Quadratic Realizability of Palindromic Matrix Polynomials[☆]

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Abstract

Let $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$ be a list consisting of a sublist $\mathcal{L}_1$ of *powers of irreducible (monic) scalar polynomials* over an algebraically closed field $\mathbb{F}$, and a sublist $\mathcal{L}_2$ of *nonnegative integers*. For an arbitrary such list $\mathcal{L}$, we give easily verifiable necessary and sufficient conditions for $\mathcal{L}$ to be the list of elementary divisors and minimal indices of some *T-palindromic quadratic* matrix polynomial with entries in the field $\mathbb{F}$. For $\mathcal{L}$ satisfying these conditions, we show how to explicitly construct a T-palindromic quadratic matrix polynomial having $\mathcal{L}$ as its structural data; that is, we provide a *T-palindromic quadratic realization of $\mathcal{L}$*. Our construction of T-palindromic realizations is accomplished by taking a direct sum of low bandwidth T-palindromic blocks, closely resembling the Kronecker canonical form of matrix pencils. An immediate consequence of our in-depth study of the structure of T-palindromic quadratic polynomials is that all even grade T-palindromic matrix polynomials have a T-palindromic strong quadratification. Finally, using a particular Möbius transformation, we show how all of our results can be easily extended to quadratic matrix polynomials with T-even structure.

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1. Introduction

An $m \times n$ *matrix polynomial* $P(\lambda)$ of *degree* k over a field $\mathbb{F}$ is of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \tag{1.1}$$

where $A_i \in \mathbb{F}^{m\times n}$ for $i = 0, 1, \ldots, k$, and $A_k \neq 0$. Matrix polynomials arise in a variety of scientific and engineering problems, and consequently they often have special algebraic structures stemming from the underlying applications – examples of such structures include *symmetric*, *Hermitian*, *T-alternating* and *T-palindromic* [1, 6, 19, 24, 25, 26, 29, 33, 40, 46]. This work is primarily focused on T-palindromic matrix polynomials; in its most basic form, these are polynomials (1.1) satisfying $A_{k-i}^T = A_i$, for $i = 0, \ldots, k$ (see Definition 2.5 for the more general notion of T-palindromicity using *grade* instead of degree).

The main objective of this paper is to contribute to the theory of *inverse problems* for matrix polynomials. The general type of inverse problem we wish to consider has two main aspects: to find simple criteria to decide whether a given list $\mathcal{L}$ of structural data can be realized by some matrix polynomial P, and to show how to explicitly construct such a realization whenever it exists, preferably one that transparently displays

[☆] An important portion of this work is contained in Chapter 8 in the Ph.D. dissertation of the fourth author [42].

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all of the structural data. It is well known that the Kronecker canonical form provides a complete solution to this inverse problem for polynomials P of degree at most one [22], at least when the underlying field $\mathbb{F}$ is algebraically closed. However, the problem becomes much more interesting if one requires P to have a particular size or a degree larger than one, or to have some additional algebraic structure (e.g., symmetric, Hermitian, alternating, palindromic, etc).

Inverse problems of this kind have been of interest at least since the 1970s [39, Thm. 5.2], and have also been considered in the classical reference [23]. In the last few years, there has been renewed interest in this problem, not only due to its theoretical importance, but also because of connections with other problems (e.g., the stratification of orbits of matrix polynomials [27]). Some of the more recent developments include:

- a characterization for a list of scalar polynomials and nonnegative integers to be the elementary divisors and minimal indices, respectively, of some *full rank* matrix polynomial of *fixed degree* [27, Thm. 5.2]. An important extension of this result to matrix polynomials of *fixed size and degree* (over any infinite field), but not necessarily of full rank, has been given in [13, Thm. 3.12].
- necessary conditions on the Smith forms of T-alternating [34], T-palindromic [35], and skew-symmetric matrix polynomials [36]. These conditions for T-palindromic and T-alternating polynomials have also been shown (under mild additional assumptions) to be sufficient in the regular case [2, Thm. 3.1].
- a characterization for a pair of matrices to be the Jordan structure of a *quadratic real symmetric* matrix polynomial [28, Thm. 9]. A characterization is also provided for polynomials having a positive definite leading and/or trailing coefficient [28, Thms. 13, 14, 17].

In this paper we focus on an inverse problem that we refer to as the *Quadratic Realizability Problem* (QRP) for a class $\mathscr{C}$ of structured matrix polynomials, consisting of two subproblems (SPs):

(SP-1) *Characterization* of those structural data lists $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)$, where $\mathcal{L}_1$ comprises the desired spectral structure (elementary divisors) and $\mathcal{L}_2$ the desired singular structure (minimal indices), of some *quadratic* matrix polynomial in the given class $\mathscr{C}$.

(SP-2) For each such realizable list $\mathcal{L}$, show how to *concretely construct* a quadratic matrix polynomial in $\mathscr{C}$ whose structural data is exactly the list $\mathcal{L}$. It is also desirable for this concrete realization to be as *simple*, *transparent*, and *canonical*[1] as possible.

The work [13, Thm. 3.12] mentioned earlier provides a solution of the QRP for the class $\mathscr{C}$ of *all* $m \times n$ matrix polynomials (over an arbitrary infinite field), whereas [27, Thm. 5.2] solves the QRP for the smaller class $\mathscr{C}$ of all $m \times n$ quadratic polynomials of full rank. However, the solutions of (SP-2) provided in [13] and [27] are neither simple, transparent, nor canonical.

The main part of the present work provides a complete and transparent solution to the QRP for the class $\mathscr{C}$ of T-palindromic matrix polynomials, where $\mathbb{F}$ is an algebraically closed field. This is achieved by developing a Kronecker-like *quasi-canonical form* for quadratic T-palindromic matrix polynomials. By "Kronecker-like" we mean a matrix polynomial that is built up as a *direct sum of canonical blocks*, each of which realizes the structural data corresponding to a certain (small) portion of the given list $\mathcal{L}$, in the same kind of transparent way that the blocks in the Kronecker canonical form reveal the elementary divisors and minimal indices of matrix pencils [22, Ch. XII, §4]. Note that these "direct-sum-of-structured-blocks" constructions of quadratic matrix polynomials are very much in the spirit of the solutions of the inverse problems for palindromic and even matrix *pencils* found in [43, 44, 45]. The phrase "quasi-canonical" here refers to the lack of uniqueness that can sometimes occur in this realization. It is possible for two essentially different direct sums of quadratic Kronecker-like blocks (i.e., direct sums *not* related by mere permutation of blocks) to have the same structural data, thus showing that our quadratic realizations are not always unique.

[1] By "canonical" we mean that the realization is unique for a fixed list $\mathcal{L}$.

It is worth noting that the work in this paper is closely connected to, and indeed has grown out of, several other ongoing QRP projects [10, 30, 38]. In particular, [30] provides an analogous but simpler Kronecker-like quasi-canonical form for (unstructured) *regular* quadratic matrix polynomials, which is extended in [10] to the general case of all quadratic matrix polynomials (regular and singular). In [38], a Kronecker-like quasi-canonical form for Hermitian quadratic matrix polynomials is obtained.

Our interest in T-palindromic quadratic matrix polynomials is twofold. From the theoretical standpoint, it is desirable to have a systematic in-depth study of the spectral and singular structure of T-palindromic quadratic polynomials. From the practical viewpoint, though, T-palindromic quadratic polynomials can also play a role in obtaining solutions to structured polynomial eigenvalue problems of higher degree [25]. Recall that for a given matrix polynomial $P(\lambda)$ the standard way to solve the associated polynomial eigenvalue problem, or more generally to compute the complete spectral and singular structure of $P(\lambda)$, is by means of *strong linearizations*, i.e., matrix pencils that have the same finite and infinite elementary divisors, and the same number of left and right minimal indices as P. The most commonly used strong linearizations are the Frobenius companion forms [11, 23], though many other examples of strong linearizations have also been studied, e.g., Fiedler-like linearizations [3, 4, 8], linearizations from the ansatz spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ [32], and the recently introduced block minimal basis pencils [16]. With a strong linearization in hand one can employ existing numerical methods for computing the structural data of matrix pencils [14, 15], and thus determine the structural data of the underlying matrix polynomial.

When a matrix polynomial $P(\lambda)$ has additional algebraic structure, its spectral and singular structures also enjoy certain symmetries – see for example [7], [33, Table 2.2], or more specifically Remark 2.6 for the T-palindromic structure considered in this paper. When solving a *structured* polynomial eigenvalue problem it is desirable to employ a strong linearization with the *same algebraic structure*. Using a structure-preserving algorithm on a structured linearization then ensures that the *computed* eigenvalues have the same symmetry as the exact eigenvalues of the underlying structured matrix polynomial. This is important from a practical standpoint, since the symmetry in the exact spectral and singular data of a structured matrix polynomial can often be traced back to an intrinsic property of the underlying physical problem modeled by such a polynomial. In the early 2000's, the fact that most of the commonly used linearizations did not preserve any algebraic structure of a matrix polynomial inspired an intensive search for structured linearizations, in particular for T-alternating and T-palindromic polynomials [5, 9, 33, 34].

Despite success in identifying structured linearizations in many cases, others remained more elusive. In [35] the authors showed that *all* T-palindromic matrix polynomials of odd degree have a T-palindromic strong linearization, but that there exist T-palindromic matrix polynomials of *even* degree for which it is impossible to find a structure-preserving linearization. The existence of such problematic even-degree T-palindromic polynomials $W(\lambda)$ suggested a new structure-preserving approach: look instead for a *strong quadratification* of $W(\lambda)$, i.e., a quadratic T-palindromic matrix polynomial $Q(\lambda)$ with the same finite and infinite elementary divisors and the same number of left and right minimal indices as $W(\lambda)$. A family of T-palindromic strong quadratifications for all even degree T-palindromic matrix polynomials was presented in [25, Thm. 3][2], demonstrating an important existence result. The development in this paper also gives that result (see Corollary 6.3), and much more. For example, our in-depth analysis allows the characterization of all *odd* degree T-palindromic matrix polynomials that have a T-palindromic strong quadratification (see Corollary 6.4).

We begin the development in Section 2 by establishing notation and reviewing basic concepts about matrix polynomials that are relevant to this paper, including the already known spectral and singular properties of T-palindromic matrix polynomials. Section 3 introduces a number of new concepts that are useful in addressing the T-palindromic QRP, establishes the properties of these concepts, and then uses them to lay out the overall strategy for our solution of the T-palindromic QRP. Via a series of technical lemmas, Section 4 then carries out the first step of this strategy, showing how to partition lists of structural data into shorter and more tractable *irreducible* sublists. In Section 5 we show how each of these irreducible sublists can be concretely realized by a canonical quadratic T-palindromic block, which transparently reveals

[2] In [25] the authors use a different notion of quadratification, though in the end the constructed family does consist of strong quadratifications in the sense of our Definition 6.2.

the structural data of that sublist. The results of Sections 3, 4, and 5 are then combined in Section 6 to give the complete solution of the T-palindromic QRP problem (Theorem 6.1), in a neat and concise fashion. Section 6 also derives some consequences of Theorem 6.1 for the existence of T-palindromic quadratifications of arbitrary T-palindromic matrix polynomials. In Section 7, Möbius transformations are used to leverage the solution of the T-palindromic QRP into a solution of the QRP for T-even matrix polynomials, including results about the existence of T-even quadratifications. Finally, Section 8 summarizes the main contributions of this paper and their significance, and also discusses the prospects for extending these results (and this argument) to higher degree matrix polynomials.

2. Background

In this section we introduce the notation and all necessary background to be used throughout the paper. Most of these notions can be found in recent papers [11, 35] or in the classical monographs [22, 23]; we include them here for the sake of completeness, as well as to establish a unified notation and terminology.

We use $\mathbb{F}$ to denote an arbitrary field and $\overline{\mathbb{F}}$ its algebraic closure. The set of polynomials in the variable λ with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[\lambda]$, while $\mathbb{F}(\lambda)$ denotes the field of fractions of $\mathbb{F}[\lambda]$ (i.e., $\mathbb{F}(\lambda)$ is the field of rational functions over $\mathbb{F}$). An $m \times n$ matrix polynomial $P(\lambda)$ over $\mathbb{F}$ is of the form

$$P(\lambda) \;=\; \sum_{i=0}^{k} A_i \lambda^i \,, \tag{2.1}$$

where $A_i \in \mathbb{F}^{m\times n}$ for $i = 0, 1, \ldots, k$. For the sake of brevity, in many cases when referring to a matrix polynomial $P(\lambda)$ we drop the dependence on λ and simply write P.

A matrix polynomial $P(\lambda)$ as in (2.1) is said to have *grade* k, which we denote by grade(P). The *degree* of P, denoted by $\deg(P)$, is the largest integer j such that coefficient of λ^j in $P(\lambda)$ is nonzero. When $A_k \neq 0$, the degree and grade are equal, otherwise grade is strictly larger than the degree. Even though the classical references on matrix polynomials only consider the notion of degree, several recent papers [11, 37, 41] show multiple advantages of working with the grade of a matrix polynomial instead of its degree.

A matrix polynomial $P(\lambda)$ from (2.1) is said to be *regular* if $P(\lambda)$ is square (i.e., $A_i \in \mathbb{F}^{n\times n}$) and its determinant is non-identically zero (i.e., the scalar polynomial $\det(P)$ has at least one nonzero coefficient); otherwise $P(\lambda)$ is said to be *singular*. Equivalently, $P(\lambda)$ is singular when at least one of the vector spaces over the field $\mathbb{F}(\lambda)$

$$\begin{aligned} \mathcal{N}_r(P) \;&:=\; \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0_{m\times 1} \right\}, \\ \mathcal{N}_\ell(P) \;&:=\; \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times m} \,:\, y(\lambda)^T P(\lambda) \equiv 0_{1\times n} \right\}, \end{aligned}$$

is nontrivial. The spaces $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$ are referred to as the *right* and *left nullspaces* of $P(\lambda)$, respectively.

A matrix polynomial $P(\lambda)$ is said to be *unimodular* if $\det(P)$ is a nonzero constant in $\mathbb{F}$. In other words, a unimodular matrix polynomial is an invertible matrix polynomial whose inverse (over the field $\mathbb{F}(\lambda)$) is again a matrix polynomial. Two matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are *unimodularly equivalent*, denoted by $P(\lambda) \sim Q(\lambda)$, if there are two unimodular matrix polynomials $U(\lambda), V(\lambda)$ such that $U(\lambda)P(\lambda)V(\lambda) = Q(\lambda)$. The *rank* of $P(\lambda)$ is the size of the largest non-identically zero minor of $P(\lambda)$, and is denoted by rank P; this notion is sometimes also referred to as the *normal rank* [7, 8] of P.

Definition 2.1. (j-reversal, [35, Def. 3.3]) *Let P be a nonzero matrix polynomial of degree d. For any $j \geq d$, the* j-reversal *of P is the matrix polynomial* $\mathrm{rev}_j P$ *given by* $(\mathrm{rev}_j P)(\lambda) \;:=\; \lambda^j P(1/\lambda)$. *In the special case when $j = d$, i.e., when degree and grade are equal, the j-reversal of P is called the* reversal *of P and is denoted by* $\mathrm{rev}P$.

Theorem 2.2. (Smith form, [21]) *Let $P(\lambda)$ be an $m \times n$ matrix polynomial with $r = \mathrm{rank}\, P$. Then $P(\lambda)$ is unimodularly equivalent to*

$$D(\lambda)_{m\times n} \;:=\; \mathrm{diag}\left(d_1(\lambda), \ldots, d_{\min\{m,n\}}(\lambda)\right), \tag{2.2}$$

where

(i) $d_1(\lambda), \ldots, d_r(\lambda)$ *are monic scalar polynomials* (*i.e., with leading coefficient equal to* 1),

(ii) $d_{r+1}(\lambda), \ldots, d_{\min\{m,n\}}(\lambda)$ *are identically zero scalar polynomials,*

(iii) $d_1(\lambda), \ldots, d_r(\lambda)$ *form a* divisibility chain, *i.e.,* $d_j(\lambda)$ *is a divisor of* $d_{j+1}(\lambda)$, *for* $j = 1, \ldots, r-1$,

(iv) *the polynomials* $d_1(\lambda), d_2(\lambda), \ldots, d_r(\lambda)$ *are uniquely determined by the multiplicative relations*

$$d_1(\lambda)d_2(\lambda)\cdots d_j(\lambda) \;=\; \gcd\{\, \textit{all } j\times j \textit{ minors of } P(\lambda)\,\}\,, \quad \textit{for}\;\; j = 1, \ldots, r\,.$$

The diagonal matrix $D(\lambda)$ *in* (2.2) *is thus unique, and is known as the* Smith form *of* $P(\lambda)$.

The nonzero diagonal elements $d_j(\lambda)$ for $j = 1, \ldots, r$ in the Smith form of P are called the *invariant factors* or *invariant polynomials* of P. The roots $\lambda_0 \in \overline{\mathbb{F}}$ of the product $d_1(\lambda)\cdots d_r(\lambda)$ in (2.2) are the (finite) *eigenvalues* of P. We say that $\lambda_0 = \infty$ is an eigenvalue of P whenever 0 is an eigenvalue of $\mathrm{rev}_j P$. Note that this definition depends on the choice of the grade j. When P is viewed as having grade j equal to $\deg P$, then $\lambda_0 = \infty$ may or may not be an eigenvalue of P, while if P is viewed as having some grade j strictly larger than the degree, then $\lambda_0 = \infty$ will necessarily be an eigenvalue of P [11, Rem. 6.6].

Definition 2.3. (Partial multiplicities). *Let* $P(\lambda)$ *be an* $m \times n$ *matrix polynomial of grade* k *over a field* $\mathbb{F}$.

(i) (Finite partial multiplicities). *For* $\lambda_0 \in \overline{\mathbb{F}}$, *we can factor the invariant polynomials* $d_i(\lambda)$ *of* P *for* $1 \le i \le r$ *as*

$$d_i(\lambda) = (\lambda - \lambda_0)^{\alpha_i} q_i(\lambda)\,, \quad \textit{with}\;\; \alpha_i \ge 0 \;\; \textit{and}\;\; q_i(\lambda_0) \ne 0.$$

By the divisibility chain property of the Smith form, the sequence of exponents $(\alpha_1, \ldots, \alpha_r)$ *satisfies the condition* $0 \le \alpha_1 \le \ldots \le \alpha_r$, *and is called the* partial multiplicity sequence of P at λ_0.

(ii) (Infinite partial multiplicities). *The* infinite partial multiplicity sequence of P *is the partial multiplicity sequence of* $\mathrm{rev}_k P$ *at* 0.

A *vector polynomial* is a vector whose entries are polynomials in the variable λ. For any subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$ it is always possible to find a basis consisting entirely of vector polynomials (simply take an arbitrary basis and multiply each vector by the denominators of its entries). The *degree* of a vector polynomial is the greatest degree of its components, and the *order* of a polynomial basis is defined as the sum of the degrees of its vectors [20, p. 494]. A *minimal basis* of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$. It can be shown that for any given subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These uniquely defined degrees are then called the *minimal indices* of $\mathcal{V}$ [20].

The following definition, which follows [13, Def. 2.17], introduces the most relevant quantities that appear in the classification of "realizable" T-palindromic quadratic matrix polynomials.

Definition 2.4. (Structural data of a matrix polynomial). *Let* $P(\lambda)$ *be an* $m \times n$ *matrix polynomial with grade* k *over a field* $\mathbb{F}$.

(i) (Spectral structure). *The* (finite) elementary divisors *of* P *are the collection of non-trivial irreducible factors* (*with their corresponding exponents*) *of the invariant polynomials of* P, *including repetitions. In particular, the* elementary divisors at a finite eigenvalue $\lambda_0 \in \overline{\mathbb{F}}$ *are the collection of factors* $(\lambda - \lambda_0)^{\alpha_i}$ *of the invariant polynomials, with* $\alpha_i > 0$. *The* elementary divisor chain *at a finite eigenvalue* $\lambda_0 \in \overline{\mathbb{F}}$ *is the list* $((\lambda - \lambda_0)^{\alpha_{r-g+1}}, \ldots, (\lambda - \lambda_0)^{\alpha_r})$ *of elementary divisors at* λ_0, *where* $\alpha_1 = \ldots = \alpha_{r-g} = 0$ *and* $0 < \alpha_{r-g+1} \le \ldots \le \alpha_r$.

The infinite elementary divisors *of* P *correspond to the elementary divisors at* 0 *of* $\mathrm{rev}_k P$. *More specifically, if* $\lambda^{\beta_1}, \ldots, \lambda^{\beta_\ell}$ *with* $0 < \beta_1 \le \cdots \le \beta_\ell$ *are the elementary divisors at* 0 *for* $\mathrm{rev}_k P$, *then* P *has* ℓ *corresponding elementary divisors at* ∞, *denoted* $\omega^{\beta_1}, \ldots, \omega^{\beta_\ell}$. *The list* $\omega^{\beta_1}, \ldots, \omega^{\beta_\ell}$ *is also referred to as the* infinite elementary divisor chain *of* P.

