 & 0 & 0 & A_0 & \lambda I \\ 0 & 0 & I & -\lambda I & 0 \\ 0 & \lambda I & \lambda A_3+A_2 & A_1 & -I \\ \lambda A_5 & -I & \lambda A_4 & 0 & 0 \\ I & 0 & -\lambda I & 0 & 0 \end{bmatrix}$
$\{0,1,3\}$	$(1,2,3)$	$\operatorname{diag}(I,I,-I,I,-I)$	$\begin{bmatrix} 0 & 0 & 0 & A_0 & \lambda I \\ 0 & 0 & \lambda I & \lambda A_2+A_1 & -I \\ 0 & I & 0 & -\lambda I & 0 \\ \lambda A_5 & \lambda A_4+A_3 & -I & 0 & 0 \\ I & -\lambda I & 0 & 0 & 0 \end{bmatrix}$
$\{0,1,3\}$	$(3,2,1)$	$\operatorname{diag}(-I,I,-I,I,I)$	$\begin{bmatrix} 0 & 0 & 0 & I & -\lambda I \\ 0 & 0 & \lambda I & \lambda A_2+A_1 & A_0 \\ 0 & I & 0 & -\lambda I & 0 \\ \lambda I & \lambda A_4+A_3 & -I & 0 & 0 \\ -I & \lambda A_5 & 0 & 0 & 0 \end{bmatrix}$
$\{0,3,4\}$	$(1,2,3)$	$\operatorname{diag}(I,-I,-I,I,I)$	$\begin{bmatrix} 0 & 0 & \lambda I & \lambda A_2 & \lambda A_1+A_0 \\ 0 & 0 & 0 & I & -\lambda I \\ I & 0 & 0 & -\lambda I & 0 \\ A_3 & \lambda I & -I & 0 & 0 \\ \lambda A_5+A_4 & -I & 0 & 0 & 0 \end{bmatrix}$
$\{0,3,4\}$	$(3,2,1)$	$\operatorname{diag}(I,I,-I,-I,I)$	$\begin{bmatrix} 0 & 0 & 0 & \lambda I & \lambda A_1+A_0 \\ 0 & 0 & \lambda I & -I & \lambda A_2 \\ 0 & I & 0 & 0 & -\lambda I \\ I & -\lambda I & 0 & 0 & 0 \\ \lambda A_5+A_4 & A_3 & -I & 0 & 0 \end{bmatrix}$
$\{0,2,4\}$	$(1,2,3)$	$\operatorname{diag}(I,-I,I,-I,I)$	$\begin{bmatrix} 0 & 0 & 0 & \lambda I & \lambda A_1+A_0 \\ 0 & 0 & I & 0 & -\lambda I \\ 0 & \lambda I & \lambda A_3+A_2 & -I & 0 \\ I & 0 & -\lambda I & 0 & 0 \\ \lambda A_5+A_4 & -I & 0 & 0 & 0 \end{bmatrix}$

Table 4.1: This table shows all of the nine distinct palindromic companion forms for polynomials of degree 5 that are constructible using Theorem 4.8. For each admissible index set $C = \{j_1, j_2, j_3\}$, the bijections $\tau : C \to \{1,2,3\}$ are described as $(\tau(j_1), \tau(j_2), \tau(j_3))$.

consecution (resp. inversion) at j if and only if τ has an inversion (resp. consecution) at j. Then `Algorithm 1` can be used to prove that $\left(\mathbb{M}_\tau(P)\right)^{\mathcal{B}} = \mathbb{M}_{\mathrm{rev}\,\tau}(P)$ via induction on the sequence of matrices $W_0, W_1, \ldots, W_{k-2}$ produced by `Algorithm 1`. This result for the zero degree terms can then be combined with the relationships proved in Lemma 4.3 to deduce the analogous result for the first degree terms of the pencils $\left(L_\tau(\lambda)\right)^{\mathcal{B}}$ and $L_{\mathrm{rev}\,\tau}(\lambda)$. $\qquad\square$

Lemma 4.12 tells us that, up to the change of signs given by the matrices S_τ and $S_{\mathrm{rev}\,\tau}$, the pencils constructed in Theorem 4.8 are paired up by the operation $R\,(\cdot)^{\mathcal{B}}\,R$, which can be viewed as a "block anti-transpose", i.e., a block transpose across the main block anti-diagonal. Notice that when $C = \{2j \,:\, j = 0, 1, \ldots, (k-1)/2\}$, due to the commutativity relations (2.5), the pencils $L_\tau(\lambda)$ are equal for *all* bijections $\tau \,:\, C \to \{1, 2, \ldots, h\}$. In particular, we have $L_\tau(\lambda) = L_{\mathrm{rev}\,\tau}(\lambda)$ for this index set C. Consequently this unique pencil satisfies the identity $R\left(R \cdot L_\tau(\lambda)\right)^{\mathcal{B}} \cdot R = R \cdot L_\tau(\lambda)$, that is, $R \cdot L_\tau(\lambda)$ is block symmetric through the main block anti-diagonal, and hence is self-paired by the operation $R\,(\cdot)^{\mathcal{B}}\,R$. For $k = 5$ this corresponds to the pencil at the bottom of Table 4.1. As a consequence, given any k odd, the number of different pencils constructed in Theorem 4.8 will always be odd.

Theorem 4.8 asserts, in particular, that each palindromic polynomial with odd degree has a palindromic strong linearization. It is worthwhile stating this as a separate fact.

Corollary 4.13. *Let k be an odd number and $P(\lambda)$ be an $n \times n$ palindromic matrix polynomial of degree k. Then there exists an $nk \times nk$ palindromic strong linearization of $P(\lambda)$.*

We want to stress that Corollary 4.13 is simply not true for palindromic polynomials of even degree, as illustrated by the example in (1.2) discussed in Section 1.

5 Anti-palindromic companion forms for odd degree

We remarked in Section 1 that anti-palindromic matrix polynomials, i.e., those satisfying $\mathrm{rev}\,P(\lambda) = -P(\lambda)^T$, have some interest in applications. Therefore, it is also natural to look for anti-palindromic linearizations of anti-palindromic polynomials. We show in this section that for polynomials with *odd degree*, any method for constructing palindromic linearizations of palindromic matrix polynomials can be very easily adapted to construct anti-palindromic linearizations of anti-palindromic polynomials. This is the content of Theorem 5.3, which can be applied to the linearizations introduced in Theorem 4.8 to provide a whole family of *anti-palindromic companion forms*. First we prove several simple preliminary lemmas.

Lemma 5.1. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial and define $Q(\lambda) := P(-\lambda)$. If $L(\lambda)$ is a linearization (resp., strong linearization) of $P(\lambda)$, then $\widetilde{L}(\lambda) := L(-\lambda)$ is a linearization (resp., strong linearization) of $Q(\lambda)$.*

Proof. Assume that the degree of $P(\lambda)$ is k. If $L(\lambda)$ is a linearization of $P(\lambda)$, then by definition there exist unimodular $U(\lambda)$ and $V(\lambda)$ such that $U(\lambda)\,L(\lambda)\,V(\lambda) = \mathrm{diag}(I_{(k-1)n}, P(\lambda))$. So $U(-\lambda)\,L(-\lambda)\,V(-\lambda) = \mathrm{diag}(I_{(k-1)n}, P(-\lambda))$, which shows that $\widetilde{L}(\lambda)$ is a linearization of $Q(\lambda)$, because $U(-\lambda)$ and $V(-\lambda)$ are unimodular.

The result for strong linearizations requires more attention. Observe that

$$
\begin{aligned}
(\mathrm{rev}\,Q)\,(\lambda) &= \lambda^k Q(1/\lambda) = \lambda^k P(-1/\lambda) = (-1)^k \left((-\lambda)^k P(-1/\lambda)\right) \\
&= (-1)^k \,(\mathrm{rev}\,P)\,(-\lambda).
\end{aligned} \tag{5.1}
$$

If $L(\lambda)$ is a strong linearization of $P(\lambda)$, then we also have that $Y(\lambda)\,(\operatorname{rev}L)(\lambda)\,Z(\lambda) = \operatorname{diag}(I_{(k-1)n}, (\operatorname{rev}P)(\lambda))$ for some unimodular matrices $Y(\lambda)$ and $Z(\lambda)$. As a consequence, $Y(-\lambda)\,(\operatorname{rev}L)(-\lambda)\,Z(-\lambda) = \operatorname{diag}(I_{(k-1)n}, (\operatorname{rev}P)(-\lambda))$, and from (5.1)

$$Y(-\lambda)\left(-(\operatorname{rev}\widetilde{L})(\lambda)\right)Z(-\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & (-1)^k\,(\operatorname{rev}Q)\,(\lambda) \end{bmatrix}.$$

This implies

$$E(\lambda)\,(\operatorname{rev}\widetilde{L})(\lambda)\,Z(-\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & (\operatorname{rev}Q)\,(\lambda) \end{bmatrix},$$

with $E(\lambda) = -\operatorname{diag}(I_{(k-1)n}, (-1)^k I)\,Y(-\lambda)$. Note that $E(\lambda)$ and $Z(-\lambda)$ are unimodular matrices, and therefore $(\operatorname{rev}\widetilde{L})(\lambda)$ is a linearization of $(\operatorname{rev}Q)\,(\lambda)$. $\qquad\square$

Lemma 5.2. *Let $P(\lambda)$ be any $n \times n$ matrix polynomial with* odd degree *and define $Q(\lambda) := P(-\lambda)$. Then $P(\lambda)$ is anti-palindromic if and only if $Q(\lambda)$ is palindromic. Also, $P(\lambda)$ is palindromic if and only if $Q(\lambda)$ is anti-palindromic.*

Proof. This follows directly from (5.1). $\qquad\square$

Next we state Theorem 5.3, the main result of this section. It is an immediate consequence of Lemmas 5.1 and 5.2, so its proof is omitted. Note that Lemma 5.2 has to be applied here both to polynomials and to linearizations.