The finite and infinite elementary divisors together comprise the spectral structure *of* P.

(ii) (Singular structure). *The* left *and* right minimal indices *of P are the minimal indices of $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P)$, respectively, and together comprise the* singular structure of P.

(iii) (Structural data). *The* structural data *of P consists of the elementary divisors* (spectral structure) *of P, together with the left and right minimal indices* (singular structure) *of P.*

There are several observations worth highlighting about Definition 2.4(i). More specifically, the quantity g is the *geometric multiplicity* of the eigenvalue λ_0 of P, that is, the number of elementary divisors of $P(\lambda)$ at λ_0. Further, since the definition of infinite elementary divisors depends on P having a specified grade k, we indicate that by referring to the *infinite elementary divisors of grade k*. Finally, to avoid any possible confusion between the elementary divisors at zero and those at ∞, we will be denoting the latter ones with the special notation ω^β.

2.1. Spectral and singular structure of T-palindromic matrix polynomials

In this section we recall some well-known results about T-palindromic matrix polynomials that are needed throughout the rest of this paper. We begin with the definition of this type of structured polynomial.

Definition 2.5. [33, Table 2.1] (T-palindromic). *A nonzero $n \times n$ matrix polynomial P of degree $k \geq 0$ is said to be* T-palindromic *if* $(\mathrm{rev}_j P)(\lambda) = P^T(\lambda)$*, for some integer j with $j \geq k$.*

Before we continue, it is worth mentioning that some references have also included under the name "T-palindromic polynomials" those $P(\lambda)$ satisfying the condition $(\mathrm{rev}_j P)(\lambda) = -P^T(\lambda)$ [35]. More recently, such matrix polynomials are referred to as T-anti-palindromic [37], and are not studied in this paper.

There are two important observations regarding Definition 2.5. First, matrix polynomials that are T-palindromic must be square, and second, the T-palindromicity is defined "with respect to grade." For instance, the degree–two scalar polynomial $p(\lambda) = \lambda^2 + \lambda$ is T-palindromic with respect to grade 3, since $(\mathrm{rev}_3\, p)(\lambda) = p(\lambda)$. However, $(\mathrm{rev}_2\, p)(\lambda) \neq p(\lambda)$, and so $p(\lambda)$ is *not* T-palindromic with respect to its degree. This important fact has been already observed in [35], where the authors proved that if a degree k polynomial P is T-palindromic, then there is exactly one $j \geq k$ such that $\mathrm{rev}_j P = P^T$ [35, Prop. 4.3]. This j is known as the *grade of palindromicity of P*.

In this paper, we adopt a convention that when we refer to a T-palindromic matrix polynomial P with grade k, we are considering k to be its unique grade of palindromicity. Again, this grade is intrinsic to P, it is not a choice, and is not necessarily the same as the degree of P.

We finish this section by recalling some important facts from [35] and [7] about the spectral and singular structure of T-palindromic matrix polynomials that are relevant to our work in this paper.

Remark 2.6. *Let $P(\lambda)$ be a matrix polynomial over $\mathbb{F}$, with* $\mathrm{char}(\mathbb{F}) \neq 2$*, and assume that P is T-palindromic with grade of palindromicity k. Then the following statements are true*:

(i) *If $p(\lambda) = (\lambda+1)^\alpha(\lambda-1)^\beta q(\lambda)$, with $q(1) \neq 0 \neq q(-1)$, is any invariant polynomial of $P(\lambda)$, then $q(\lambda)$ is palindromic* [35, Thm. 7.6].

(ii) *If k is even, then any odd degree elementary divisor of $P(\lambda)$ associated with either of the eigenvalues $\lambda_0 = \pm 1$ has even multiplicity* [35, Cor. 8.2].

(iii) *For any $\beta \geq 1$, the elementary divisors λ^β and ω^β have the same multiplicity* (*i.e., they appear the same number of times*) [35, Cor. 8.1].

(iv) *The left and right minimal indices of $P(\lambda)$ coincide. Namely, if $\eta_1 \geq \eta_2 \geq \ldots \geq \eta_q$ and $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq \varepsilon_p$ are the left and right minimal indices of $P(\lambda)$, respectively, then $p = q$ and $\eta_i = \varepsilon_i$, for $i = 1, \ldots, p$* [7, Thm. 3.6].

When $\mathbb{F}$ is an algebraically closed field with $\mathrm{char}\,\mathbb{F} \neq 2$*, the scalar palindromic polynomial $q(\lambda)$ in* (*i*) *can be factored as $q(\lambda) = \lambda^\nu \prod_{i=1}^m (\lambda - a_i)(\lambda - a_i^{-1})$* [35, Cor. 5.9]. *Thus, for an algebraically closed field $\mathbb{F}$, the finite elementary divisors of $P(\lambda)$ associated with eigenvalues $a \neq 0, \pm 1$ are paired in the form $(\lambda - a)^\beta, (\lambda - \frac{1}{a})^\beta$.*

3. Solution strategy for the T-palindromic QRP

In this section, we lay out the whole strategy for the solution of the T-Palindromic QRP. Though part of the content from the previous sections is valid for arbitrary fields, from now on we assume that $\mathbb{F}$ is an *algebraically closed field* with $\operatorname{char}\mathbb{F} \neq 2$. The case when $\mathbb{F} = \mathbb{R}$ has been considered in [42, Ch. 9], whereas the case of other non-algebraically closed fields is a subject for future investigation.

We start by introducing some basic concepts about lists of elementary divisors and minimal indices.

Definition 3.1. (Lists of elementary divisors and minimal indices).

(i) *A* list of finite elementary divisors *is a list of the form*

$$\mathcal{L}_{fin} = \Big\{ (\lambda - a_1)^{\alpha_{1,1}}, \dots, (\lambda - a_1)^{\alpha_{1,g_1}}, \dots, (\lambda - a_s)^{\alpha_{s,1}}, \dots, (\lambda - a_s)^{\alpha_{s,g_s}} \Big\},$$

where $a_1, \dots, a_s \in \mathbb{F}$, with $a_i \neq a_j$ for $i \neq j$, and the $\alpha_{i,j}$'s are positive *integers.*

(ii) *An* elementary divisor chain of length g *associated with $a \in \mathbb{F}$ is a list of the form*

$$\Big((\lambda - a)^{\alpha_1}, \dots, (\lambda - a)^{\alpha_g} \Big),$$

with $0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_g$.

(iii) *An* elementary divisor chain of length g *associated with $\lambda_0 = \infty$ is of the form* $\mathcal{L}_\infty = \Big(\omega^{\beta_1}, \dots, \omega^{\beta_g} \Big)$, *with $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_g$.*

(iv) *A* list $\mathcal{L}$ of elementary divisors and minimal indices *is of the form*

$$\mathcal{L} = \Big\{ \mathcal{L}_{fin}\,;\, \mathcal{L}_\infty\,;\, \mathcal{L}_{left}\,;\, \mathcal{L}_{right} \Big\}, \tag{3.1}$$

where $\mathcal{L}_{fin}$ is a list of finite elementary divisors, $\mathcal{L}_\infty$ is an elementary divisor chain associated with ∞, and $\mathcal{L}_{left} = \{\eta_1, \dots, \eta_q\}$ and $\mathcal{L}_{right} = \{\varepsilon_1, \dots, \varepsilon_p\}$ are lists of nonnegative integers.

Definition 3.2. (Partial multiplicities). *The exponents $\alpha_{i,1}, \dots, \alpha_{i,g_i}$ corresponding to all of the elementary divisors $(\lambda - a_i)^{\alpha_{i,1}}, \dots, (\lambda - a_i)^{\alpha_{i,g_i}}$ associated with a certain $a_i \in \mathbb{F}$ in $\mathcal{L}_{fin}$ are called the (nonzero)* partial multiplicities of a_i in $\mathcal{L}$. *Collectively for all $a_i \in \mathbb{F}$, they are the* finite partial multiplicities of $\mathcal{L}$. *Similarly, the exponents $\beta_1, \dots, \beta_g$ corresponding to the exponents in the list $\mathcal{L}_\infty$ are the (nonzero)* infinite partial multiplicities of $\mathcal{L}$.

Throughout this paper we say that $\mathcal{L}$ is the list of elementary divisors and minimal indices of a given matrix polynomial P if the elementary divisors and minimal indices of P are precisely those in $\mathcal{L}$. When necessary for emphasis, we denote this list by $\mathcal{L}(P)$.

The following definition introduces some key quantities associated with a list $\mathcal{L}$ of elementary divisors and minimal indices, that will appear throughout the entire paper.

Definition 3.3. *Let $\mathcal{L}$ be a list of elementary divisors and minimal indices as in (3.1).*

(i) *The* total finite degree *and the* total infinite degree *of $\mathcal{L}$, denoted by $\delta_{fin}(\mathcal{L})$ and $\delta_\infty(\mathcal{L})$, respectively, are defined by*

$$\delta_{fin}(\mathcal{L}) := \sum_{i=1}^{s} \sum_{j=1}^{g_i} \alpha_{i,j}\,, \qquad \text{and} \qquad \delta_\infty(\mathcal{L}) := \beta_1 + \cdots + \beta_g\,,$$

where $\alpha_{i,1}, \dots, \alpha_{i,g_i}$, for $i = 1, \dots, s$, are the (nonzero) finite partial multiplicities of $\mathcal{L}$, and $\beta_1, \dots, \beta_g$ are the (nonzero) infinite partial multiplicities of $\mathcal{L}$.

(ii) *The* total degree of $\mathcal{L}$ *is the number given by $\delta(\mathcal{L}) := \delta_{fin}(\mathcal{L}) + \delta_\infty(\mathcal{L})$.*

(iii) *The* sum of all minimal indices of $\mathcal{L}$ *is the number given by* $\mu(\mathcal{L}) := \sum_{i=1}^{p} \varepsilon_i + \sum_{j=1}^{q} \eta_j$.

(iv) *The* length of the longest elementary divisor chain in $\mathcal{L}$ *is denoted by* $\gamma(\mathcal{L})$.

For simplicity, when there is no risk of confusion about which list $\mathcal{L}$ is under consideration, we adopt the convention that the quantities (ii)–(iv) from Definition 3.3 will be denoted by δ, μ, and γ, respectively. Also, note that if $\mathcal{L} = \mathcal{L}(P)$ for some matrix polynomial P, then $\gamma(\mathcal{L})$ is the largest geometric multiplicity of any finite or infinite eigenvalue of P.

There is an elementary relationship between the quantities $\delta(\mathcal{L}), \mu(\mathcal{L})$, $\text{grade}(P)$, and $\text{rank}(P)$, for any matrix polynomial $P(\lambda)$ whose list of elementary divisors and minimal indices is $\mathcal{L}$; this fundamental relationship is known as the *Index Sum Theorem*.

Theorem 3.4. [11, Thm. 6.5] (Index Sum Theorem). *Let $P(\lambda)$ be an arbitrary matrix polynomial over an arbitrary field, and let $\mathcal{L}$ denote the list of elementary divisors and minimal indices of P. Then:*

$$\delta(\mathcal{L}) \,+\, \mu(\mathcal{L}) \;=\; \text{grade}(P) \cdot \text{rank}\, P. \tag{3.2}$$

Now that we have concretely established notions of a list of elementary divisors and minimal indices and related quantities, we are ready to discuss the main topic of this paper. Recall that by "solving the QRP" we mean the following:

> *Given a list $\mathcal{L}$ of elementary divisors and minimal indices, determine whether or not there exists a quadratic matrix polynomial whose elementary divisors and minimal indices are precisely the ones in $\mathcal{L}$. In the affirmative case, construct such a matrix polynomial.*

Note that if Q is a quadratic matrix polynomial whose elementary divisors and minimal indices are that of $\mathcal{L}$, then we say that *"Q is a (quadratic) realization of $\mathcal{L}$"* or that *"Q realizes $\mathcal{L}$."*

In this paper (up to Section 7) we solve the T-palindromic QRP, which means that the quadratic polynomial Q that realizes $\mathcal{L}$ is to be T-palindromic. Moreover, we show how to construct a *quasi-canonical* quadratic realization for each realizable list $\mathcal{L}$, consisting of the direct sum of canonical quadratic T-palindromic blocks, each associated to simple combinations of elementary divisors and minimal indices elements in the list. The first natural notion that we will need is the following.

Definition 3.5. (p-quad Realizability). *A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be* p-quad realizable over the field $\mathbb{F}$ *if there exists some T-palindromic quadratic matrix polynomial over $\mathbb{F}$, with grade of palindromicity* 2*, whose elementary divisors and minimal indices are exactly the ones in $\mathcal{L}$.*

Based on the properties of the spectral and singular structure of T-palindromic matrix polynomials described in Section 2.1, we introduce the following concept.

Definition 3.6. (p-quad Symmetry). *A list $\mathcal{L}$ of elementary divisors and minimal indices over an algebraically closed field $\mathbb{F}$ with* $\text{char}\,\mathbb{F} \neq 2$ *is said to have* p-quad symmetry *if the following conditions are satisfied:*

(1) (a) *For any $a \neq 0, \pm 1$ and any $\beta \geq 1$, the elementary divisor $(\lambda - a)^\beta$ appears in $\mathcal{L}$ with the same multiplicity as $(\lambda - \frac{1}{a})^\beta$ (i.e., they appear exactly the same number of times, perhaps zero).*

(b) *For any $\beta \geq 1$, the elementary divisors λ^β and ω^β appear in $\mathcal{L}$ with the same multiplicity.*

(c) *Any odd degree elementary divisor in $\mathcal{L}$ associated with eigenvalue $a = \pm 1$ has even multiplicity.*

(2) *The ordered sublists $\mathcal{L}_{left}$ and $\mathcal{L}_{right}$ of left and right minimal indices are identical.*

The notion in Definition 3.7 regarding lists of elementary divisors and minimal indices plays a central role in this work. As we will see in Theorem 6.1, it comprises the necessary and sufficient conditions for a list of elementary divisors and minimal indices to be p-quad realizable.

Definition 3.7. (p-quad Admissibility). *A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be* p-quad admissible *if the following conditions are satisfied:*

(a) $\gamma \leq \frac{1}{2}\big(\delta + \mu\big)$,

(b) *$\mathcal{L}$ has p-quad symmetry.*

Remark 3.8. *A consequence of the p-quad symmetry in* Definition 3.7(b) *is that δ and μ are both even, and hence*

(c) *$\delta + \mu$ is even.*

It has been already observed that conditions (a) *and* (c) *play a key role in the unstructured* quadratic *realizability problem* [10, 30]*, whereas condition* (b) *is needed here only to accommodate the additional T-palindromic structure. Moreover, the slightly modified conditions* (a) *and* (c)*, namely*

(a') $\gamma \leq \frac{1}{d}\,(\delta + \mu)$,

(c') *$\delta + \mu$ is a multiple of d,*

are in fact the necessary and sufficient conditions for a list of elementary divisors and minimal indices to be realizable by a matrix polynomial of grade d [13].

It is important to observe that Definition 3.7 imposes all of the previously known necessary conditions for a list $\mathcal{L}$ to be p-quad realizable. In other words, any p-quad realizable list is p-quad admissible. In particular, condition (a) in Definition 3.7 is a consequence of Theorem 3.4, while the condition (b) for p-quad symmetry comes from [35, Cor. 8.1–8.2] for the elementary divisors and [7, Thm. 3.6] for the minimal indices. In terms of Definitions 3.5 and 3.7, the main result of this paper states that a list $\mathcal{L}$ is p-quad realizable if and only if $\mathcal{L}$ is p-quad admissible (c.f., Theorem 6.1).

Given two lists $\mathcal{L} = \{\mathcal{L}_{fin}; \mathcal{L}_\infty; \mathcal{L}_{left}; \mathcal{L}_{right}\}$ and $\widehat{\mathcal{L}} = \Big\{\widehat{\mathcal{L}}_{fin}; \widehat{\mathcal{L}}_\infty; \widehat{\mathcal{L}}_{left}; \widehat{\mathcal{L}}_{right}\Big\}$ of elementary divisors and minimal indices, the *concatenation* of $\mathcal{L}$ and $\widehat{\mathcal{L}}$, denoted by $c(\mathcal{L}, \widehat{\mathcal{L}})$, is the list of elementary divisors and minimal indices

$$c\big(\mathcal{L}, \widehat{\mathcal{L}}\big) := \Big\{\{\mathcal{L}_{fin}, \widehat{\mathcal{L}}_{fin}\}; \{\mathcal{L}_\infty, \widehat{\mathcal{L}}_\infty\}; \{\mathcal{L}_{left}, \widehat{\mathcal{L}}_{left}\}; \{\mathcal{L}_{right}, \widehat{\mathcal{L}}_{right}\}\Big\}\,, \tag{3.3}$$

obtained by simply adjoining the corresponding lists, retaining all repetitions. The extension of this notion to any finite number of lists is immediate.

The following result can be obtained by direct verification of the conditions in Definition 3.7.

Lemma 3.9. *The concatenation of any finite number of p-quad admissible lists is also a p-quad admissible list.*

Note that the key fact one needs to prove Lemma 3.9 is that the length of the longest chain of elementary divisors γ is subadditive under concatenation of lists, namely $\gamma\Big(c\big(\mathcal{L}, \widehat{\mathcal{L}}\big)\Big) \leq \gamma(\mathcal{L}) + \gamma(\widehat{\mathcal{L}})$.

As a consequence of Lemma 3.9, we can now build new p-quad admissible lists from other p-quad admissible lists simply by concatenation. A more interesting question is whether we can take a p-quad admissible list and split it into smaller p-quad admissible lists. To be more precise, a *partition* of a list of elementary divisors and minimal indices $\mathcal{L} = \{\mathcal{L}_{fin}; \mathcal{L}_\infty; \mathcal{L}_{left}; \mathcal{L}_{right}\}$ consists of m lists $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_m$, with $m > 1$, satisfying

$$\begin{aligned} \mathcal{L}_{fin} &= \big\{(\mathcal{L}_1)_{fin}, \cdots, (\mathcal{L}_m)_{fin}\big\}, & \mathcal{L}_\infty &= \big\{(\mathcal{L}_1)_\infty, \cdots, (\mathcal{L}_m)_\infty\big\}, \\ \mathcal{L}_{right} &= \big\{(\mathcal{L}_1)_{right}, \cdots, (\mathcal{L}_m)_{right}\big\}, & \mathcal{L}_{left} &= \big\{(\mathcal{L}_1)_{left}, \cdots, (\mathcal{L}_m)_{left}\big\}, \end{aligned}$$

where some of these lists may be empty. If at least two of these lists $\mathcal{L}_1, \ldots, \mathcal{L}_m$ are non-empty, then the partition is said to be *nontrivial*. This leads to the following notion.

Definition 3.10. (p-quad Irreducibility). *A list $\mathcal{L}$ is* p-quad irreducible *if it is p-quad admissible, and there is no nontrivial partition of $\mathcal{L}$ into p-quad admissible sublists.*

Note that because of Lemma 3.9, when checking whether a list $\mathcal{L}$ is p-quad irreducible it suffices to consider partitions into two nonempty sublists.

One of the main contributions of this section is isolating the notion of a p-quad irreducible list, and giving a *complete set* of all the p-quad irreducible lists in Tables 1 and 2. In the next section we show that all p-quad admissible lists can be built as a concatenation of copies of the p-quad irreducible lists in Tables 1-2 (c.f., Theorem 3.16).

Type	Subtype	Elementary Divisors/Minimal Indices	Conditions
$\boxed{\mathcal{X}}$	$\mathcal{X}_1$	$(\lambda-a)^m, (\lambda-\frac{1}{a})^m$	$m \geq 1, a \neq 0, \pm 1$
	$\mathcal{X}_2$	λ^m, ω^m	$m \geq 1$
$\boxed{\mathcal{Y}}$	$\mathcal{Y}_1$	$(\lambda-1)^{2m}$	$m \geq 1$
	$\mathcal{Y}_1'$	$(\lambda+1)^{2m}$	$m \geq 1$
	$\mathcal{Y}_2$	$(\lambda-1)^{2m+3}, (\lambda-1)^{2m+3}$	$m \geq 0$
	$\mathcal{Y}_2'$	$(\lambda+1)^{2m+3}, (\lambda+1)^{2m+3}$	$m \geq 0$
$\boxed{\mathcal{S}}$	$\mathcal{S}_1$	$\varepsilon = 2k\,,\, \eta = 2k$	$k \geq 0$
	$\mathcal{S}_2$	$\varepsilon = 2k+1\,,\, \eta = 2k+1$	$k \geq 0$

Table 1: The irreducible NoDO lists

Remark 3.11. *Note that list $\widetilde{\mathcal{C}}_1$ is a particular case of both $\mathcal{C}_1$ and $\mathcal{C}_1'$ lists (when $m = n = 1$) and, moreover, is the only overlap of these two kind of lists. Even though this introduces a redundancy in* Table 2*, we have isolated this particular case because it makes the beginning of the proof of* Theorem 3.16 *much cleaner.*

Remark 3.12. *It is interesting to observe the relationship between the primed and the unprimed lists in* Tables 1 and 2*. Each primed list can, at least symbolically, be obtained from its unprimed counterpart by simply interchanging the roles of $(\lambda-1)$ with $(\lambda+1)$. Note that the only list for which such an interchange does not affect its elementary divisor structure is $\widetilde{\mathcal{C}}_1$, hence there is no gain in considering its primed counterpart. Due to this duality between $(\lambda-1)$ and $(\lambda+1)$, we will design T-palindromic quadratic matrix polynomials (blocks) that realize each of the unprimed lists in such a way that when the roles of $(\lambda-1)$ and $(\lambda+1)$ are interchanged, the new blocks become T-palindromic quadratic realizations for the primed counterparts of each of the lists. To get a simple matching between the blocks that realize the unprimed and the corresponding primed lists, we will simply replace λ by $-\lambda$ (additional details are given in* Tables 3–6 *and* Lemma 5.10*).*

Proposition 3.13. *Each list in* Table 1 *and* Table 2 *is p-quad irreducible.*

Proof. The proof is by direct verification for each list in the Tables, checking first that the given list is p-quad admissible, and second that any partition of it into two sublists violates at least one of the conditions in Definition 3.7. □

From Definition 3.10 we see that the simplest p-quad admissible lists are the p-quad irreducible ones. It now follows from Proposition 3.13 that many examples of such irreducible lists can be found in Tables 1 and 2. The following definition considers p-quad admissible lists that can be partitioned into p-quad irreducible lists from these tables.

Type	Subtype	Elementary Divisors/Minimal Indices	Conditions
$\boxed{\mathcal{A}}$	$\mathcal{A}_1$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, (\lambda-a)^n, (\lambda-\frac{1}{a})^n$	$n\geq m>0, a\neq 0,\pm 1$
	$\mathcal{A}_1'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, (\lambda-a)^n, (\lambda-\frac{1}{a})^n$	$n\geq m>0, a\neq 0,\pm 1$
	$\mathcal{A}_2$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, \lambda^n, \omega^n$	$n\geq m>0$
	$\mathcal{A}_2'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, \lambda^n, \omega^n$	$n\geq m>0$
$\boxed{\mathcal{B}}$	$\mathcal{B}_1$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, (\lambda+1)^{2n}$	$n\geq m>0$
	$\mathcal{B}_1'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, (\lambda-1)^{2n}$	$n\geq m>0$
	$\mathcal{B}_2$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, (\lambda-1)^{2n}$	$n>m>0$
	$\mathcal{B}_2'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, (\lambda+1)^{2n}$	$n>m>0$
$\boxed{\mathcal{C}}$	$\mathcal{C}_1$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, (\lambda+1)^n, (\lambda+1)^n$	n odd, $0<m\leq n$
	$\mathcal{C}_1'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, (\lambda-1)^n, (\lambda-1)^n$	n odd, $0<m\leq n$
	$\widetilde{\mathcal{C}}_1$	$\lambda-1, \lambda-1, \lambda+1, \lambda+1$	
	$\mathcal{C}_2$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, (\lambda-1)^n, (\lambda-1)^n$	n odd, $m\geq 0$ $2n-2m\geq 4$
	$\mathcal{C}_2'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, (\lambda+1)^n, (\lambda+1)^n$	n odd, $m\geq 0$ $2n-2m\geq 4$
$\boxed{\mathcal{M}}$	$\mathcal{M}_1$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, \varepsilon=2k, \eta=2k$	$2k\geq m>0$
	$\mathcal{M}_1'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, \varepsilon=2k, \eta=2k$	$2k\geq m>0$
	$\mathcal{M}_2$	$\underbrace{\lambda-1,\dots,\lambda-1}_{2m}, \varepsilon=2k+1, \eta=2k+1$	$2k+1\geq m>0$
	$\mathcal{M}_2'$	$\underbrace{\lambda+1,\dots,\lambda+1}_{2m}, \varepsilon=2k+1, \eta=2k+1$	$2k+1\geq m>0$

Table 2: The irreducible "degree-one" lists

Definition 3.14. (p-quad Partitionability). *A list of elementary divisors and minimal indices is* p-quad partitionable *if it can be partitioned into p-quad irreducible sublists of the types appearing in* Table 1 *and* Table 2. *This includes p-quad irreducible lists obtained via trivial partitioning.*

A direct consequence of the previous results is that any p-quad partitionable list is p-quad admissible.

Lemma 3.15. *Let $\mathcal{L}$ be a list of elementary divisors and minimal indices. If $\mathcal{L}$ is p-quad partitionable, then $\mathcal{L}$ is p-quad admissible.*

Proof. For a p-quad partitionable list $\mathcal{L}$, Definition 3.14 implies that $\mathcal{L}$ is a concatenation of lists in Tables 1 and 2. Then by Proposition 3.13 we know that $\mathcal{L}$ is a concatenation of p-quad admissible lists, and by Lemma 3.9 that $\mathcal{L}$ as a whole is p-quad admissible. □

The converse of Lemma 3.15 is also true, and we include that fact in the Theorem 3.16 characterization of p-quad partitionability.

Theorem 3.16. (Palindromic Quadratic Partitioning Theorem). *Let $\mathcal{L}$ be any list of elementary divisors and minimal indices. Then $\mathcal{L}$ is p-quad partitionable* if and only if *$\mathcal{L}$ is p-quad admissible.*

To prove the remaining implication (the "if" part) of Theorem 3.16, we will proceed constructively in Section 4. More precisely, we describe a procedure that always produces a partitioning with the desired properties, starting from any p-quad admissible list. It is important to note that the partitioning algorithm does not always provide a unique partitioning of a p-quad admissible list $\mathcal{L}$, but it does guarantee that there always exists at least one suitable partitioning.

Another important consequence of Theorem 3.16 is that Tables 1 and 2 constitute a complete enumeration of all p-quad irreducible lists of elementary divisors and minimal indices. In particular there are only finitely many qualitatively distinct types of p-quad irreducible lists, namely, the ones described in these two tables.

Combining the results of the next two sections will show that any p-quad admissible list is not only p-quad partitionable, but actually p-quad realizable. Since we have already seen that the converse is true (see the paragraph right after Remark 3.8), this will give us the main result of this paper (Theorem 6.1), a simple characterization of p-quad realizability in terms of p-quad admissibility. Our strategy to construct a T-palindromic realization of any particular p-quad admissible list will be as follows:

- **Step 1.** Partition the given p-quad admissible list into p-quad *irreducible* sublists.
- **Step 2.** Concretely realize each p-quad irreducible sublist by a "canonical" T-palindromic quadratic block.
- **Step 3.** Take the direct sum of all of the "canonical" quadratic blocks from Step 2.

Step 1 is the object of Section 4, whereas Step 2 will be carried out in Section 5. In particular, we display in Section 5 a T-palindromic quadratic realization (consisting of a single block) for each of the p-quad irreducible lists of elementary divisors and minimal indices given in Tables 1 and 2. Note that the direct sum of the blocks in Step 3 gives a structured Kronecker-like quasi-canonical form for T-palindromic quadratic matrix polynomials, and this direct sum will have the desired finite and infinite elementary divisors and minimal indices by Lemma 5.1.

Remark 3.17. *It is worth emphasizing that the p-quad irreducible lists from* Table 1 *can be viewed as "degenerate" cases of the lists in* Table 2. *More precisely, any list of type $\mathcal{A}$ with $m = 0$, becomes a corresponding list of type $\mathcal{X}$. Analogously, by setting $m = 0$, the lists of type $\mathcal{Y}_1$ (resp., $\mathcal{Y}_1'$) are degenerate cases of lists of types $\mathcal{B}_1'$ and $\mathcal{B}_2$ (resp., $\mathcal{B}_1$ and $\mathcal{B}_2'$), type $\mathcal{Y}_2$ (resp., $\mathcal{Y}_2'$) lists are degenerate cases of type $\mathcal{C}_1'$ and $\mathcal{C}_2$ (resp., $\mathcal{C}_1$ and $\mathcal{C}_2'$) lists, and type $\mathcal{S}$ lists are degenerate cases of type $\mathcal{M}$ lists. The reason why we isolate the lists of types $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{S}$ in* Table 1 *has to do with the structure of the proof of* Theorem 3.16 *given in* Section 4.

4. The partitioning algorithm

The goal of this section is to prove the "if" implication in Theorem 3.16, i.e., that every p-quad admissible list of elementary divisors and minimal indices is p-quad partitionable. To do this we present a "partitioning algorithm" that produces a partition of any given p-quad admissible list into p-quad irreducible lists from Tables 1 and 2. Note that for some admissible lists this partition is unique, while for others it is not.

We start by establishing some preliminary special partitioning results. In the following, a list of just minimal indices (that is, a list with *No E*lementary *D*ivisors) is termed a *NoED list*.

Lemma 4.1. (The NoED Lemma). *Any NoED list $\mathcal{L}$ with p-quad symmetry is p-quad admissible, and uniquely p-quad partitionable into lists from* Table 1*.*

Proof. Any list $\mathcal{L}$ of minimal indices with p-quad symmetry has $\gimel(\mathcal{L}) = 0$, so $\mathcal{L}$ is p-quad admissible by Definition 3.7. Such a p-quad symmetric list must be of the form $\mathcal{L} = \{\varepsilon_i, \eta_i\}_{i=1}^k$, where $\varepsilon_i = \eta_i$ for all $1 \leq i \leq k$. Each pair of minimal indices $\{\varepsilon_i, \eta_i\}$ is then a p-quad irreducible sublist of type $\mathcal{S}_1$ or $\mathcal{S}_2$ in Table 1; hence $\mathcal{L}$ is p-quad partitionable by Definition 3.14. Since in Table 1 any right (left) minimal index can *only* be paired up with a left (right) minimal index of equal value, this partitioning is unique. □

Definition 4.2. (NoDO list). *A list of elementary divisors and minimal indices with* No D*egree*-O*ne elementary divisors for the eigenvalues $\lambda_0 = \pm 1$ is called a* NoDO list*.*

Lemma 4.3. (The NoDO Lemma). *Any NoDO list $\mathcal{L}$ with p-quad symmetry is uniquely p-quad partitionable into lists from* Table 1*.*

Proof. Begin the partitioning by splitting $\mathcal{L}$ into two sublists $\mathcal{E}$ and $\mathcal{T}$, where $\mathcal{E}$ contains all of the elementary divisors and $\mathcal{T}$ contains all of the minimal indices; this preliminary splitting is necessary, since no list in Table 1 contains both an elementary divisor and a minimal index. Clearly each of the lists $\mathcal{E}$ and $\mathcal{T}$ inherit the p-quad symmetry property from $\mathcal{L}$. Indeed, we claim that $\mathcal{E}$ and $\mathcal{T}$ are separately p-quad partitionable, and uniquely so. For $\mathcal{T}$ this follows immediately from the NoED Lemma 4.1. All that remains is to see that $\mathcal{E}$ is uniquely p-quad partitionable.

Since $\mathcal{E}$ is a NoDO list with p-quad symmetry, we may do an initial partitioning of the elementary divisors in $\mathcal{E}$ into the following three groups:

$$\begin{aligned} &\text{(i)} \quad \text{all } (\lambda - a)^\beta \text{ with } \beta \geq 1 \text{ and } a \neq 0, \pm 1, \\ &\text{(ii)} \quad \text{all } \lambda^\alpha \text{ and } \omega^\beta \text{ with } \alpha, \beta \geq 1, \\ &\text{(iii)} \quad \text{all } (\lambda \pm 1)^\beta \text{ with } \beta \geq 2. \end{aligned} \tag{4.1}$$

Again this is necessary, since there is no interaction between these three elementary divisor groups in any of the Table 1 lists. By the p-quad symmetry condition (1a) in Definition 3.6, all of the elementary divisors in group (i) can be paired up to form lists of type $\mathcal{X}_1$ in Table 1, while condition (1b) implies that all of the elementary divisors in group (ii) can be paired up to form lists of type $\mathcal{X}_2$. This collection of pairings is clearly unique, since in both cases only elementary divisors of the *same* degree can be associated with each other. Finally, elementary divisors in group (iii) of even degree each individually forms a list of type $\mathcal{Y}_1$ or $\mathcal{Y}_1'$, while p-quad symmetry condition (1c) guarantees that the odd degree elementary divisors in group (iii) will always exactly (and uniquely) pair up to form lists of type $\mathcal{Y}_2$ and $\mathcal{Y}_2'$. This completes the unique p-quad partitioning of the list $\mathcal{E}$, and hence also of $\mathcal{L}$. □

It is important to note that there is no loss of generality in restricting attention to Table 1 lists in Lemmas 4.1 and 4.3. In these two scenarios the elementary divisors $(\lambda \pm 1)$ are completely absent from the list $\mathcal{L}$ that is being partitioned, but at the same time they are part of every Table 2 list.

The final tool needed for the implementation of the partitioning algorithm in the proof of Theorem 3.16 is the following somewhat technical lemma.

Lemma 4.4. (The Single Eigenvalue Lemma). *Let $\mathcal{L}$ be an elementary divisor chain for either the eigenvalue $\lambda_0 = 1$ or $\lambda_0 = -1$. Then $\mathcal{L}$ is p-quad partitionable* if and only if *$\mathcal{L}$ is p-quad admissible.*

Proof. Without loss of generality, we may assume $\lambda_0 = 1$. The proof for $\lambda_0 = -1$ follows exactly the same argument using lists of types $\mathcal{B}_2'$ and $\mathcal{C}_2'$ instead of $\mathcal{B}_2$ and $\mathcal{C}_2$, respectively.

If $\mathcal{L}$ is a p-quad partitionable list, then Lemma 3.15 implies that $\mathcal{L}$ is p-quad admissible.

To prove the converse, assume that $\mathcal{L}$ is a p-quad admissible list, and let k be the number of degree-one elementary divisors in $\mathcal{L}$. The proof proceeds by induction on k. Note that the p-quad symmetry of $\mathcal{L}$ (see Definition 3.6), implies that k is even, so the induction is over even numbers only.

<u>Base case:</u> If $k = 0$ the list has no degree-one elementary divisors, so the desired conclusion follows from the NoDO Lemma 4.3.

<u>Inductive hypothesis:</u> Assume that any p-quad admissible list having at most an even number $k \leq N$ of degree-one elementary divisors is p-quad partitionable, where N is a *positive even* integer.

Now let $\mathcal{L}$ by an arbitrary p-quad admissible list with $k = N + 2$. The fact that $\mathcal{L}$ is p-quad admissible implies that $\gamma(\mathcal{L}) \leq \frac{1}{2}\delta(\mathcal{L})$ (see Definition 3.7), and consequently that there must be at least one elementary divisor $(\lambda - 1)^\alpha$ in $\mathcal{L}$ with $\alpha \geq 3$. To see this, we write the list $\mathcal{L}$ as

$$\mathcal{L} = \left\{ \overbrace{(\lambda - 1), \ldots, (\lambda - 1)}^{k}, (\lambda - 1)^{\alpha_1}, \ldots, (\lambda - 1)^{\alpha_p} \right\},$$

with $\alpha_i \geq 2$. We know by the p-quad admissibility of $\mathcal{L}$ that

$$\gamma(\mathcal{L}) \;=\; k + p \;\leq\; \frac{1}{2}\,(k + \alpha_1 + \cdots + \alpha_p) \;=\; \delta(\mathcal{L})\,,$$

which is equivalent to $\; k \leq (\alpha_1 - 2) + \cdots + (\alpha_p - 2)$. Since $k > 0$, it must be $\alpha_i - 2 > 0$, for at least one $1 \leq i \leq p$.

Then we have the following four subcases, that we analyze separately:

(s1) α is even and $k \geq \alpha - 2$. In this case, the type $\mathcal{B}_2$ sublist $\overbrace{\lambda - 1, \ldots, \lambda - 1}^{\alpha-2}, (\lambda - 1)^\alpha$ can be partitioned away from $\mathcal{L}$. The remaining sublist $\mathcal{L}'$ has $\gamma(\mathcal{L}') = \gamma(\mathcal{L}) - (\alpha - 1)$ and $\delta(\mathcal{L}') = \delta(\mathcal{L}) - (\alpha - 2) - \alpha = \delta(\mathcal{L}) - (2\alpha - 2)$. The assumption that $\mathcal{L}$ is a p-quad admissible list implies that $\gamma(\mathcal{L}) \leq \frac{1}{2}\delta(\mathcal{L})$, and consequently, that

$$\gamma(\mathcal{L}') \;=\; \gamma(\mathcal{L}) - (\alpha - 1) \;\leq\; \frac{1}{2}\delta(\mathcal{L}) - (\alpha - 1) \;=\; \frac{1}{2}\Big(\delta(\mathcal{L}) - 2(\alpha - 1)\Big) = \frac{1}{2}\delta(\mathcal{L}')\,. \tag{4.2}$$

Relation (4.2) together with the fact that $\mathcal{L}'$ has p-quad symmetry inherited from $\mathcal{L}$, imply that $\mathcal{L}'$ is a p-quad admissible list with at most N degree-one elementary divisors. Partitioning of $\mathcal{L}'$ can now be completed by the induction hypothesis.

(s2) α is even and $k < \alpha - 2$. In this case, partition off the type $\mathcal{B}_2$ sublist with $(\lambda - 1)^\alpha$ and all of the available $\lambda - 1$ elementary divisors $\overbrace{\lambda - 1, \ldots, \lambda - 1}^{k}, (\lambda - 1)^\alpha$. The remaining sublist $\mathcal{L}'$ is a NoDO list with p-quad symmetry, and its partitioning can then be completed using the NoDO Lemma 4.3.

(s3) α is odd and $k \geq 2\alpha - 4$. Note that $2\alpha - 4 \geq 2$. By p-quad symmetry of the list $\mathcal{L}$, there must be a second copy of $(\lambda - 1)^\alpha$ in $\mathcal{L}$. Then we can partition off the type $\mathcal{C}_2$ sublist $\overbrace{\lambda - 1, \ldots, \lambda - 1}^{2\alpha-4}, (\lambda - 1)^\alpha, (\lambda - 1)^\alpha$. The remaining sublist $\mathcal{L}'$ inherits p-quad symmetry from $\mathcal{L}$ and has

$$\begin{aligned} \gamma(\mathcal{L}') &= \gamma(\mathcal{L}) - (2\alpha - 4) - 2 \;=\; \gamma(\mathcal{L}) - (2\alpha - 2)\,, \\ \delta(\mathcal{L}') &= \delta(\mathcal{L}) - (2\alpha - 4) - 2\alpha \;=\; \delta(\mathcal{L}) - (4\alpha - 4)\,. \end{aligned} \tag{4.3}$$

Finally, (4.3) together with $\gamma(\mathcal{L}) \leq \frac{1}{2}\delta(\mathcal{L})$ imply that $\gamma(\mathcal{L}') \leq \frac{1}{2}\delta(\mathcal{L}')$, and therefore $\mathcal{L}'$ is a p-quad admissible list with at most N degree-one elementary divisors. Applying the induction hypothesis finishes off the partitioning of $\mathcal{L}'$.

(s4) α is odd and $k < 2\alpha - 4$. In this final case, we partition off the following sublist of type $\mathcal{C}_2$ $\overbrace{\lambda-1,\ldots,\lambda-1}^{k}, (\lambda-1)^\alpha, (\lambda-1)^\alpha$. The remaining sublist $\mathcal{L}'$ has p-quad symmetry and no degree-one elementary divisors. Hence $\mathcal{L}'$ is p-quad partitionable by the NoDO Lemma 4.3. □

With all the necessary auxiliary results established, we proceed with the proof of Theorem 3.16.

Proof. (of the Palindromic Quadratic Partitioning Theorem 3.16)
($\Rightarrow$) This implication is Lemma 3.15.
($\Leftarrow$) We will show algorithmically how a p-quad admissible list $\mathcal{L}$ of elementary divisors and minimal indices can be p-quad partitioned. First note that if $\mathcal{L}$ contains any zero minimal indices, then they can be partitioned off right away into lists of type $\mathcal{S}_1$, leaving a remaining sublist that is clearly still p-quad admissible. Thus, without loss of generality we will from now on assume that $\mathcal{L}$ contains no zero minimal indices.