Theorem 5.3. *Let $P(\lambda)$ be any $n \times n$ anti-palindromic matrix polynomial with* odd degree *and define $Q(\lambda) := P(-\lambda)$. Let $\widetilde{L}(\lambda)$ be any strong palindromic linearization of the palindromic polynomial $Q(\lambda)$. Then $L(\lambda) := \widetilde{L}(-\lambda)$ is a strong anti-palindromic linearization of $P(\lambda)$.*

6 The recovery of minimal indices

We noted in Section 1 that minimal indices are intrinsic quantities associated with singular matrix polynomials that are relevant in many control problems [8, 16]. In this section we show how to easily recover the minimal indices of a polynomial from those of any of the linearizations introduced in Theorem 4.8. The results in this section are consequences of results in [7].

Let us recall very briefly the concept of minimal indices (see [6, Section 2] or [7, Section 2] for more complete summaries). A *vector polynomial* is a vector whose entries are polynomials in the variable λ, and its *degree* is the greatest degree of its components. For any subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$ it is always possible to find a basis consisting entirely of vector polynomials. Then the *order* of a polynomial basis of $\mathcal{V}$ is the sum of the degrees of its vectors [8, p. 494], and a *minimal basis* of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$. It can be shown [8] that for any subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any two minimal bases of $\mathcal{V}$ are always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. The left (resp., right) minimal indices of a singular matrix polynomial $P(\lambda)$ are the minimal indices of its left (resp., right) null spaces (see Section 2). Observe that any square matrix polynomial $P(\lambda)$ has the same number of left and right minimal indices; recall also that if $P(\lambda)$ is palindromic, then its left minimal indices are equal to its right minimal indices [6, Theorem 3.6].

As mentioned in Section 2, strong linearizations preserve the elementary divisors of a polynomial $P(\lambda)$ and also the *number* of left and right minimal indices, but they do not in general preserve the *values* of the minimal indices. Therefore the recovery of the minimal indices of $P(\lambda)$ from those of one of its linearization is, in general, a non-trivial task [6, 7]. However, Theorem 6.1 shows that this recovery is very simple from any of the linearizations constructed in Theorem 4.8.

Theorem 6.1. *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with odd degree $k \geq 3$, and let $S_\tau \cdot R \cdot L_\tau(\lambda)$ be one of the strong linearizations of $P(\lambda)$ introduced in Theorem 4.8. Let $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ and $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ be, respectively, the left and right minimal indices of $P(\lambda)$. Then the following statements hold.*

(a) *The left and right minimal indices of $S_\tau \cdot R \cdot L_\tau(\lambda)$ are, respectively,*

$$\eta_1 + \frac{k-1}{2} \leq \eta_2 + \frac{k-1}{2} \leq \cdots \leq \eta_p + \frac{k-1}{2} \quad and$$
$$\varepsilon_1 + \frac{k-1}{2} \leq \varepsilon_2 + \frac{k-1}{2} \leq \cdots \leq \varepsilon_p + \frac{k-1}{2}.$$

(b) *If $P(\lambda)$ is palindromic, then $\eta_i = \varepsilon_i$ for $i = 1, \ldots, p$, and the left and right minimal indices of $S_\tau \cdot R \cdot L_\tau(\lambda)$ are both equal to $\varepsilon_1 + \frac{k-1}{2} \leq \varepsilon_2 + \frac{k-1}{2} \leq \cdots \leq \varepsilon_p + \frac{k-1}{2}$.*

Proof. As discussed in Section 4, recall that every pencil $S_\tau \cdot R \cdot L_\tau(\lambda)$ is strictly equivalent to some Fiedler pencil of $P(\lambda)$. We denote this Fiedler pencil by $F_\sigma(P)$, following the notation in [7]. In this proof we also denote $S_\tau \cdot R \cdot L_\tau(\lambda)$ by $S_\tau \cdot R \cdot L_\tau(P)$, in order to make explicit the dependence on $P(\lambda)$, dropping the dependence on λ for brevity. Since minimal indices are preserved by strict equivalence, therefore the minimal indices of $S_\tau \cdot R \cdot L_\tau(P)$ are equal to those of $F_\sigma(P)$.