We proceed by defining some key quantities. Let r and s be the number of *degree-one* elementary divisors $(\lambda-1)$ and $(\lambda+1)$, respectively; recall that p-quad symmetry implies that both r and s are *even*. As a warm-up, let us consider the case where $r = s$. If $r = s = 0$, then p-quad partitionability of $\mathcal{L}$ follows immediately from the NoDO Lemma. On the other hand, if $r = s$ is nonzero, then the list $\mathcal{L}$ can be partitioned first into $r/2$ sublists of type $\widetilde{\mathcal{C}}_1$, and a remaining sublist $\mathcal{L}'$ that has p-quad symmetry and is NoDO. Therefore, the NoDO Lemma implies that $\mathcal{L}'$ is p-quad partitionable.

Now we consider the case where $r \neq s$, starting with $r - s = \ell > 0$, i.e., there are more degree-one $(\lambda-1)$ than $(\lambda+1)$ elementary divisors. The partitioning of $\mathcal{L}$ begins by combining all s of the $(\lambda+1)$ elementary divisors with s of the $(\lambda-1)$ elementary divisors into $s/2$ sublists of type $\widetilde{\mathcal{C}}_1$, leaving a remaining sublist $\mathcal{L}'$ that is p-quad symmetric, has *no degree-one* $(\lambda+1)$ elementary divisors at all, and exactly ℓ degree-one $(\lambda-1)$ elementary divisors.

Next we try to take *as many as possible* of the remaining ℓ degree-one $(\lambda-1)$ elementary divisors and combine them together with elementary divisors in $\mathcal{L}'$ that are *not* associated with the eigenvalue $\lambda_0 = 1$, and also together with (nonzero) minimal indices in $\mathcal{L}'$, forming lists of type

$$\mathcal{A}_1\,,\ \mathcal{A}_2\,,\ \mathcal{B}_1\,,\ \mathcal{C}_1\,,\ \mathcal{M}_1\,,\ \text{ or }\ \mathcal{M}_2\,. \tag{4.4}$$

How many degree-one $(\lambda-1)$ elementary divisors can be "absorbed" into such lists? Observe that each of these list types has a certain "capacity" to absorb $(\lambda-1)$ elementary divisors. For example, an $\mathcal{A}_1$ list can contain a maximum of $2n$ copies of $(\lambda-1)$, where $2n$ is exactly the total degree of the elementary divisors $(\lambda-a)^n$ and $\left(\lambda-\frac{1}{a}\right)^n$ in the $\mathcal{A}_1$ list that are *not* associated with the eigenvalue $\lambda_0 = 1$. Analogous statements can be made about each of the list types $\mathcal{A}_2, \mathcal{B}_1$, and $\mathcal{C}_1$. Similarly, an $\mathcal{M}_1$ or $\mathcal{M}_2$ list has a maximum "capacity" to absorb $(\lambda-1)$ elementary divisors that is exactly the sum of the minimal indices in the list. Thus we see that the "total capacity" to absorb $(\lambda-1)$ elementary divisors into the six types of list in (4.4) is $\widetilde{\delta}(\mathcal{L}') + \mu(\mathcal{L}')$, where $\widetilde{\delta}(\mathcal{L}')$ is the sum of the degrees of all elementary divisors in $\mathcal{L}'$ that are *not* associated with the eigenvalue $\lambda_0 = 1$. We then have two subcases to consider:

(a) $\ell \leq \widetilde{\delta}(\mathcal{L}') + \mu(\mathcal{L}')$,

(b) $\ell > \widetilde{\delta}(\mathcal{L}') + \mu(\mathcal{L}')$.

In case (a), *all* of the ℓ degree-one elementary divisors $(\lambda-1)$ in $\mathcal{L}'$ can be partitioned off into lists as in (4.4), leaving a remaining sublist $\mathcal{L}''$ that has p-quad symmetry and is NoDO; the p-quad partitioning of $\mathcal{L}''$ is then completed by the NoDO Lemma 4.3.

On the other hand, in case (b), after $\widetilde{\delta}(\mathcal{L}') + \mu(\mathcal{L}')$ of the degree-one elementary divisors $(\lambda-1)$ are partitioned away (uniquely) into lists of type (4.4), the remaining p-quad symmetric sublist $\mathcal{L}'''$ has *only* elementary divisors associated with the eigenvalue $\lambda_0 = 1$. In other words, $\mathcal{L}'''$ is an elementary divisor chain for $\lambda_0 = 1$, and $\mu(\mathcal{L}''') = 0$. All that remains is to show that $\mathcal{L}'''$ is p-quad admissible, in particular that $\mathcal{L}'''$ satisfies condition (a) of Definition 3.7; we would then be done by the Single Eigenvalue Lemma 4.4.

To see why Definition 3.7(a) holds for $\mathcal{L}'''$, observe that if $\mathcal{T}$ denotes a sublist of type $\widetilde{\mathcal{C}}_1$ or any of the types in (4.4) that has been partitioned off from $\mathcal{L}$ so far, then:

(i) *At least half* of the elementary divisors in $\mathcal{T}$ are degree-one $(\lambda-1)$ elementary divisors.

(ii) If $t=\delta(\mathcal{T})+\mu(\mathcal{T})$, then $\mathcal{T}$ contains exactly $t/2$ degree-one elementary divisors $(\lambda-1)$.

Now (i) implies that the longest elementary divisor chain in the *original* list $\mathcal{L}$ must have been the one associated with eigenvalue $\lambda_0=1$. Letting d denote the number of degree-one elementary divisors $(\lambda-1)$ that have been removed from $\mathcal{L}$ in order to get to $\mathcal{L}'''$, then $\gamma(\mathcal{L}''')=\gamma(\mathcal{L})-d$ and $\delta(\mathcal{L}''')=\delta(\mathcal{L}''')+\mu(\mathcal{L}''')=\delta(\mathcal{L})+\mu(\mathcal{L})-2d$. But from the p-quad admissibility of $\mathcal{L}$ we know that $\delta(\mathcal{L})+\mu(\mathcal{L})\geq 2\gamma(\mathcal{L})$, so

$$\delta(\mathcal{L})+\mu(\mathcal{L})-2d \,\geq\, 2\gamma(\mathcal{L})-2d\,,$$

which implies $\delta(\mathcal{L}''')+\mu(\mathcal{L}''') \,\geq\, 2(\gamma(\mathcal{L})-d) \,=\, 2\gamma(\mathcal{L}''')$, showing that $\mathcal{L}'''$ is p-quad admissible as desired. This completes the proof of Theorem 3.16 in case $r-s>0$, that is, the case when there are more $(\lambda-1)$ than $(\lambda+1)$ elementary divisors.

The case $s-r>0$, with more $(\lambda+1)$ than $(\lambda-1)$ elementary divisors, is handled similarly; instead of using the lists from (4.4) we use their *primed* counterparts, i.e.,

$$\mathcal{A}_1'\,,\,\mathcal{A}_2'\,,\,\mathcal{B}_1'\,,\,\mathcal{C}_1'\,,\,\mathcal{M}_1'\,,\ \text{ or }\ \mathcal{M}_2'\,, \tag{4.5}$$

in order to "absorb" as many as possible of the extra $(\lambda+1)$ elementary divisors. The proof now continues to the end, mutatis mutandis, in a fashion completely analogous to the case $r-s>0$. ☐

Remark 4.5. *The partitioning algorithm demonstrates the existence of a p-quad partition for every p-quad admissible structural data list $\mathcal{L}$. However, as was mentioned earlier, the uniqueness of this p-quad partition is not always guaranteed. A key condition that usually (but not always) leads to a non-unique partition is the presence in $\mathcal{L}$ of degree-one elementary divisors $(\lambda\pm1)$; the necessity of this for non-uniqueness is a consequence of* Lemma 4.3*. Consider, for example, the list*

$$\mathcal{L} := \Big\{\,\lambda-1,\,\lambda-1,\,\lambda+1,\,\lambda+1,\,\varepsilon_1=1,\,\varepsilon_2=1,\,\eta_1=1,\,\eta_2=1\,\Big\}\,.$$

This list can be p-quad partitioned in two qualitatively distinct ways, i.e., into either

$$\{\,\lambda-1,\,\lambda-1,\,\lambda+1,\,\lambda+1\,\}\,,\,\{\,\varepsilon_1=1,\,\eta_1=1\,\}\,,\,\{\,\varepsilon_2=1,\,\eta_2=1\,\}\,, \tag{4.6}$$

or

$$\{\,\lambda-1,\,\lambda-1,\,\varepsilon_1=1,\,\eta_1=1\,\}\,,\,\{\,\lambda+1,\,\lambda+1,\,\varepsilon_2=1,\,\eta_2=1\,\}\ . \tag{4.7}$$

Note that the sublists in the first partition are of type $\widetilde{C}_1$, $\mathcal{S}_2$, and $\mathcal{S}_2$, while in the second partition are of type $\mathcal{M}_2$ and $\mathcal{M}_2'$ (our partitioning algorithm produces the partition (4.6)).

On the other hand, the presence of degree-one elementary divisors $(\lambda\pm1)$ is not by itself sufficient to guarantee the non-uniqueness of p-quad partitioning. The list

$$\mathcal{L}' := \Big\{\,\lambda-1,\,\lambda-1,\,\lambda+1,\,\lambda+1,\,(\lambda+1)^4\,\Big\} \tag{4.8}$$

contains the problematic elementary divisors $(\lambda\pm1)$, but is nonetheless uniquely p-quad partitionable into lists of type $\widetilde{C}_1$ and $\mathcal{Y}_1'$.

The results of this paper provide a canonical *T-palindromic quadratic realization for any p-quad admissible list that is* uniquely *p-quad partitionable. Unfortunately, characterizing exactly which p-quad admissible lists are uniquely p-quad partitionable is not obvious; we leave that problem for future research.*

5. T-palindromic quadratic realizations for p-quad irreducible lists

In Sections 3 and 4 we have first introduced and studied the simplest of all p-quad admissible lists of elementary divisors and minimal indices, i.e., p-quad irreducible lists. Then, we have shown that every p-quad admissible list can be partitioned into finitely many p-quad irreducible sublists. We now proceed by concretely constructing T-palindromic quadratic matrix polynomials that realize each of the p-quad irreducible lists from Tables 1-2.

5.1. *Tools for designing and analyzing blocks*

In this section we introduce some notation and establish several fundamental results that will be used throughout the rest of this paper. With $\widetilde{I}_k$ and $\widetilde{N}_k$ we denote the $k \times k$ constant matrices given by

$$\widetilde{I}_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k\times k} \quad \text{and} \quad \widetilde{N}_k := \begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \iddots & \iddots & \\ 0 & 1 & & \end{bmatrix}_{k\times k}. \tag{5.1}$$

Lemma 5.1. (Spectral and singular structures of a direct sum). *Let $P(\lambda)$ and $Q(\lambda)$ be two matrix polynomials over an algebraically closed field $\mathbb{F}$, each of grade k. Further let $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ be the lists of elementary divisors and minimal indices of P and Q, respectively. Then the list of elementary divisors and minimal indices of the grade k matrix polynomial* $\mathrm{diag}(P, Q)$ *is the concatenation of the lists $\mathcal{L}(P)$ and $\mathcal{L}(Q)$, i.e., $c(\mathcal{L}(P), \mathcal{L}(Q))$ as in (3.3).*

Proof. We need to show two things:

(a) The list of elementary divisors of $\mathrm{diag}(P, Q)$ is the concatenation of the lists of elementary divisors of P and Q,

(b) The list of minimal indices of $\mathrm{diag}(P, Q)$ is the concatenation of the lists of minimal indices of P and Q.

The proof of claim (a) can be found in [23, Prop. S1.5] when $\mathbb{F} = \mathbb{C}$, for the case of finite elementary divisors. For the infinite ones, just apply the result for the elementary divisors associated with zero in $\mathrm{rev}_k(\mathrm{diag}(P, Q)) = \mathrm{diag}(\mathrm{rev}_k P, \mathrm{rev}_k Q)$. Note however that the same proof can be easily adapted so that it holds for arbitrary fields, simply by replacing the elementary divisors of the form $(\lambda - \lambda_0)^\alpha$ by the powers of $\mathbb{F}$-irreducible scalar polynomials. The proof of claim (b) can be found in [31]. □

The next two lemmas are workhorses of this entire section. They allow us to easily determine the elementary divisors of special anti-triangular matrix polynomials. Before proceeding we remind the reader that the notation $P(\lambda) \sim Q(\lambda)$ means that the matrix polynomials $P(\lambda)$ and $Q(\lambda)$ are unimodularly equivalent, and that such $P(\lambda)$ and $Q(\lambda)$ have identical finite elementary divisors.

In the following, we use the notation Row_i and Col_j to denote the ith row and jth column of a general matrix. The notation $A \to B$ corresponds to the elementary row (resp., column) operation that replaces the row (resp., column) A by the row (resp., column) B, and $A \leftrightarrow B$ denotes row (resp., column) transposition between A and B. For the sake of uniqueness, the gcd of two scalar polynomials is considered to be monic.

Lemma 5.2. *Let f, g, h be scalar polynomials over an arbitrary field, and let $r := \gcd(f, h)$. Then:*

(a) $\begin{bmatrix} 0 & g \\ f & h \end{bmatrix} \cdot U = \begin{bmatrix} t & s \\ r & 0 \end{bmatrix}$*, where U is a unimodular matrix, both s and t are polynomial multiples of g, and the relation $rs = fg$ holds.*

(b) *Let r, s, t be scalar polynomials such that r divides t. Then* $\begin{bmatrix} t & s \\ r & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & s \\ r & 0 \end{bmatrix}$*, using exactly one elementary row operation of the form* $\mathrm{Row}_1 \to \mathrm{Row}_1 + k \cdot \mathrm{Row}_2$*, where k is a scalar polynomial.*

(c) *Let r, s, t be scalar polynomials such that* $\gcd(r, s) = 1$. *Then* $\begin{bmatrix} t & s \\ r & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & s \\ r & 0 \end{bmatrix}$, *using exactly one elementary column operation of the form* $\mathrm{Col}_1 \to \mathrm{Col}_1 + \beta \cdot \mathrm{Col}_2$ *and one elementary row operation of the form* $\mathrm{Row}_1 \to \mathrm{Row}_1 + k \cdot \mathrm{Row}_2$, *where β and k are scalar polynomials.*

Proof. (a) The fact that $\mathbb{F}[\lambda]$ is a Euclidean domain implies that there exist $a, b \in \mathbb{F}[\lambda]$ such that $fa + hb = \gcd(f, h) = r$. Also, $f = r\widehat{f}$ and $h = r\widehat{h}$ for some $\widehat{f}, \widehat{h} \in \mathbb{F}[\lambda]$. By a direct computation,

$$\begin{bmatrix} 0 & g \\ f & h \end{bmatrix} \cdot U = \begin{bmatrix} 0 & g \\ f & h \end{bmatrix} \cdot \begin{bmatrix} a & -\widehat{h} \\ b & \widehat{f} \end{bmatrix} = \begin{bmatrix} gb & g\widehat{f} \\ r & 0 \end{bmatrix} =: \begin{bmatrix} t & s \\ r & 0 \end{bmatrix}.$$

Clearly $t := gb$ and $s := g\widehat{f}$ are both polynomial multiples of g. Finally, the fact that $\det(U) = 1$ implies that U is a unimodular matrix and that $gf = rs$, as desired.

(b) Since r divides t, there exists a polynomial p such that $t = pr$. Consequently, performing the elementary row operation $\mathrm{Row}_1 \to \mathrm{Row}_1 - p \cdot \mathrm{Row}_2$ gives us the desired equivalence.

(c) If $\gcd(r, s) = 1$, then there exist scalar polynomials a, b such that $ar + bs = 1$, and hence also $tar + tbs = t$. Performing a single row and a single column operation ($\mathrm{Row}_1 \to \mathrm{Row}_1 - (ta) \cdot \mathrm{Row}_2$ and $\mathrm{Col}_1 \to \mathrm{Col}_1 - (tb) \cdot \mathrm{Col}_2$, respectively) gives us the desired equivalence. □

Remark 5.3. *It is possible to state and prove a modified version of* Lemma 5.2(a) *where* $r := \gcd(g, h)$, *the roles of r and s are exchanged, and multiplication by a unimodular matrix on the right is replaced by a corresponding multiplication on the left. Similarly, for* Lemma 5.2(b)-(c) *the roles of elementary column and row operations are interchanged.*

Lemma 5.4. (Bi-antidiagonal Collapsing Lemma). *Let $B(\lambda)$ be an $n \times n$ matrix polynomial over an arbitrary field of the form*

$$B(\lambda) = \begin{bmatrix} & & & & a_n(\lambda) \\ & & & a_{n-1}(\lambda) & b_{n-1}(\lambda) \\ & & \iddots & \iddots & \\ & a_2(\lambda) & b_2(\lambda) & & \\ a_1(\lambda) & b_1(\lambda) & & & \end{bmatrix}.$$

Let $r(\lambda) := \gcd(a_1, b_1)$ *and assume the following:*

(a) *r divides each of the polynomials $a_1, a_2, \ldots, a_n$, and*

(b) $\gcd\left(\dfrac{a_1 a_2 \cdots a_j}{r^{j-1}}, b_j\right) = r$, *for $j = 1, \ldots, n-1$.*

Then $B(\lambda)$ is unimodularly equivalent to the anti-diagonal matrix $W(\lambda) := \widetilde{I}_n \cdot \mathrm{diag}\Big(\underbrace{r(\lambda), \ldots, r(\lambda)}_{n-1}, p(\lambda)\Big)$,

where

$$p(\lambda) := r(\lambda) \cdot \left(\frac{a_1(\lambda) a_2(\lambda) \cdots a_n(\lambda)}{r^n(\lambda)}\right) = \frac{a_1(\lambda) a_2(\lambda) \cdots a_n(\lambda)}{r^{n-1}(\lambda)}.$$

Moreover, the unimodular equivalence $B(\lambda) \sim W(\lambda)$ can be achieved in such a way that the only elementary row operation involving the first row is of the form $\mathrm{Row}_1 \to \mathrm{Row}_1 + h(\lambda) \cdot \mathrm{Row}_2$, *for some polynomial $h(\lambda)$.*

Proof. We begin by introducing several auxiliary matrices that appear in our unimodular reduction of $B(\lambda)$ to $W(\lambda)$. Let $B_1(\lambda) = B(\lambda)$, and for $i = 2, 3, \ldots, n-1$ define $n \times n$ matrix polynomials $B_i(\lambda)$ by

$$B_i(\lambda) := \begin{bmatrix} & & & & & a_n(\lambda) \\ & & & & a_{n-1}(\lambda) & b_{n-1}(\lambda) \\ & & & \iddots & \iddots & \\ & 0 & a_{i+1}(\lambda) & b_{i+1}(\lambda) & & \\ & \frac{1}{r^{i-1}(\lambda)} \prod_{k=1}^{i} a_k(\lambda) & b_i(\lambda) & & & \\ r(\lambda)\widetilde{I}_{i-1} & & & & & \end{bmatrix}.$$

Furthermore, for all $j = 1, 2, \ldots, n-1$, let U_j be the 2×2 unimodular matrix obtained by applying Lemma 5.2(a) to the 2×2 submatrix of B_j living in rows $n-j$ and $n-j+1$ and in columns j and $j+1$. Then using the $n \times n$ unimodular matrix $\widetilde{U}_j := \operatorname{diag}\left(I_{j-1}, U_j, I_{n-j-1}\right)$, we see that

$$\widetilde{B}_{j+1} := B_j \cdot \widetilde{U}_j = \begin{bmatrix} & & & & & a_n(\lambda) \\ & & & & a_{n-1}(\lambda) & b_{n-1}(\lambda) \\ & & & \iddots & \iddots & \\ & r(\lambda)\cdot \widehat{t}_{j+1}(\lambda) & \frac{1}{r^{j}(\lambda)} \prod_{k=1}^{j+1} a_k(\lambda) & b_{j+1}(\lambda) & & \\ & r(\lambda) & 0 & & & \\ r(\lambda)\widetilde{I}_{j-1} & & & & & \end{bmatrix}, \tag{5.2}$$

for some $\widehat{t}_{j+1}(\lambda)$ (see Lemma 5.2(a)). Performing one additional elementary row operation gives

$$\left(I_n - e_{n-j} \cdot \widehat{t}_{j+1} \cdot e_{n-j+1}^T\right) \cdot \widetilde{B}_{j+1} = \left(I_n - e_{n-j} \cdot \widehat{t}_{j+1} \cdot e_{n-j+1}^T\right) \cdot B_j \cdot \widetilde{U}_j = B_{j+1}, \tag{5.3}$$

and so $B_j \sim B_{j+1}$. Since relations (5.2) and (5.3) hold for all $j = 1, 2, \ldots, n-1$, we obtain that

$$B(\lambda) = B_1(\lambda) \sim B_2(\lambda) \sim \cdots \sim B_{n-1}(\lambda) \sim B_n(\lambda),$$

where $B_n(\lambda) := (I_n - e_1 \cdot \widehat{t}_n \cdot e_2^T) \cdot B_{n-1} \cdot \widetilde{U}_{n-1}$ is exactly $W(\lambda)$, so the proof is complete. □

Remark 5.5. *Using* Remark 5.3*, we can also obtain a "downwards version" of* Lemma 5.4*, where the reduction starts instead from the upper right and proceeds down to the lower left of the matrix. In this version* $r(\lambda) := \gcd(a_n, b_{n-1})$*, and condition* (b) *is replaced by*

$$\gcd\left(\frac{a_n a_{n-1} \cdots a_{n-j+1}}{r^{j-1}}, b_{n-j}\right) = r(\lambda), \quad \textit{for} \quad j = 1, \ldots, n-1.$$

Further, the matrix $W(\lambda)$ *is replaced by* $\operatorname{diag}(r(\lambda), \ldots, r(\lambda), p(\lambda)) \cdot \widetilde{I}_n = W^T(\lambda)$*. Accordingly, in this case the only elementary column operation in the unimodular reduction that involves the first column of* $B(\lambda)$ *is of the form* $\mathrm{Col}_1 \to \mathrm{Col}_1 + h(\lambda) \cdot \mathrm{Col}_2$*, for some polynomial* $h(\lambda)$*.*

Remark 5.6. *Careful examination of the proofs of* Lemmas 5.2 *and* 5.4 *shows that these results hold more generally for matrices with entries in any Bezout domain* [18]*;* $\mathbb{F}[\lambda]$ *is just one example of such a domain.*

5.2. Building blocks

Before we start building T-palindromic quadratic realizations for each of the p-quad irreducible lists of elementary divisors and/or minimal indices from Tables 1 and 2, we first establish some notation. Based on everything done so far, it is evident that the scalar polynomials $\lambda + 1$ and $\lambda - 1$ (as well as their reversals) play an important role in the p-quad realizability problem. Hence, for the sake of brevity, we will sometimes use the notation:

$$\varphi(\lambda) := \lambda + 1, \qquad \theta(\lambda) := \lambda - 1.$$

Note that $\mathrm{rev}_1 \varphi = \varphi$, $\mathrm{rev}_1 \theta = -\theta$, $\mathrm{rev}_2 \varphi = \lambda\varphi$, and $\mathrm{rev}_2 \theta = -\lambda\theta$. Other pieces of notation concern matrices.

Two types of matrices will frequently occur; we refer to them as "quadratic" blocks and "splitter" blocks. The *quadratic* $k \times k$ block, $\mathcal{Q}_k$, is defined as follows:

$$\begin{aligned} \mathcal{Q}_k : \mathcal{P}_1 \times \mathcal{P}_1 \times \mathcal{P}_2 &\;\to\; \mathbb{F}[\lambda]^{k\times k} \\ (p(\lambda), q(\lambda), r(\lambda)) &\;\mapsto\; (p\cdot q)(\lambda)\cdot \widetilde{I}_k + r(\lambda)\cdot \widetilde{N}_k\,, \end{aligned} \tag{5.4}$$

where $\mathcal{P}_j := \{x(\lambda) \in \mathbb{F}[\lambda] : \deg x(\lambda) \le j\}$, for some nonnegative integer j. From (5.4) it is obvious that each entry of $\mathcal{Q}_k$ is a scalar polynomial of degree at most two, hence the term *quadratic* block is suitable. Here are some concrete examples of quadratic blocks:

$$\mathcal{Q}_k(\lambda-a, \lambda-a, \lambda) \;=\; (\lambda-a)^2\widetilde{I}_k + \lambda\widetilde{N}_k \;=\; \begin{bmatrix} & & & (\lambda-a)^2 \\ & & (\lambda-a)^2 & \lambda \\ & \iddots & \iddots & \\ (\lambda-a)^2 & \lambda & & \end{bmatrix}_{k\times k},$$

$$\mathcal{Q}_k(1, 1, \lambda^2) \;=\; 1\cdot\widetilde{I}_k + \lambda^2\widetilde{N}_k \;=\; \begin{bmatrix} & & & 1 \\ & & 1 & \lambda^2 \\ & \iddots & \iddots & \\ 1 & \lambda^2 & & \end{bmatrix}_{k\times k}.$$

The following lemma collects some useful and easily provable properties of $\mathcal{Q}_k$.

Lemma 5.7. *Let $\mathcal{Q}_k$ be defined as in (5.4). The following statements are true:*

(a) $\mathcal{Q}_k(p,q,r) = \mathcal{Q}_k(p,q,r)^T$.

(b) $\mathcal{Q}_k(p,q,r) = \mathcal{Q}_k(q,p,r)$*, i.e., $\mathcal{Q}_k$ is symmetric in its first two arguments.*

(c) $\mathrm{rev}_2\mathcal{Q}_k\big(p,q,r\big) = \mathcal{Q}_k\big(\mathrm{rev}_1 p, \mathrm{rev}_1 q, \mathrm{rev}_2 r\big)$.

Proof. We only prove (c), since (a) and (b) are immediate. Set $p(\lambda) = p_0 + \lambda p_1$ and $q(\lambda) = q_0 + \lambda q_1$, so that, $(pq)(\lambda) = p_0q_0 + \lambda(p_1q_0 + p_0q_1) + \lambda^2 p_1q_1$. Then

$$\mathrm{rev}_2(pq) \;=\; p_1q_1 + \lambda(p_1q_0+p_0q_1) + \lambda^2 p_0q_0 \;=\; (p_1+\lambda p_0)(q_1+\lambda q_0) \;=\; (\mathrm{rev}_1 p)(\mathrm{rev}_1 q)\,,$$

and the desired result easily follows. □

The other blocks that are of interest, the so called called *splitter* blocks H, K, and L, are defined as:

$$\begin{aligned} K(\lambda) &:= \begin{bmatrix} & & (\lambda-1)^2 \\ & (\lambda-1)^2 & \lambda(1-\lambda) \\ (\lambda-1)^2 & (\lambda-1) & -\lambda \end{bmatrix} = \begin{bmatrix} & & \theta^2 \\ & \theta^2 & -\lambda\theta \\ \theta^2 & \theta & -\lambda \end{bmatrix}, \\ H(\lambda) &:= \begin{bmatrix} & & 1-\lambda^2 \\ & (\lambda+1)^2 & \lambda(1+\lambda) \\ \lambda^2-1 & (\lambda+1) & \lambda \end{bmatrix} = \begin{bmatrix} & & -\varphi\theta \\ & \varphi^2 & \lambda\varphi \\ \varphi\theta & \varphi & \lambda \end{bmatrix}, \end{aligned} \qquad L(\lambda) := \begin{bmatrix} \lambda & \lambda^2 \\ 1 & \lambda \end{bmatrix}. \tag{5.5}$$

Clearly, all splitter blocks are quadratic and T-palindromic with grade 2. The role of these blocks is not immediately obvious, and neither is the reason for giving them such a name. Answers to these questions will become clear throughout this entire section, but an initial insight can be obtained by looking into the Smith forms of each of these blocks. For example, the Smith form of K is $\mathrm{diag}\big(1, (\lambda-1)^3, (\lambda-1)^3\big)$, which follows from Theorem 2.2(d) and the fact that the gcd of all 1×1 minors of K is 1, the gcd of all 2×2 minors is $(\lambda-1)^3$, and $\det K = -(\lambda-1)^6$. One could speculate that the $(2,2)$ entry of K is somehow being "split" between the two corner entries on the anti-diagonal of K to give its Smith form. To see that this is exactly what is happening, consider the 3×3 matrix polynomial

$$F(\lambda) := \begin{bmatrix} & & (\lambda-1)^{2m} \\ & (\lambda-1)^2 & \lambda(1-\lambda) \\ (\lambda-1)^{2k} & (\lambda-1) & -\lambda \end{bmatrix}, \tag{5.6}$$

obtained by a slight modification of the anti-diagonal entries of K. Assuming that $1 \leq k \leq m$, again it is easy to see that the Smith form of F is $\operatorname{diag}\left(1, (\lambda-1)^{2k+1}, (\lambda-1)^{2m+1}\right)$ (the gcd of all 1×1 minors of F is 1, the gcd of all 2×2 minors is $(\lambda-1)^{2k+1}$ and $\det F = -(\lambda-1)^{2m+2k+2}$), i.e., the middle entry of F is "split" between the other two anti-diagonal entries $(\lambda-1)^{2k}$ and $(\lambda-1)^{2m}$. A similar conclusion follows from examining the Smith form of H, which is $\operatorname{diag}(1, (\lambda-1)(\lambda+1)^2, (\lambda-1)(\lambda+1)^2)$, as can be seen, again, by looking at the gcd of the $1 \times 1, 2 \times 2$, and 3×3 minors. Finally, note that the block L is singular, with rank one and no finite or infinite elementary divisors, and with $\varepsilon = \eta = 1$.

5.3. "Canonical" palindromic blocks

In Tables 3–6 we define several types of matrix polynomials, and in Section 5.4 we show that each of them is a T-palindromic quadratic realization for a corresponding type of the p-quad irreducible list from Tables 1 and 2. Indeed, the notation for different types of polynomials given in Tables 3–6 is chosen so the types of polynomials X, Y, S, A, B, C, and M, are T-palindromic quadratic realizations for the types of p-quad irreducible lists $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{S}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{M}$, respectively, from Tables 1 and 2.

Now recall that any p-quad admissible list of elementary divisors and minimal indices can be partitioned into p-quad irreducible sublists, i.e., the ones in Tables 1 and 2 (c.f. Theorem 3.16). Assuming that the matrix polynomials from Tables 3–6 are in fact T-palindromic quadratic realizations for corresponding p-quad irreducible lists, it is reasonable, taking into account Lemma 5.1, to use these polynomials as building blocks for a T-palindromic quadratic realization of an arbitrary p-quad admissible list. Since this turns out to be the case, we choose to refer to the polynomials in Tables 3–6 simply as *blocks*. Furthermore, the low anti-bandwidth of these blocks makes them resemble the blocks arising in the Kronecker canonical form of matrix pencils.

In Tables 3-6, the entries $*, \bullet, \diamond$ and $\odot$ appearing in the corners between adjacent anti-diagonal blocks are always assumed to be located in the upper left corner. More precisely, if the first column of the upper block is the j^{th} column of the whole matrix and the first row of the lower block is the i^{th} row of the whole matrix, then the entries $*, \bullet, \diamond, \odot$ are in the (i, j) position.

Note that the blocks in Tables 5 and 6 have been divided in two cases, denoted with an additional subindex "a" or "b". The reason for this is the need to consider separately the realization of lists of types $\mathcal{C}$ and $\mathcal{M}$ in Table 2 for the different cases: $n-m$ even (subindex a) or $n-m$ odd (subindex b) for the type $\mathcal{C}$ lists, and m even (subindex a) and m odd (subindex b) for type $\mathcal{M}$ lists.

Remark 5.8. (Limit cases). *There are several "limit cases" for some of the block types in* Tables 3–6 *where the definition can be ambiguous due to some of the inner blocks having null size. All these limit cases are defined after removing the blocks of size* 0 *and "collapsing" or removing the glueing entries. The following cases make this idea more concrete.*

- The cases $m = 0$ for block Y_2 and $k = 0$ in both S_1 and S_2 blocks in Table 3. *In all these cases, only the central block remains, and the glueing entries do not appear. More precisely,* $Y_2 = K, S_1 = 0,$ *and* $S_2 = L$.

- The case $m = n$ for block A_1 in Table 4. *Here block* A_1 *is defined as follows*

$$\mathrm{A}_1 = \begin{bmatrix} & \boxed{\mathcal{Q}_m(-\theta, \widehat{p}, \theta^2)} \\ \boxed{\mathcal{Q}_m(\theta, p, \theta^2)} & * \end{bmatrix}, \qquad \textit{with} \qquad * = (\lambda-1)^2. \tag{5.7}$$

Type	Subtype	Block	Conditions ($\theta = \lambda - 1$)
X	X_1	$\mathcal{Q}_m\left(\lambda - a, \lambda - \frac{1}{a}, \theta^2\right)$	$a \neq 0, \pm 1$
	X_2	$\mathcal{Q}_m(1, \lambda, \theta^2)$	
Y	Y_1	$\mathcal{Q}_m\left(\theta, \theta, \lambda\right)$	$m \geq 1$
	Y_1'	Replace λ by $-\lambda$ in the block Y_1	
	Y_2	$\left[\begin{array}{ccc} & & \boxed{\mathcal{Q}_m(\theta,\theta,\lambda)} \\ & \boxed{K} & * \\ \boxed{\mathcal{Q}_m(\theta,\theta,\lambda)} & * & \end{array}\right]_{(2m+3)\times(2m+3)}$	$* = \lambda$ $m \geq 0$ K as in (5.5)
	Y_2'	Replace λ by $-\lambda$ in the block Y_2	
S	S_1	$\left[\begin{array}{ccc} & & \boxed{\mathcal{Q}_k(\lambda,\lambda,1)} \\ & \boxed{0} & * \\ \boxed{\mathcal{Q}_k(1,1,\lambda^2)} & \bullet & \end{array}\right]_{(2k+1)\times(2k+1)}$	$* = 1$ $\bullet = \lambda^2$ $k \geq 0$
	S_2	$\left[\begin{array}{ccc} & & \boxed{\mathcal{Q}_k(\lambda,\lambda,1)} \\ & \boxed{L} & * \\ \boxed{\mathcal{Q}_k(1,1,\lambda^2)} & \bullet & \end{array}\right]_{(2k+2)\times(2k+2)}$	$* = 1$ $\bullet = \lambda^2$ $k \geq 0$ L as in (5.5)

Table 3: Blocks of type X, Y and S

- The case $m = n$ for block A_2 in Table 4. *Here block A_2 is defined as follows*

$$A_2 = \left[\begin{array}{cc} & \boxed{\mathcal{Q}_m(\lambda, -\theta, \theta^2)} \\ \boxed{\mathcal{Q}_m(1, \theta, \theta^2)} & * \end{array}\right], \qquad \textit{with} \qquad * = (\lambda - 1)^2. \tag{5.8}$$

- The case $m = n$ for block B_1 in Table 4. *In this case, block B_1 is defined analogously to A_1.*
- The case $\ell = 0$ (i.e., $m = n$) in block C_{1a} in Table 5. *Here C_{1a} block is defined as*

$$C_{1a} = \left[\begin{array}{cc} & \boxed{\mathcal{Q}_m(\varphi, -\theta, \theta^2)} \\ \boxed{\mathcal{Q}_m(\varphi, \theta, \theta^2)} & \end{array}\right]. \tag{5.9}$$

Note that, in this case, the glueing entries $*$ are missing.

Type	Subtype	Block	Conditions
			$(\theta = \lambda - 1)$
A	A_1	$\begin{bmatrix} & & \mathcal{Q}_m(-\theta, \widehat{p}, \theta^2) \\ & \mathcal{Q}_{n-m}(p, q, \theta^2) & * \\ \mathcal{Q}_m(\theta, p, \theta^2) & * & \end{bmatrix}$	$* = \theta^2$ $q = \lambda - \frac{1}{a}$ $p = \lambda - a$ $a \neq 0, \pm 1$ $\widehat{p} = 1 - a\lambda = \mathrm{rev}_1(\lambda - a)$ $0 < m \leq n$
	A_1'	Replace λ by $-\lambda$ and a by $-a$ in the block A_1	
	A_2	$\begin{bmatrix} & & \mathcal{Q}_m(\lambda, -\theta, \theta^2) \\ & \mathcal{Q}_{n-m}(1, \lambda, \theta^2) & * \\ \mathcal{Q}_m(1, \theta, \theta^2) & * & \end{bmatrix}$	$* = \theta^2$ $0 < m \leq n$
	A_2'	Replace λ by $-\lambda$ in the block A_2	
B	B_1	Set $a = -1$ in the block A_1	$0 < m \leq n$
	B_1'	Set $a = 1$ in the block A_1'	$0 < m \leq n$
	B_2	$\begin{bmatrix} & & \mathcal{Q}_m(\theta, \theta, -\lambda\theta) \\ & \mathcal{Q}_{n-m}(\theta, \theta, \lambda) & * \\ \mathcal{Q}_m(\theta, \theta, \theta) & \bullet & \end{bmatrix}$	$\bullet = \theta$ $* = \mathrm{rev}_2\theta = \lambda - \lambda^2 = -\lambda\theta$ $0 < m < n$
	B_2'	Replace λ by $-\lambda$ in the block B_2	

Table 4: Blocks of types A and B

- The case $h = 0$ (i.e., $k = \ell$) in blocks M_{1a} and M_{1b} in Table 6. *Here* M_{1a} *is defined as follows*

$$\mathrm{M}_{1a} = \begin{bmatrix} & & \mathcal{Q}_m(\lambda, -\theta, \theta) \\ & \Psi & * \\ \mathcal{Q}_m(1, \theta, \mathrm{rev}_2\theta) & \bullet & \end{bmatrix}, \quad \textit{with} \quad \begin{array}{rcl} * & = & (\lambda - 1), \\ \bullet & = & \mathrm{rev}_2(\lambda - 1) \\ & = & -\lambda(1 - \lambda) \end{array}, \tag{5.10}$$

  while M_{1b} *is defined analogously.*

- The case $h = 0$ (i.e., $\ell = k + 1$) in blocks M_{2a} and M_{2b} in Table 6. *Here* M_{2a} *and* M_{2b} *blocks are defined in a similar way as* M_{1a} *and* M_{1b} *above.*

The corresponding primed blocks are defined in an analogous way.

Subtype	Block	Conditions
		$(\varphi = \lambda + 1\,,\, \theta = \lambda - 1)$
C_{1a}	$\begin{bmatrix} & & & \mathcal{Q}_m(\varphi,-\theta,\theta^2) \\ & & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & * \\ & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & & \\ \mathcal{Q}_m(\varphi,\theta,\theta^2) & * & & \end{bmatrix}$	$\begin{array}{rcl} n-m & = & 2\ell \\ m & > & 0 \\ n & \text{is} & \text{odd} \\ * & = & \theta^2 \end{array}$
C_{1b}	$\begin{bmatrix} & & & & \mathcal{Q}_{m-1}(\varphi,-\theta,\theta^2) \\ & & & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & * \\ & & H & * & \\ & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & * & & \\ \mathcal{Q}_{m-1}(\varphi,\theta,\theta^2) & * & & & \end{bmatrix}$	$\begin{array}{rcl} n-m & = & 2\ell+1 \\ m & > & 0 \\ n & \text{is} & \text{odd} \\ * & = & \theta^2 \end{array}$ H as in (5.5)
C'_{1a}	Replace λ by $-\lambda$ in the block C_{1a}	
C'_{1b}	Replace λ by $-\lambda$ in the block C_{1b}	
$\widetilde{C}_1$	$\begin{bmatrix} 0 & -\varphi\theta \\ \varphi\theta & 0 \end{bmatrix}$	
C_{2a}	$\begin{bmatrix} & & & \mathcal{Q}_m(\theta,\theta,-\lambda\theta) \\ & & \mathcal{Q}_\ell(\theta,\theta,\lambda) & * \\ & \mathcal{Q}_\ell(\theta,\theta,\lambda) & & \\ \mathcal{Q}_m(\theta,\theta,\theta) & \bullet & & \end{bmatrix}$	$\begin{array}{rcl} n-m & = & 2\ell > 0 \\ \bullet & = & \theta \\ * & = & \mathrm{rev}_2\theta \\ & = & -\lambda\theta \end{array}$
C_{2b}	$\begin{bmatrix} & & & & \mathcal{Q}_m(\theta,\theta,-\lambda\theta) \\ & & & \mathcal{Q}_{\ell-1}(\theta,\theta,\lambda) & * \\ & & K & \diamond & \\ & \mathcal{Q}_{\ell-1}(\theta,\theta,\lambda) & \diamond & & \\ \mathcal{Q}_m(\theta,\theta,\theta) & \bullet & & & \end{bmatrix}$	$\begin{array}{rcl} n-m & = & 2\ell+1 \\ \ell & \geq & 1 \\ \diamond & = & \lambda \\ \bullet & = & \theta \\ * & = & \mathrm{rev}_2\theta \\ & = & -\lambda\theta \end{array}$ K as in (5.5)
C'_{2a}	Replace λ by $-\lambda$ in the block C_{2a}	
C'_{2b}	Replace λ by $-\lambda$ in the block C_{2b}	

Table 5: Blocks of type C

Remark 5.9. *The blocks of types X, Y, and S in* Table 3 *are degenerate cases of the blocks of types A, B, C, and M in* Tables 4-6*, in the same way as the lists in* Table 1 *are degenerate cases of the lists in* Table 2 *(see* Remark 3.17*). More precisely, blocks of type A_1 and A'_1 degenerate onto type X_1 blocks when $m = 0$. Similarly with blocks of type A_2, A'_2 and X_2. In the same way, type Y_1 (resp., Y'_1) blocks are degenerate cases of type B_2 (resp., B'_2) blocks, whereas blocks of type Y_2 (resp., Y'_2) are degenerate cases of type C_{2b} (resp., C'_{2b}) blocks, when $m = 0$. Finally, M_{1a} and M_{2a} blocks degenerate onto S_1 blocks, and M_{1b}, M_{2b} degenerate on S_2 blocks when $m = 0$. Note, however, that there is no redundancy between blocks within* Tables 3–6*, except the case of lists $\mathcal{C}_1$ and $\mathcal{C}'_1$, which overlap in list $\widetilde{\mathcal{C}}_1$ (see* Remark 3.11*).*

Subtype	Block	Conditions
		$(\theta = \lambda - 1)$
$\mathrm{M}_{1\mathrm{a}}$	$\begin{bmatrix} & & & & \mathcal{Q}_m(\lambda,-\theta,\theta) \\ & & & \mathcal{Q}_h(\lambda,\lambda,1) & * \\ & & \Psi & \odot & \\ & \mathcal{Q}_h(1,1,\lambda^2) & \diamond & & \\ \mathcal{Q}_m(1,\theta,\mathrm{rev}_2\theta) & \bullet & & & \end{bmatrix}$	$m > 0$ $2k \geq m$ $m = 2\ell$ $h = k-\ell$ $\Psi = 0_{1\times 1}$ $* = \theta$ $\bullet = \mathrm{rev}_2\theta$ $= -\lambda\theta$ $\diamond = \lambda^2$ $\odot = \mathrm{rev}_2(\diamond)$ $= 1$
$\mathrm{M}_{1\mathrm{b}}$	Same as the above block except (see Conditions to the right)	$m = 2\ell - 1$ $\Psi = L$ L as in (5.5)
$\mathrm{M}'_{1\mathrm{a}}$	Replace λ by $-\lambda$ in the block $\mathrm{M}_{1\mathrm{a}}$	
$\mathrm{M}'_{1\mathrm{b}}$	Replace λ by $-\lambda$ in the block $\mathrm{M}_{1\mathrm{b}}$	
$\mathrm{M}_{2\mathrm{a}}$	Same as block $\mathrm{M}_{1\mathrm{a}}$ except (see Conditions to the right)	$2k+1 \geq m$ $m = 2\ell$ $h = k-\ell+1$
$\mathrm{M}_{2\mathrm{b}}$	Same as the above block except (see Conditions to the right)	$2k+1 \geq m$ $m = 2\ell - 1$ $\Psi = L$ L as in (5.5)
$\mathrm{M}'_{2\mathrm{a}}$	Replace λ by $-\lambda$ in the block $\mathrm{M}_{2\mathrm{a}}$	
$\mathrm{M}'_{2\mathrm{b}}$	Replace λ by $-\lambda$ in the block $\mathrm{M}_{2\mathrm{b}}$	

Table 6: Blocks of type M

5.4. Spectral and singular structure of canonical palindromic blocks

In Tables 3–6 we have introduced different types of "canonical" blocks, which we claim have the structural data given by the corresponding lists of elementary divisors and minimal indices in Tables 1 and 2; i.e., each type of block is a T-palindromic quadratic realization for the corresponding p-quad irreducible list. Here we prove that this is indeed the case.