The pencil $F_\sigma(P)$ is a function of $P(\lambda)$, and can be considered for any other $n \times n$ matrix polynomial $Q(\lambda)$ with degree k; we denote that pencil by $F_\sigma(Q)$. If $0 \leq \eta'_1 \leq \eta'_2 \leq \cdots \leq \eta'_q$ and $0 \leq \varepsilon'_1 \leq \varepsilon'_2 \leq \cdots \leq \varepsilon'_q$ are, respectively, the left and right minimal indices of $Q(\lambda)$, then the left and right minimal indices of $F_\sigma(Q)$ are given [7, Corollaries 5.8 and 5.11], respectively, by

$$\eta'_1 + \mathfrak{c}(\sigma) \leq \eta'_2 + \mathfrak{c}(\sigma) \leq \cdots \leq \eta'_q + \mathfrak{c}(\sigma) \quad \text{and} \tag{6.1}$$
$$\varepsilon'_1 + \mathfrak{i}(\sigma) \leq \varepsilon'_2 + \mathfrak{i}(\sigma) \leq \cdots \leq \varepsilon'_q + \mathfrak{i}(\sigma). \tag{6.2}$$

The quantities $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$ are defined in [7], but only two properties of them are of interest here: (1) $\mathfrak{c}(\sigma) + \mathfrak{i}(\sigma) = k - 1$; and, (2) they are the same for any $n \times n$ singular polynomial $Q(\lambda)$ of degree k. Therefore we can determine $\mathfrak{c}(\sigma)$ and $\mathfrak{i}(\sigma)$ by applying (6.1) and (6.2) to any particular matrix polynomial $Q(\lambda)$. Let us then assume that $Q(\lambda)$ is singular and palindromic, so that $\eta'_i = \varepsilon'_i$ for $i = 1, \ldots, q$ [6, Theorem 3.6]. Moreover, the minimal indices of $F_\sigma(Q)$ are equal to those of $S_\tau \cdot R \cdot L_\tau(Q)$; but this linearization of $Q(\lambda)$ is palindromic by Theorem 4.8, so $\eta'_i + \mathfrak{c}(\sigma) = \varepsilon'_i + \mathfrak{i}(\sigma)$ for $i = 1, \ldots, q$. Thus $\mathfrak{c}(\sigma) = \mathfrak{i}(\sigma) = (k - 1)/2$, and Theorem 6.1 follows from applying (6.1) and (6.2) to $P(\lambda)$. $\square$

Our last result is a corollary of Theorem 6.1 that asserts that all pencils constructed in Theorem 4.8 are strictly equivalent.

Corollary 6.2. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with odd degree $k \geq 3$. Then all pencils constructed in Theorem 4.8 for $P(\lambda)$ are strictly equivalent.*

Proof. By Theorem 6.1, all pencils constructed in Theorem 4.8 for $P(\lambda)$ have the same minimal indices. On the other hand, all these pencils are strong linearizations of $P(\lambda)$, so they all have the same finite and infinite elementary divisors. Since two matrix pencils are strictly equivalent if and only if they have the same elementary divisors and minimal indices [9], the result follows. □

Remark 6.3. As a further consequence of results in [7], it can be shown that if $P(\lambda)$ is singular, then *none* of the pencils constructed in Theorem 4.8 for $P(\lambda)$ are ever strictly equivalent to either the classical first or second Frobenius companion form of $P(\lambda)$.

We finally mention that the recovery of eigenvectors and minimal bases of an odd degree matrix polynomial from those of the linearizations constructed in Theorem 4.8 can be obtained as a consequence of the general results presented in [2].

7   Conclusions and future work

We have presented a symbolic procedure to construct a large family of palindromic companion forms for odd degree matrix polynomials. These companion forms provide uniform templates for producing strong linearizations of square matrix polynomials, which are valid for all polynomials of odd degree $k \geq 3$ over an arbitrary field, and are palindromic whenever the polynomial is palindromic. These linearizations are easily constructible from the coefficients of the polynomial, and can be simply modified to obtain anti-palindromic companion forms for each odd degree. Finally, for singular polynomials $P(\lambda)$ we have shown that the minimal indices of these linearizations are very simply related to the minimal indices of $P(\lambda)$.

The results in this paper are in sharp contrast with the situation for even degree palindromic polynomials, as described in Section 1 of this paper. Since there are palindromic matrix polynomials of even degree that have no palindromic linearizations of any kind, palindromic companion forms cannot exist for any even degree. Thus the natural continuation of the present paper is to address the even degree case in more detail, and try to obtain necessary and sufficient conditions for the existence of palindromic linearizations. This topic will be the subject of future work.

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