Our first result is a technical lemma that leverages the knowledge of the spectral and singular structure of unprimed blocks in order to determine that of their primed counterparts.

Lemma 5.10. *Assume that any unprimed block in* Tables 3–6 *realizes the corresponding list in* Tables 1–2*. Then any primed block in* Tables 3–6 *has the same spectral and singular structure as the corresponding unprimed block, except for the elementary divisors associated with $\lambda = \pm 1$. In particular, the degrees of elementary divisors associated with $\lambda = -1$ (resp., $\lambda = 1$) in the primed block are the same as the degrees of the elementary divisors associated with $\lambda = 1$ (resp., $\lambda = -1$) in the unprimed block.*

Proof. Our proof depends on several facts that relate the structural data of an arbitrary polynomial $P(\lambda)$ to that of the new polynomial $Q(\lambda) := P(-\lambda)$. More specifically, the following statements hold:

(a) For any $b \in \mathbb{F}$, $(\lambda - b)^\alpha$ is an elementary divisor of $P(\lambda)$ if and only if $(\lambda + b)^\alpha$ is an elementary divisor of $Q(\lambda)$.

(b) ω^β is an infinite elementary divisor of $P(\lambda)$ if and only if ω^β is an infinite elementary divisor of $Q(\lambda)$.

(c) $P(\lambda)$ and $Q(\lambda)$ have exactly the same left and right minimal indices.

Note that one can prove the above statements either directly, or by observing that $Q(\lambda)$ is just obtained from $P(\lambda)$ by a special Möbius transformation [37]. In the latter case, statements (a)-(b) follow from [37, Thm. 5.3] and (c) from [37, Thm. 7.3].

Now observe that the majority of the primed blocks in Tables 3–6 (i.e., all primed blocks except A_1'), are obtained by replacing λ with $-\lambda$ in the corresponding unprimed blocks. Since the structural data of each of those unprimed blocks contains only minimal indices and elementary divisors associated with $0, \pm 1$ and ∞, applying statements (a)–(c) gives the desired conclusion for the corresponding primed blocks.

The proof will be complete once we establish the relationship between the structural data of A_1 and A_1', which we do in two steps. For this, we introduce the notation $A_1(\lambda, a)$ to emphasize the dependence of A_1 on the variable λ and the parameter a.

(i) From the assumption we know that the block $A_1(\lambda, a)$ has $\overbrace{(\lambda-1),\ldots,(\lambda-1)}^{2m}, (\lambda-a)^n, (\lambda-\frac{1}{a})^n$ as its elementary divisors, and has no minimal indices. Consequently, the block $A_1(\lambda, -a)$ also has no minimal indices, and its only elementary divisors are $\overbrace{(\lambda-1),\ldots,(\lambda-1)}^{2m}, (\lambda+a)^n, (\lambda+\frac{1}{a})^n$.

(ii) Applying statement (a) and (c) to $A_1(\lambda, -a)$ shows that $A_1(-\lambda, -a) = A_1'$ has no minimal indices, and its only elementary divisors are $\overbrace{(\lambda+1),\ldots,(\lambda+1)}^{2m}, (\lambda-a)^n, (\lambda-\frac{1}{a})^n$. Hence the Lemma holds for the block A_1', and the proof is complete. $\square$

Remark 5.11. *Let* $P(\lambda) = \sum_{i=0}^k A_i\lambda^i$ *be a matrix polynomial of grade* k *and consider the new polynomial* $Q(\lambda) := P(-\lambda) = \sum_{i=0}^k (-1)^i A_i \lambda^i$ *as in the proof of* Lemma 5.10*. Then*

$$(\mathrm{rev}_k Q)(\lambda) = \sum_{i=0}^{k} (-1)^{k-i} A_{k-i}\lambda^i = (-1)^k \sum_{i=0}^{k} A_{k-i}(-\lambda)^i = (-1)^k (\mathrm{rev}_k P)(-\lambda)\,.$$

Now if $P(\lambda)$ *is* T*-palindromic with grade of palindromicity* k*, then* $(\mathrm{rev}_k Q)(\lambda) = (-1)^k P(-\lambda)^T = (-1)^k Q(\lambda)^T$*. Furthermore, if* k *is even, then* $(\mathrm{rev}_k Q)(\lambda) = Q(\lambda)^T$*, so* $Q(\lambda)$ *is also* T*-palindromic with grade of palindromicity* k*.*

Theorem 5.12. *Any p-quad irreducible list* $\mathcal{L}$ *from* Tables 1 *and* 2 *is p-quad realizable. In particular,* $\mathcal{L}$ *is realizable by the corresponding block in* Tables 3–6*.*

Proof. The proof is carried out by analyzing separately each type of blocks in Tables 3–6. We only focus on the unprimed blocks. Once these blocks have been analyzed, the result for the primed ones follows directly from Lemma 5.10. Note that our proof is quite thorough and somewhat repetitive, but we have included most of the arguments for the sake of completeness.

First of all, due to a very simple low-bandwidth anti-diagonal structure of the blocks from Tables 3–6, one can verify directly that only the blocks of type S and M are *singular*, i.e., all other types of blocks have a *trivial singular structure*. Second, it is also straightforward to see that all unprimed and primed blocks are T-palindromic with grade of palindromicity 2. For the unprimed blocks, this is a consequence of Lemma 5.7 and the fact that all splitter blocks in (5.5) are T-palindromic with grade of palindromicity 2. For the primed blocks, it is a consequence of the result for the unprimed blocks and Remark 5.11.

We now investigate the spectral and singular structure of each block, showing that in each case it is exactly the same as that of the corresponding structural data list in Table 1 or 2.

X Blocks: Applying Lemma 5.4 to type X_1 blocks gives

$$\mathcal{Q}_m\left(\lambda-a,\lambda-\frac{1}{a},(\lambda-1)^2\right)\;\sim\;\begin{bmatrix} & (\lambda-a)^m\left(\lambda-\frac{1}{a}\right)^m \\ \widetilde{I}_{m-1} & \end{bmatrix}.$$

A permutation of the columns shows that the Smith form of $\mathcal{Q}_m\left(\lambda-a,\lambda-\frac{1}{a},(\lambda-1)^2\right)$ is equal to $\operatorname{diag}\left(I_{m-1},\,(\lambda-a)^m\left(\lambda-\frac{1}{a}\right)^m\right)$. The Smith form, together with Theorem 3.4, implies that the structural data of any type X_1 block consists of exactly two elementary divisors $(\lambda-a)^m$ and $\left(\lambda-\frac{1}{a}\right)^m$, i.e., is exactly the same as that described in Table 1 for p-quad irreducible lists of type $\mathcal{X}_1$.

For type X_2 blocks, we again use Lemma 5.4 to obtain

$$\mathcal{Q}_m\Big(1,\lambda,(\lambda-1)^2\Big)\;\sim\;\begin{bmatrix} & \lambda^m \\ \widetilde{I}_{m-1} & \end{bmatrix},$$

and conclude that a type X_2 block has a single finite elementary divisor λ^m. The fact that an X_2 block is T-palindromic, together with Remark 2.6(iii), implies that X_2 also has a single infinite elementary divisor ω^m. Thus the structural data of a type X_2 block is exactly the same as that described in Table 1 for a p-quad irreducible list of type $\mathcal{X}_2$.

Y Blocks: Applying Lemma 5.4 to type Y_1 block gives

$$Y_1=\mathcal{Q}_m\Big(\lambda-1,\lambda-1,\lambda\Big)\;\sim\;\begin{bmatrix} & (\lambda-1)^{2m} \\ \widetilde{I}_{m-1} & \end{bmatrix}.$$

This equivalence, together with Theorem 3.4 and the fact that Y_1 is regular, implies that the structural data of a Y_1 block consists of just a single non-trivial elementary divisor $(\lambda-1)^{2m}$. Hence it is exactly the same as that described in Table 1 for a list of type $\mathcal{Y}_1$.

Now consider blocks of type Y_2. Applying Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $m\times m$ blocks of Y_2, respectively, we obtain:

$$Y_2\sim\left[\begin{array}{ccccccc}
 & & & & & & \widetilde{I}_{m-1} \\
 & & & & 0 & (\lambda-1)^{2m} & \\
 & & & & (\lambda-1)^2 & \lambda & \\
 & & & (\lambda-1)^2 & \lambda(1-\lambda) & 0 & \\
 & 0 & (\lambda-1)^2 & \lambda-1 & -\lambda & 0 & \\
 & (\lambda-1)^{2m} & \lambda & 0 & 0 & 0 & \\
\widetilde{I}_{m-1} & & & & & &
\end{array}\right]=:Y_2^{(1)}.$$

Applying Lemma 5.2 and Remark 5.3 to the 2×2 blocks on the antidiagonal of $Y_2^{(1)}$, we obtain the following unimodular equivalence:

$$Y_2\;\sim\;Y_2^{(1)}\;\sim\;\left[\begin{array}{ccccc}
 & & & & \widetilde{I}_m \\
 & & & (\lambda-1)^{2(m+1)} & \\
 & & (\lambda-1)^2 & \lambda(1-\lambda) & \\
 & (\lambda-1)^{2(m+1)} & \lambda-1 & -\lambda & \\
\widetilde{I}_m & & & &
\end{array}\right]=:Y_2^{(2)}.$$

Observe that the central 3×3 block of $Y_2^{(2)}$ is a block of type $F(\lambda)$ as in (5.6), whose finite spectral structure consists of two finite elementary divisors $(\lambda-1)^{2m+3},(\lambda-1)^{2m+3}$. This observation, the fact that Y_2 is regular, and Theorem 3.4 now imply that the structural data of Y_2 is the same as that described in Table 1 for a list of type $\mathcal{Y}_2$.

S blocks: Applying Collapsing Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $k \times k$ submatrices of S_1, respectively, gives

$$S_1 \sim \begin{bmatrix} & & & & \widetilde{I}_{k-1} \\ & & & \lambda^{2k} & \\ & & 0 & 1 & \\ & 1 & \lambda^2 & 0 & \\ \widetilde{I}_{k-1} & & & & \end{bmatrix} \sim \begin{bmatrix} & & \widetilde{I}_k \\ & 0 & \\ \widetilde{I}_k & & \end{bmatrix} \sim \begin{bmatrix} I_{2k} & \\ & 0 \end{bmatrix}. \tag{5.11}$$

Note that the second unimodular equivalence in (5.11) is obtained by performing the following elementary row and column operations in the specified order:

$$\begin{array}{rcl} \mathrm{Col}_{k+1} & \longrightarrow & \mathrm{Col}_{k+1} - \lambda^2 \cdot \mathrm{Col}_k \\ \mathrm{Row}_k & \longrightarrow & \mathrm{Row}_k - \lambda^{2k} \cdot \mathrm{Row}_{k+1} \\ \mathrm{Row}_{k+1} & \longleftrightarrow & \mathrm{Row}_k\,. \end{array}$$

The unimodular equivalence (5.11) implies that S_1 has no finite elementary divisors. This, together with the fact that S_1 is T-palindromic, also implies that it has no infinite elementary divisors (see Remark 2.6(iii)). On the other hand, from (5.11) we observe that S_1 is singular and that it has exactly one left η and one right ε minimal index. Now using the Index Sum Theorem 3.4 we obtain the following chain of equalities:

$$\begin{array}{rcl} \mathrm{rank}(S_1) \cdot \mathrm{grade}(S_1) & = & \delta_{fin}(S_1) + \delta_\infty(S_1) + \mu(S_1) \\ 2k \cdot 2 & = & 0 + 0 + \varepsilon + \eta\,. \end{array} \tag{5.12}$$

The fact that $\varepsilon = \eta$ (see Remark 2.6(iv)) and (5.12) imply that $\varepsilon = \eta = 2k$, i.e., the structural data of S_1 is identical to that described in Table 1 for a list of type $\mathcal{S}_1$.

For the S_2 block, we apply Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $k \times k$ submatrices of S_2, respectively, to obtain the following unimodular equivalence:

$$S_2 \sim \begin{bmatrix} & & & & & \widetilde{I}_{k-1} \\ & & & & \lambda^{2k} & \\ & & \lambda & \lambda^2 & 1 & \\ & & 1 & \lambda & & \\ & 1 & \lambda^2 & & & \\ \widetilde{I}_{k-1} & & & & & \end{bmatrix} =: S_2^{(1)}\,.$$

Applying the following sequence of elementary unimodular operations to $S_2^{(1)}$

$$\begin{array}{rcl} \mathrm{Col}_{k+1} & \longrightarrow & \mathrm{Col}_{k+1} - \lambda^2 \cdot \mathrm{Col}_k\,, \\ \mathrm{Row}_{k+1} & \longrightarrow & \mathrm{Row}_{k+1} - \lambda \cdot \mathrm{Row}_{k+2}\,, \\ \mathrm{Row}_k & \longrightarrow & \mathrm{Row}_k - \lambda^{2k} \cdot \mathrm{Row}_{k+1}\,, \\ \mathrm{Col}_{k+2} & \longrightarrow & \mathrm{Col}_{k+2} - \lambda \cdot \mathrm{Col}_{k+1}\,, \end{array}$$

leads to the unimodular equivalence

$$S_2 \sim S_2^{(1)} \sim \begin{bmatrix} & & & & & \widetilde{I}_{k-1} \\ & & & & 0 & \\ & & 0 & 0 & 1 & \\ & & 1 & 0 & & \\ & 1 & 0 & & & \\ \widetilde{I}_{k-1} & & & & & \end{bmatrix} =: S_2^{(2)}\,.$$

A straightforward row and column permutation of $S_2^{(2)}$ shows that the Smith form of S_2 is $\mathrm{diag}(I_{2k+1}, 0)$. Consequently, we conclude that S_2 has no finite elementary divisors, which together with the fact that S_2 is T-palindromic also implies that it has no infinite elementary divisors either (see Remark 2.6(iii)).

Regarding the singular structure of S_2, observe that S_2 has exactly one left η and one right ε minimal index. Again, S_2 being T-palindromic and Remark 2.6(iv) imply that $\eta = \varepsilon$. Finally, a calculation analogous to (5.12) shows that $\varepsilon = \eta = 2k+1$, showing that the structural data of the type S_2 block is the one described in Table 1 for a list of type $\mathcal{S}_2$.

A blocks: Applying Collapsing Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $m \times m$ submatrices of A_1, respectively, gives the unimodular equivalence

$$A_1 \sim A_1^{(1)} := \left[\begin{array}{ccc} & & \begin{array}{cc} & (\lambda-1)\widetilde{I}_{m-1} \\ (-1)^m(\lambda-1)(\mathrm{rev}_1 p)^m & \\ (\lambda-1)^2 & \end{array} \\ & \begin{array}{cccc} & & & pq \\ & & pq & (\lambda-1)^2 \\ & \iddots & \iddots & \\ pq & (\lambda-1)^2 & & \\ (\lambda-1)^2 & & & \end{array} & \\ \begin{array}{cc} & (\lambda-1)p^m \\ (\lambda-1)\widetilde{I}_{m-1} & \end{array} & & \end{array}\right],$$

where $p = \lambda - a$ and $q = \lambda - 1/a$. Pre-multiplying $A_1^{(1)}$ by $\mathrm{diag}(I_{m-1}, (-1)^m, I_n)$, and using Lemma 5.2 and Remark 5.3, gives the following unimodular equivalence

$$A_1 \sim A_1^{(1)} \sim A_1^{(2)} := \left[\begin{array}{ccc} & & (\lambda-1)\widetilde{I}_m \\ & \begin{array}{cccc} & & & pq(\mathrm{rev}_1 p)^m \\ & & pq & (\lambda-1)^2 \\ & \iddots & \iddots & \\ & pq & (\lambda-1)^2 & \\ p^{m+1}q & (\lambda-1)^2 & & \end{array} & \\ (\lambda-1)\widetilde{I}_m & & \end{array}\right].$$

Next, applying Lemma 5.4 to the central block of $A_1^{(2)}$ gives the following unimodular equivalence:

$$A_1 \sim \left[\begin{array}{ccc} & & (\lambda-1)\widetilde{I}_m \\ & \begin{array}{cc} & p^n q^{n-m}(\mathrm{rev}_1 p)^m \\ \widetilde{I}_{n-m-1} & \end{array} & \\ (\lambda-1)\widetilde{I}_m & & \end{array}\right] =: A_1^{(3)}$$

Since $\mathrm{rev}_1 p = -aq$, we conclude that $A_1^{(3)}$, and consequently A_1, has two finite elementary divisors p^n and q^n, together with $2m$ finite elementary divisors $(\lambda - 1)$. The Index Sum Theorem 3.4 and the fact that A_1 is regular, imply that A_1 has no infinite elementary divisors, and that its spectral structure is the one corresponding to the list of type $\mathcal{A}_1$ in Table 2.

In case when $n = m$ (see (5.7)), one can show using transformations similar as above that

$$A_1 \sim A_1^{(1)} = \left[\begin{array}{ccc} & & (\lambda-1)\widetilde{I}_{m-1} \\ & \begin{array}{cc} 0 & (-1)^m(\lambda-1)(\mathrm{rev}_1 p)^m \\ (\lambda-1)p^m & (\lambda-1)^2 \end{array} & \\ (\lambda-1)\widetilde{I}_{m-1} & & \end{array}\right]. \tag{5.13}$$

After realizing that the the Smith form of the central 2×2 block of $A_1^{(1)}$ in (5.13) is $\mathrm{diag}(\lambda-1, (\lambda-1)p^m q^m)$, the desired result follows analogously.

For blocks of type A_2, the unimodular reduction process is similar. We start by collapsing the lower-left and the upper-right $m \times m$ blocks of A_2 by using Lemma 5.4 and Remark 5.5, respectively, and obtain the following equivalence:

$$A_2 \sim \left[\begin{array}{ccc|cccc|cc}
 & & & & & & & & (\lambda-1)\widetilde{I}_{m-1} \\
 & & & & & & & \lambda^m(\lambda-1) & \\ \hline
 & & & & & & \lambda & (\lambda-1)^2 & \\
 & & & & & \lambda & (\lambda-1)^2 & & \\
 & & & & \iddots & \iddots & & & \\
 & & & \lambda & (\lambda-1)^2 & & & & \\ \hline
 & & (\lambda-1) & (\lambda-1)^2 & & & & & \\
 (\lambda-1)\widetilde{I}_{m-1} & & & & & & & &
\end{array}\right] =: A_2^{(1)}.$$

Applying Lemma 5.2 and Remark 5.3 to $A_2^{(1)}$ gives

$$A_2 \sim A_2^{(1)} \sim \left[\begin{array}{c|cccc|c}
 & & & & & (\lambda-1)\widetilde{I}_m \\ \hline
 & & & & \lambda^{m+1} & \\
 & & & \lambda & (\lambda-1)^2 & \\
 & & \iddots & \iddots & & \\
 & \lambda & (\lambda-1)^2 & & & \\ \hline
 (\lambda-1)\widetilde{I}_m & & & & &
\end{array}\right] =: A_2^{(2)}.$$

Further, collapsing the central block of $A_2^{(2)}$ with Lemma 5.4 shows that

$$A_2 \sim A_2^{(1)} \sim A_2^{(2)} \sim \left[\begin{array}{c|cc|c}
 & & & (\lambda-1)\widetilde{I}_m \\ \hline
 & & \lambda^n & \\
 & \widetilde{I}_{n-m-1} & & \\ \hline
 (\lambda-1)\widetilde{I}_m & & &
\end{array}\right] =: A_2^{(3)}\,, \tag{5.14}$$

implying that $A_2^{(3)}$, and consequently A_2, has $2m$ degree-one elementary divisors $(\lambda-1)$ and exactly one elementary divisor λ^n. But since A_2 is T-palindromic, from Remark 2.6(iii) we know that A_2 must also have exactly one infinite elementary divisor ω^n. Further, (5.14) shows that A_2 has trivial singular structure, because A_2 is regular, and therefore, the structural data of the type A_2 block is exactly the same as the one in Table 2 for lists of type $\mathcal{A}_2$.

In the limit case $m = n$ (see (5.8)), one can use similar transformations to obtain

$$A_2 \sim A_2^{(1)} = \left[\begin{array}{c|cc|c}
 & & & (\lambda-1)\widetilde{I}_{m-1} \\ \hline
 & 0 & (\lambda-1)\lambda^m & \\
 & (\lambda-1) & (\lambda-1)^2 & \\ \hline
 (\lambda-1)\widetilde{I}_{m-1} & & &
\end{array}\right]. \tag{5.15}$$

Now the desired result follows from the fact that the Smith form of the central 2×2 block of $A_2^{(1)}$ in (5.15) is given by $\operatorname{diag}(\lambda-1, (\lambda-1)\lambda^m)$.

B blocks: Applying Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $m\times m$ submatrices of B_1, respectively, gives the unimodular equivalence $B_1 \sim B_1^{(1)}$, where

$$B_1^{(1)} := \left[\begin{array}{cc|cccc|cc}
 & & & & & & & (\lambda-1)\widetilde{I}_{m-1} \\
 & & & & & & (-1)^m(\lambda+1)^m(\lambda-1) & \\ \hline
 & & & & & (\lambda+1)^2 & (\lambda-1)^2 & \\
 & & & & (\lambda+1)^2 & (\lambda-1)^2 & & \\
 & & & \iddots & \iddots & & & \\
 & & (\lambda+1)^2 & (\lambda-1)^2 & & & & \\ \hline
 & (\lambda+1)^m(\lambda-1) & (\lambda-1)^2 & & & & & \\
 (\lambda-1)\widetilde{I}_{m-1} & & & & & & &
\end{array}\right].$$

Pre-multiplying $B_1^{(1)}$ by $\operatorname{diag}(I_{m-1}, (-1)^m, I_n)$, and using Lemma 5.2 and Remark 5.3, give the unimodular equivalence $B_1 \sim B_1^{(1)} \sim B_1^{(2)}$, where

$$B_1^{(2)} := \left[\begin{array}{cccccc} & & & & & (\lambda-1)\widetilde{I}_m \\ & & & & (\lambda+1)^{m+2} & \\ & & & (\lambda+1)^2 & (\lambda-1)^2 & \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & \\ & (\lambda+1)^2 & (\lambda-1)^2 & & & \\ & (\lambda+1)^{m+2} \quad (\lambda-1)^2 & & & & \\ (\lambda-1)\widetilde{I}_m & & & & & \end{array}\right].$$

The last step in our unimodular reduction of B_1 consists of applying Lemma 5.4 to the central block of $B_1^{(2)}$, and it produces

$$B_1 \sim B_1^{(1)} \sim B_1^{(2)} \sim \left[\begin{array}{cccc} & & & (\lambda-1)\widetilde{I}_m \\ & & (\lambda+1)^{2n} & \\ & \widetilde{I}_{n-m-1} & & \\ (\lambda-1)\widetilde{I}_m & & & \end{array}\right] =: B_1^{(3)}\,. \tag{5.16}$$

From (5.16) it is clear that the finite spectral structure of $B_1^{(3)}$, and consequently of B_1, consists of $2m$ degree-one elementary divisors $(\lambda-1)$ and exactly one elementary divisor $(\lambda+1)^{2n}$. Now the Index Sum Theorem 3.4 and the regularity of B_1 (see (5.16)) imply that B_1 has no infinite elementary divisors. Therefore the structural data of type B_1 blocks is exactly the same as that described in Table 2 for lists of type $\mathcal{B}_1$.

The limit case $m = n$ (see Remark 5.8) follows directly from $B_1^{(1)}$, without the need of any extra unimodular transformations, and the observation that the Smith form of the central 2×2 block

$$\left[\begin{array}{cc} 0 & (-1)^m(\lambda+1)^m(\lambda-1) \\ (\lambda+1)^m(\lambda-1) & (\lambda-1)^2 \end{array}\right],$$

is given by $\operatorname{diag}(\lambda-1, (\lambda-1)(\lambda+1)^{2m})$.

For blocks of type B_2 we proceed as follows. First, we use Lemma 5.4 and Remark 5.5 to collapse the lower-left and the upper-right $m \times m$ submatrices of B_2, respectively, to obtain the equivalence

$$B_2 \sim B_2^{(1)} := \left[\begin{array}{ccccccc} & & & & & & (\lambda-1)\widetilde{I}_{m-1} \\ & & & & & (\lambda-1)^{m+1} & \\ & & & & (\lambda-1)^2 & \lambda-\lambda^2 & \\ & & & (\lambda-1)^2 & \lambda & & \\ & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & & & \\ & & (\lambda-1)^2 \quad \lambda & & & & \\ & (\lambda-1)^{m+1} & \lambda-1 & & & & \\ (\lambda-1)\widetilde{I}_{m-1} & & & & & & \end{array}\right].$$

Next use Lemma 5.2 and Remark 5.3 to unimodularly reduce $B_2^{(1)}$ to the matrix polynomial $B_2^{(2)}$ defined by

$$B_2^{(2)} := \left[\begin{array}{ccccccc} & & & & & & (\lambda-1)\widetilde{I}_m \\ & & & & & (\lambda-1)^{m+2} & \\ & & & & (\lambda-1)^2 & \lambda & \\ & & & \iddots & \iddots & & \\ & & (\lambda-1)^2 & \lambda & & & \\ & (\lambda-1)^{m+2} & \lambda & & & & \\ (\lambda-1)\widetilde{I}_m & & & & & & \end{array}\right].$$

Finally, applying Lemma 5.4 to the central block of $B_2^{(2)}$ leads to the following chain of equivalences:

$$B_2 \sim B_2^{(1)} \sim B_2^{(2)} \sim \left[\begin{array}{cccc} & & & (\lambda-1)\widetilde{I}_m \\ & & (\lambda-1)^{2n} & \\ & \widetilde{I}_{n-m-1} & & \\ (\lambda-1)\widetilde{I}_m & & & \end{array}\right] =: B_2^{(3)}.$$

Clearly, the list of finite elementary divisors of $B_2^{(3)}$, and consequently of B_2, consists of $2m$ degree-one elementary divisors $(\lambda-1)$ and a single elementary divisor $(\lambda-1)^{2n}$. The fact that B_2 is regular, together with Theorem 3.4, now implies that B_2 has no infinite elementary divisors. Hence the structural data of the type B_2 block is exactly the same as that described in Table 2 for lists of type $\mathcal{B}_2$.

C blocks: Let us first consider blocks of type $C_{1\mathrm{b}}$. We use Lemma 5.4 and Remark 5.5 to first collapse the lower-left and the upper-right $m\times m$ corner blocks of $C_{1\mathrm{b}}$, respectively, and obtain the following equivalence:

$$C_{1\mathrm{b}} \sim \left[\begin{array}{cccccccc} & & & & & & & \theta\widetilde{I}_{m-2} \\ & & & & & & \theta\varphi^{m-1} & \\ & & & & & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & \theta^2 & \\ & & & & H & \theta^2 & & \\ & & & \mathcal{Q}_\ell(\varphi,\varphi,\theta^2) & \theta^2 & & & \\ & \theta\varphi^{m-1} & & \theta^2 & & & & \\ \theta\widetilde{I}_{m-2} & & & & & & & \end{array}\right] =: C_{1\mathrm{b}}^{(1)}.$$

Next we apply Lemma 5.2 and Remark 5.3 at the junctions between the top two upper-right blocks and the bottom two lower-left blocks of $C_{1\mathrm{b}}^{(1)}$, to get the unimodularly equivalent matrix polynomial:

$$C_{1\mathrm{b}}^{(2)} := \left[\begin{array}{cccccccccc} & & & & & & & & & \theta\widetilde{I}_{m-1} \\ & & & & & & & & \varphi^{m+1} & \\ & & & & & & & \varphi^2 & \theta^2 & \\ & & & & & & \iddots & \iddots & & \\ & & & & & \varphi^2 & \theta^2 & & & \\ & & & & H & \theta^2 & & & & \\ & & & \varphi^2 & \theta^2 & & & & & \\ & & \iddots & \theta^2 & & & & & & \\ & \varphi^{m+1} & \begin{array}{c}\varphi^2 \\ \theta^2\end{array} & \iddots & & & & & & \\ \theta\widetilde{I}_{m-1} & & & & & & & & & \end{array}\right].$$

Using Lemma 5.4 and Remark 5.5 to collapse the second and the fourth blocks of $C_{1b}^{(2)}$ along the anti-diagonal gives the following unimodular equivalence:

$$C_{1b}^{(2)} \sim \left[\begin{array}{ccccccccc}
 & & & & & & & & \theta\,\widetilde{I}_{m-1}\\
 & & & & & & & \widetilde{I}_{\ell-1} & \\
 & & & & & & \varphi^{m+2\ell-1} & & \\
 & & & & & -\varphi\theta & \theta^2 & & \\
 & & & & \varphi^2 & \lambda\varphi & & & \\
 & & & \varphi\theta & \varphi & \lambda & & & \\
 & & \varphi^{m+2\ell-1} & \theta^2 & & & & & \\
 & \widetilde{I}_{\ell-1} & & & & & & & \\
\theta\,\widetilde{I}_{m-1} & & & & & & & &
\end{array}\right] =: C_{1b}^{(3)}.$$

Now apply Lemma 5.2 and Remark 5.3 to the 2×2 blocks which are at the junctions of the central block H (see (5.5)) with the adjacent blocks, to get:

$$C_{1b}^{(3)} \sim \left[\begin{array}{ccccccc}
 & & & & & & \theta\,\widetilde{I}_{m-1}\\
 & & & & & \widetilde{I}_{\ell} & \\
 & & & & -\varphi^{m+2\ell}\theta & & \\
 & & & \varphi^2 & \lambda\varphi & & \\
 & & \varphi^{m+2\ell}\theta & \varphi & \lambda & & \\
 & \widetilde{I}_{\ell} & & & & & \\
\theta\,\widetilde{I}_{m-1} & & & & & &
\end{array}\right]. \tag{5.17}$$

By computing all the nonzero minors, it can now be seen that the Smith form of the central 3×3 block in (5.17) is equal to $\operatorname{diag}\big(1, \varphi^n\theta, \varphi^n\theta\big)$. Hence, the Smith form of the whole C_{1b} block is given by

$$\operatorname{diag}\Big(I_{n-m}\,, \theta\, I_{2m-2}\,, \varphi^n\theta\,, \varphi^n\theta\Big),$$

so that the finite spectral structure of the block C_{1b} is the one indicated in Table 2 for the list of type $\mathcal{C}_1$. Equivalence (5.17) implies that C_{1b} block is regular, and from Theorem 3.4 we conclude that C_{1b} has no infinite elementary divisors. Thus the structural data of type C_{1b} blocks is exactly the same as that described in Table 2 for lists of type $\mathcal{C}_1$ in the case where m is even.

When it comes to the type C_{1a} block, it is useful to observe that a C_{1a} block differs from a C_{1b} block only in the absence of the central block H and the adjacent $*, \bullet$ entries, and in the size of the lower-left and upper-right corner blocks. Thus performing the unimodular reduction of C_{1a}, analogous to the one used for the block C_{1b}, gives the following equivalence

$$C_{1a} \sim \left[\begin{array}{cccccc}
 & & & & & \theta\,\widetilde{I}_{m}\\
 & & & & \widetilde{I}_{\ell-1} & \\
 & & & \varphi^{m+2\ell} & & \\
 & & \varphi^{m+2\ell} & & & \\
 & \widetilde{I}_{\ell-1} & & & & \\
\theta\,\widetilde{I}_{m} & & & & &
\end{array}\right] =: C_{1a}^{(1)}. \tag{5.18}$$

Now (5.18), together with the relation $m + 2\ell = n$, implies that the Smith form of the type C_{1a} block is $\operatorname{diag}\big(I_{2\ell-2}\,, \theta\, I_{2m}\,, \varphi^n\,, \varphi^n\big)$. Furthermore, from (5.18) we know that C_{1a} is regular, and so Theorem 3.4 now implies that C_{1a} has no infinite elementary divisors. Hence the structural data of the block C_{1a} is exactly the same as that described in Table 2 for lists of type $\mathcal{C}_1$ when m is odd.

For the limit case $m = n$ (i.e., $\ell = 0$) for blocks of type C_{1a}, we can get (see (5.9))

$$C_{1a} \sim C_{1a}^{(1)} = \left[\begin{array}{cccc} & & & (\lambda-1)\widetilde{I}_m \\ & 0 & (\lambda-1)(\lambda+1)^m & \\ & (\lambda-1)(\lambda+1)^m & 0 & \\ (\lambda-1)\widetilde{I}_m & & & \end{array}\right],$$

and the result immediately follows.

For blocks of type C_2 the arguments are very similar to the ones for blocks of type C_1. We start with the blocks of type C_{2b}. Applying Lemma 5.4 and Remark 5.5 to the lower-left and the upper-right $m \times m$ submatrices of C_{2b}, respectively, gives the following equivalence:

$$C_{2b} \sim \left[\begin{array}{ccccc} & & & & \begin{array}{cc} & \theta\widetilde{I}_{m-1} \\ \theta^{m+1} & \end{array} \\ & & & \mathcal{Q}_{\ell-1}(\theta,\theta,\lambda) & \mathrm{rev}_2\theta \\ & & K & \lambda & \\ & \mathcal{Q}_{\ell-1}(\theta,\theta,\lambda) & \lambda & & \\ \begin{array}{cc} & \theta^{m+1} \\ \theta\,\widetilde{I}_{m-1} & \end{array} & \theta & & & \end{array}\right] =: C_{2b}^{(1)}.$$

Next we apply Lemma 5.2 and Remark 5.3 at the junctions between the top two upper right blocks and the bottom two lower left blocks of $C_{2b}^{(1)}$, to obtain

$$C_{2b}^{(1)} \sim C_{2b}^{(2)} := \left[\begin{array}{ccccc} & & & & \theta\,\widetilde{I}_m \\ & & & \begin{array}{cccc} & & & \theta^{m+2} \\ & & \theta^2 & \lambda \\ & \iddots & \iddots & \\ \theta^2 & \lambda & & \end{array} & \\ & & K & \lambda & \\ & \begin{array}{cccc} & & & \theta^2 \\ & & \iddots & \lambda \\ & \theta^2 & \iddots & \\ \theta^{m+2} & \lambda & & \end{array} & \lambda & & \\ \theta\,\widetilde{I}_m & & & & \end{array}\right].$$

Now we collapse the second and the fourth blocks along the anti-diagonal of $C_{2b}^{(2)}$ using Lemma 5.4 and Remark 5.5, to obtain the following equivalence:

$$C_{2b}^{(2)} \sim \left[\begin{array}{cccccc} & & & & & \theta\,\widetilde{I}_m \\ & & & & \begin{array}{cc} & \widetilde{I}_{\ell-2} \\ \theta^{m+2\ell-2} & \end{array} & \\ & & & \begin{array}{ccc} & & \theta^2 \\ & \theta^2 & -\lambda\theta \\ \theta^2 & \theta & -\lambda \end{array} & \lambda & \\ & \begin{array}{cc} & \theta^{m+2\ell-2} \\ \widetilde{I}_{\ell-2} & \end{array} & \lambda & & & \\ \theta\,\widetilde{I}_m & & & & & \end{array}\right] =: C_{2b}^{(3)}.$$

Finally, applying Lemma 5.2 and Remark 5.3 to the 2×2 blocks which are at the junctions of the central block K with the adjacent blocks gives

$$C_{2\mathrm{b}}^{(3)} \sim \left[\begin{array}{cccccc} & & & & & \theta\,\widetilde{I}_m \\ & & & & \widetilde{I}_{\ell-1} & \\ & & \begin{array}{ccc} & & \theta^{m+2\ell} \\ & \theta^2 & -\lambda\theta \\ \theta^{m+2\ell} & \theta & -\lambda \end{array} & & & \\ & \widetilde{I}_{\ell-1} & & & & \\ \theta\,\widetilde{I}_m & & & & & \end{array}\right]. \tag{5.19}$$

By a straightforward computation of the nonzero minors (or either by elementary row and column operations), it can be seen that the Smith form of the central 3×3 block in (5.19) is equal to $\operatorname{diag}\left(1\,,\,\theta^n\,,\,\theta^n\right)$ (recall that $n = m+2\ell+1$). Hence the Smith form of the whole $C_{2\mathrm{b}}$ block is given by $\operatorname{diag}\left(I_{2\ell-1}\,,\,\theta I_{2m}\,,\,\theta^n\,,\,\theta^n\right)$, so that the finite spectral structure of the block $C_{2\mathrm{b}}$ is the one indicated in Table 2 for the list of type $\mathcal{C}_2$ with m odd. Equivalence (5.19) implies that $C_{2\mathrm{b}}$ is regular, and from Theorem 3.4 we conclude that $C_{2\mathrm{b}}$ has no infinite elementary divisors. Thus the structural data of the block $C_{2\mathrm{b}}$ is exactly the same as that described in Table 2 for lists of type $\mathcal{C}_2$ with m odd.

The arguments for the $C_{2\mathrm{a}}$ block follow as a particular case of the ones for $C_{2\mathrm{b}}$, in a similar fashion as the ones for the $C_{1\mathrm{a}}$ block follow from the arguments used for the $C_{1\mathrm{b}}$ block.

M blocks: We start by investigating the spectral and singular structure of type $M_{1\mathrm{a}}$ blocks when $h > 0$. Performing the following elementary *row* operations on $M_{1\mathrm{a}}$

$$\begin{array}{rcl} \mathrm{Row}_{m+h} & \longrightarrow & \mathrm{Row}_{m+h} - \lambda^2 \cdot \mathrm{Row}_{m+h+1}\,, \\ \mathrm{Row}_{m+h-1} & \longrightarrow & \mathrm{Row}_{m+h-1} - \lambda^2 \cdot \mathrm{Row}_{m+h}\,, \\ & \vdots & \\ \mathrm{Row}_{m+1} & \longrightarrow & \mathrm{Row}_{m+1} - \lambda^2 \cdot \mathrm{Row}_{m+2}\,, \end{array} \tag{5.20}$$

only affects the second block along the anti-diagonal of $M_{1\mathrm{a}}$, i.e., $\mathcal{Q}(\lambda,\lambda,1)$. In fact, the effect of the unimodular transformations from (5.20) is that all anti-diagonal entries in the block $\mathcal{Q}_h(\lambda,\lambda,1)$ become 0, i.e., $\mathcal{Q}_h(\lambda,\lambda,1)$ is replaced by $\mathcal{Q}_h(0,0,1)$. We continue by performing the following elementary unimodular *row* operations

$$\begin{array}{rcl} \mathrm{Row}_{m} & \longrightarrow & \mathrm{Row}_{m} + \lambda \cdot \mathrm{Row}_{m+1}\,, \\ \mathrm{Row}_{m-1} & \longrightarrow & \mathrm{Row}_{m-1} + \lambda \cdot \mathrm{Row}_{m}\,, \\ & \vdots & \\ \mathrm{Row}_{1} & \longrightarrow & \mathrm{Row}_{1} + \lambda \cdot \mathrm{Row}_{2}\,, \end{array} \tag{5.21}$$

so that all anti-diagonal entries in the upper right block $\mathcal{Q}_m(\lambda,-\theta,\theta)$ become 0 as well. In summary, the effect of performing elementary row operations from (5.20) and (5.21) on $M_{1\mathrm{a}}$ gives the following equivalence:

$$M_{1\mathrm{a}} \sim \left[\begin{array}{ccccc} & & & & \mathcal{Q}_m(0,0,\theta) \\ & & & \mathcal{Q}_h(0,0,1) & \theta \\ & & 0 & 1 & \\ & \mathcal{Q}_h(1,1,\lambda^2) & \lambda^2 & & \\ \mathcal{Q}_m(1,\theta,-\lambda\theta) & \mathrm{rev}_2\theta & & & \end{array}\right] =: M_{1\mathrm{a}}^{(1)}.$$

We continue to reduce $M_{1\mathrm{a}}$ by performing the following elementary *column* operations on $M_{1\mathrm{a}}^{(1)}$:

$$\begin{array}{rcl} \mathrm{Col}_2 & \longrightarrow & \mathrm{Col}_2 + \lambda\,\mathrm{Col}_1\,, \\ \mathrm{Col}_3 & \longrightarrow & \mathrm{Col}_3 + \lambda\,\mathrm{Col}_2\,, \\ & \vdots & \\ \mathrm{Col}_{m+1} & \longrightarrow & \mathrm{Col}_{m+1} + \lambda\,\mathrm{Col}_m\,. \end{array} \tag{5.22}$$

The effect of these unimodular transformations is that all $-\lambda\theta$ entries below the main anti-diagonal in the lower left block $\mathcal{Q}_m(1,\theta,-\lambda\theta)$ become 0, as well as the entry $\mathrm{rev}_2\theta = -\lambda\theta$ at the junction of the fourth and the fifth block along the anti-diagonal of $M_{1\mathrm{a}}^{(1)}$. The last set of elementary *column* operations we perform is given by

$$\begin{array}{rcl} \mathrm{Col}_{m+2} & \longrightarrow & \mathrm{Col}_{m+2} - \lambda^2\,\mathrm{Col}_{m+1}\,, \\ \mathrm{Col}_{m+3} & \longrightarrow & \mathrm{Col}_{m+3}\; - \lambda^2\,\mathrm{Col}_{m+2}, \\ & \vdots & \\ \mathrm{Col}_{m+h+1} & \longrightarrow & \mathrm{Col}_{m+h+1} - \lambda^2\,\mathrm{Col}_{m+h}\,, \end{array} \tag{5.23}$$

and as their consequence, all λ^2 entries below the main anti-diagonal in the block $\mathcal{Q}_h(1,1,\lambda^2)$, as well as the entry λ^2 at the junction of the zero block and $\mathcal{Q}_h(1,1,\lambda^2)$, become zero.

Performing all of the elementary operations from (5.20) – (5.23) on $M_{1\mathrm{a}}$ gives the following equivalence

$$M_{1\mathrm{a}} \sim M_{1\mathrm{a}}^{(1)} \sim \left[\begin{array}{ccccc} & & & & (\lambda-1)\widetilde{N}_m \\ & & & \widetilde{N}_h & \lambda-1 \\ & & 0 & 1 & \\ & \widetilde{I}_h & & & \\ (\lambda-1)\,\widetilde{I}_m & & & & \end{array}\right], \tag{5.24}$$

and so the Smith form of $M_{1\mathrm{a}}$ is given by $\mathrm{diag}\Big(I_{2h}\,,\,(\lambda-1)I_{2m}\,,\,0\Big)$. Hence, the complete list of finite elementary divisors of the block $M_{1\mathrm{a}}$ consists of exactly $2m$ degree-one elementary divisors $(\lambda-1)$, and is identical to the finite elementary divisor sublist of the type $\mathcal{M}_1$ list given in Table 2 when m is even. The fact that $M_{1\mathrm{a}}$ is T-palindromic and that it has no elementary divisors of the form λ^β implies that $M_{1\mathrm{a}}$ also has no infinite elementary divisors, i.e., $\delta_\infty(M_{1\mathrm{a}}) = 0$ (see Remark 2.6(iii)).

Now we turn to the question of determining the singular structure of $M_{1\mathrm{a}}$. The Smith form of $M_{1\mathrm{a}}$ gives $\dim\mathcal{N}_r(M_{1\mathrm{a}}) = \dim\mathcal{N}_\ell(M_{1\mathrm{a}}) = 1$. Hence $M_{1\mathrm{a}}$ has just one left and one right minimal index, denoted by η and ε, respectively. From the Index Sum Theorem 3.4 we have

$$\mu(M_{1\mathrm{a}}) = 2\cdot\mathrm{rank}\,(M_{1\mathrm{a}}) - \delta_{fin}(M_{1\mathrm{a}}) - \delta_\infty(M_{1\mathrm{a}}), \quad \text{which implies} \quad \mu(M_{1\mathrm{a}}) = 4k\,. \tag{5.25}$$

Observing that $M_{1\mathrm{a}}$ is a T-palindromic matrix polynomial and recalling Remark 2.6(iv), together with (5.25), give $\varepsilon = \eta = 2k$. Thus the structural data of the block $M_{1\mathrm{a}}$ is exactly the same as that described in Table 2 for lists of type $\mathcal{M}_1$ when m is even.

The limit case $h = 0$ (i.e., $k = \ell$) for type $M_{1\mathrm{a}}$ blocks follows directly from the strict equivalence (see (5.10)) $M_{1\mathrm{a}} \sim \mathrm{diag}(\theta I_{2m}, 0)$, and the application of Theorem 3.4, which reads $\mu + 2m = 2\cdot 2m$, so $\mu = 2m$. Since $M_{1\mathrm{a}}$ is T-palindromic and it has just one right and one left minimal index, ε, η, respectively, it must be that $\varepsilon = \eta = m$.

For the type $M_{1\mathrm{b}}$ block, the analysis of the singular structure will be identical, whereas it only takes an additional one row and one column elementary unimodular operation to obtain an analog of (5.24) for the block $M_{1\mathrm{b}}$. For the sake of brevity and non-repetitiveness we omit the details.

The argument for the $M_{2\mathrm{a}}$ and $M_{2\mathrm{b}}$ blocks are nearly identical to those for the corresponding $M_{1\mathrm{a}}$ and $M_{2\mathrm{b}}$ blocks, so we omit the details for these blocks as well. □

6. Palindromic quadratic realization and some consequences

The main result of this work now easily follows from the results established in the previous sections.

Theorem 6.1. (T-Palindromic Quadratic Realization Theorem). *A list of elementary divisors and minimal indices $\mathcal{L}$ is p-quad realizable if and only if $\mathcal{L}$ is p-quad admissible.*

Proof. Let us first assume that $\mathcal{L}$ is p-quad realizable, and let Q be a quadratic T-palindromic matrix polynomial whose list of elementary divisors and minimal indices is $\mathcal{L}$. First, $\mathcal{L}$ has p-quad symmetry (see, for instance, Corollaries 8.1 and 8.2 in [35]). Secondly, $\mathcal{L}$ satisfies

$$\gamma(\mathcal{L}) \leq \operatorname{rank} Q = \frac{1}{2}(\delta(\mathcal{L}) + \mu(\mathcal{L})),$$

where the second equality is an immediate consequence of Theorem 3.4. Hence $\mathcal{L}$ is p-quad admissible.

We now prove the converse. Assume that $\mathcal{L}$ is a p-quad admissible list of elementary divisors and minimal indices. By Theorem 3.16, $\mathcal{L}$ is p-quad partitionable into p-quad irreducible sublists appearing in Tables 1 and 2. From Theorem 5.12 we have that each of those sublists are p-quad realizable, and that the direct sum of these p-quad realizations is in fact a p-quad realization for $\mathcal{L}$ (see Lemma 5.1). □

One of the practical motivations to consider p-quad realizability is to determine, given a T-palindromic matrix polynomial P with grade at least two, whether or not there exists a T-palindromic matrix polynomial of grade two that is spectrally equivalent to P. The formal definition of spectral equivalence of polynomials P and Q can be found in [11, Def. 3.2], but in the end it reduces to P and Q having the same spectral structure and the same number of minimal indices (left and right). What is important for us in this setting is the following definition.

Definition 6.2. [11, Thm. 4.1] *Let P be a matrix polynomial. A quadratic matrix polynomial Q is a* strong quadratification *of P if the following three conditions hold:*

(a) $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(Q)$, *and* $\dim \mathcal{N}_\ell(P) = \dim \mathcal{N}_\ell(Q)$,

(b) *P and Q have the same finite elementary divisors.*

(c) *P and Q have the same infinite elementary divisors.*

Unlike what happens with linearizations, there are always T-palindromic strong quadratifications for any T-palindromic polynomial of even grade. This result is in the same direction as [11, Cor. 5.9], which states that any (unstructured) matrix polynomial of even grade has a strong quadratification.

Corollary 6.3. *Any T-palindromic matrix polynomial of even (nonzero) grade has a T-palindromic strong quadratification.*

Proof. Let $P(\lambda)$ be a T-palindromic matrix polynomial of even grade ℓ, and let $\mathcal{L}$ be the list of the elementary divisors and minimal indices of P. Then $\mathcal{L}$ satisfies the following:

(a) $\gamma(\mathcal{L}) \leq \operatorname{rank} P = \frac{1}{\ell}(\delta(\mathcal{L}) + \mu(\mathcal{L})) \leq \frac{1}{2}(\delta(\mathcal{L}) + \mu(\mathcal{L}))$.

(b) $\mathcal{L}$ has p-quad symmetry.

Note that the equality in (a) is an immediate consequence of Theorem 3.4, while (b) follows from [35, Cors. 8.1-8.2] and [7, Thm. 3.6].

This shows that $\mathcal{L}$ is a p-quad admissible list and so, by Theorem 6.1, there is a quadratic T-palindromic matrix polynomial $Q(\lambda)$ whose list of elementary divisors and minimal indices is $\mathcal{L}$. Then Q is a strong quadratification of P, because both P and Q have the same finite and infinite elementary divisors, and the dimensions of the left and right nullspaces of P and Q coincide. □

By contrast with Corollary 6.3, not every T-palindromic matrix polynomial of odd grade has a T-palindromic strong quadratification. For instance, the (scalar) T-palindromic polynomial $p(\lambda) = (\lambda+1)^3$ of grade 3 has no T-palindromic strong quadratification, since the odd degree elementary divisors associated with $a = -1$ do not have even multiplicity (so that the list of elementary divisors of p does not have p-quad symmetry). However, Theorem 6.1 shows that this is the only obstruction to the existence of T-palindromic strong quadratifications.

Corollary 6.4. *Let P be a T-palindromic matrix polynomial with odd grade $\ell \geq 3$. Then, the following conditions are equivalent:*

(i) *There is a T-palindromic strong quadratification of P.*

(ii) *Any odd degree elementary divisor of P associated with $a = -1$ has even multiplicity.*

Proof. Let $\mathcal{L}(P)$ be the list of elementary divisors and minimal indices of P and recall that

$$\mathcal{L}(P) \;=\; \Big\{\, \mathcal{L}_{fin}(P)\,;\, \mathcal{L}_{\infty}(P)\,;\, \mathcal{L}_{left}(P)\,;\, \mathcal{L}_{right}(P)\,\Big\}\,.$$

We first prove the implication (i) $\Rightarrow$ (ii). Assume that Q is a T-palindromic strong quadratification of P. Then, by Theorem 6.1, the list $\mathcal{L}(Q)$ has p-quad symmetry. The fact that Q is a strong quadratification for P implies that P and Q have identical finite and infinite elementary divisors, i.e.,

$$\mathcal{L}_{fin}(Q) \;=\; \mathcal{L}_{fin}(P) \quad \text{and} \quad \mathcal{L}_{\infty}(Q) \;=\; \mathcal{L}_{\infty}(P)\,. \tag{6.1}$$

Now relation (6.1) imply that $\mathcal{L}(P)$ satisfies (1) in Definition 3.6, and so condition (ii) follows in particular from (1c) in Definition 3.6.

Next we prove the implication (ii) $\Rightarrow$ (i). By Theorem 6.1, it is enough to prove that $\mathcal{L}(P)$ is p-quad admissible. First note that $\mathcal{L}(P)$ satisfies conditions (1a), (1b) and (2) in Definition 3.6 (see [35, Cor. 8.1] and [7, Thm. 3.6]). Also, any odd degree elementary divisor of $\mathcal{L}(P)$ associated with $a = 1$ has even multiplicity (see [35, Cor. 8.2]). All these facts, together with the hypothesis (ii), imply that $\mathcal{L}$ has p-quad symmetry. Finally, using Theorem 3.4, we get:

$$\gamma(\mathcal{L}) \leq \operatorname{rank} P = \frac{1}{\ell}\Big(\delta(\mathcal{L}) + \mu(\mathcal{L})\Big) \leq \frac{1}{2}\Big(\delta(\mathcal{L}) + \mu(\mathcal{L})\Big).$$

Hence, $\mathcal{L}(P)$ is p-quad admissible. □

Remark 6.5. *Let P be a T-palindromic matrix polynomial with odd grade satisfying condition (ii) in* Corollary 6.4*. Then $\mathcal{L}(P)$ is p-quad symmetric, and this, together with* Theorem 3.4*, imply that* $\operatorname{rank} P$ *is even. If P is regular, this in turn implies that the size of P is even. Then,* Corollary 6.4 *is in accordance with the analogous result for regular unstructured polynomials in* [30].

6.1. Separability of spectral and singular structures

The Kronecker canonical form immediately displays a fundamental but often unsung property possessed by every matrix pencil — the *separability* of its spectral and singular structures. By this we mean the equivalence of every pencil $L(\lambda)$ to a direct sum of pencils $L_{\mathrm{r}}(\lambda) \oplus L_{\mathrm{s}}(\lambda)$, where L_{r} is regular and L_{s} is *completely singular*. Note that a matrix polynomial P is completely singular if its structural data consists only of minimal indices, and contains no elementary divisors at all (neither finite nor infinite); the quadratic T-palindromic matrix polynomial $L(\lambda)$ in (5.5) illustrates this phenomenon. For many other examples and further information on completely singular polynomials see [12].

The separability property is often taken for granted, since it is well known to hold for all pencils. But for any degree (or grade) d larger than 1, there are matrix polynomials whose spectral and singular structure cannot be separated in this way (note that this was shown for $d = 2$ in [10]). Examples of non-separability within the class $\mathscr{C}$ of quadratic T-palindromic matrix polynomials can be found earlier in this paper; quadratic blocks of type M in Table 6 are non-separable within $\mathscr{C}$ because they realize p-quad *irreducible* lists of type $\mathcal{M}$, containing a mixture of both minimal indices and elementary divisors. In this context, then, it is natural to try to characterize those p-quad admissible structural data lists that can be realized in a separable fashion. Thus we consider the following special subproblem of the T-palindromic QRP:

> *Given a p-quad admissible list $\mathcal{L}$ of elementary divisors and minimal indices, determine whether or not $\mathcal{L}$ can be realized by a T-palindromic quadratic matrix polynomial in which the spectral and singular structures are* separated.

To this end, we introduce the following notation and terminology.

Definition 6.6. (p-quad rs-realizability). *A list $\mathcal{L} := (\mathcal{L}_{reg}, \mathcal{L}_{sing})$, where $\mathcal{L}_{reg}$ is a list of elementary divisors and $\mathcal{L}_{sing}$ is a list of minimal indices, is said to be* p-quad rs-realizable *over a field $\mathbb{F}$ if there exists a T-palindromic quadratic matrix polynomial in direct sum form $P = P_r \oplus P_s$ such that:*

(i) *P_r is regular, with elementary divisors exactly those in $\mathcal{L}_{reg}$, and*

(ii) *P_s is completely singular, with minimal indices exactly those in $\mathcal{L}_{sing}$.*

Note that in Definition 6.6 it is implicit that $\mathcal{L}$ must be p-quad admissible (hence realizable), otherwise there is no chance for $\mathcal{L}$ to be p-quad rs-realizable. The type M blocks of Table 6 all give examples of realizations for p-quad admissible lists that are *not* p-quad rs-realizable. Clearly a necessary and sufficient condition for a list $\mathcal{L} = (\mathcal{L}_{reg}, \mathcal{L}_{sing})$ to be p-quad rs-realizable is that each of $\mathcal{L}_{reg}$ and $\mathcal{L}_{sing}$ are individually p-quad realizable. However, in Theorem 6.7 we provide a simpler characterization of p-quad rs-realizability.

Theorem 6.7. *A list $\mathcal{L} := (\mathcal{L}_{reg}, \mathcal{L}_{sing})$, where $\mathcal{L}_{reg}$ is a list of elementary divisors and $\mathcal{L}_{sing}$ is a list of minimal indices, is p-quad rs-realizable if and only if $\mathcal{L}$ satisfies the following two conditions:*

(i) *$\gamma(\mathcal{L}) \leq \frac{1}{2}\delta(\mathcal{L})$, and*

(ii) *$\mathcal{L}$ has p-quad symmetry.*

Proof. We start by observing that conditions (i)–(ii) in the statement are equivalent to saying that $\mathcal{L}$ is p-quad admissible and satisfies condition (i). Also, one can easily check that the following relations hold

$$\begin{array}{c}\delta(\mathcal{L}) = \delta(\mathcal{L}_{reg}) \\ \delta(\mathcal{L}_{sing}) = 0\end{array}, \qquad \begin{array}{c}\gamma(\mathcal{L}) = \gamma(\mathcal{L}_{reg}) \\ \gamma(\mathcal{L}_{sing}) = 0\end{array}, \qquad \text{and} \qquad \begin{array}{c}\mu(\mathcal{L}) = \mu(\mathcal{L}_{sing}) \\ \mu(\mathcal{L}_{reg}) = 0\end{array}. \tag{6.2}$$

Now we are ready to prove the "only if" part of the statement. Since $\mathcal{L}$ is p-quad realizable, it is also p-quad admissible by Theorem 6.1. Moreover, the assumption that $\mathcal{L}$ is rs-realizable implies that $\mathcal{L}_{reg}$ is p-quad realizable, which together with (6.2) give

$$\gamma(\mathcal{L}) = \gamma(\mathcal{L}_{reg}) \leq \frac{1}{2}\left(\delta(\mathcal{L}_{reg}) + \mu(\mathcal{L}_{reg})\right) = \frac{1}{2}\delta(\mathcal{L}_{reg}) = \frac{1}{2}\delta(\mathcal{L});$$

here the inequality follows again from Theorem 6.1.

In order to prove the converse it suffices to show that both $\mathcal{L}_{reg}$ and $\mathcal{L}_{sing}$ are p-quad realizable, or by Theorem 6.1, that both lists are p-quad admissible. Since $\mathcal{L}$ is p-quad admissible, it has p-quad symmetry, and so both $\mathcal{L}_{reg}$ and $\mathcal{L}_{sing}$ have p-quad symmetry as well. Now the proof will be complete if we can show that both $\mathcal{L}_{reg}$ and $\mathcal{L}_{sing}$ also satisfy condition (a) in Definition 3.7. From (6.2) it is clear that that is the case for $\mathcal{L}_{sing}$. As for $\mathcal{L}_{reg}$, we also use (6.2) and the hypothesis in the statement (i) to obtain

$$\gamma(\mathcal{L}_{reg}) = \gamma(\mathcal{L}) \leq \frac{1}{2}\delta(\mathcal{L}) = \frac{1}{2}\delta(\mathcal{L}_{reg}) = \frac{1}{2}\left(\delta(\mathcal{L}_{reg}) + \mu(\mathcal{L}_{reg})\right),$$

which completes the proof. □

7. The T-Alternating QRP

Another important family of structured matrix polynomials that arise in applications consists of *T-alternating matrix polynomials* [33, 34, 40]. A particular subset of those polynomials, the T-even matrix polynomials, is the main object of study in this section.

Definition 7.1 (T-even, [33])**.** *A nonzero $n \times n$ matrix polynomial P of grade $k \geq 0$ is said to be T–*even *if $P(\lambda)^T = P(-\lambda)$.*

In [34, Thm. 5.4] the authors showed that for any T-even matrix polynomial of odd degree, one can explicitly construct a T-even strong linearization. Furthermore, [34, Thm. 5.5] shows that not all T-even matrix polynomials of even degree have a T-even strong linearization. One of the main results in this section shows that every T-even matrix polynomial of even grade always has a T-even strong quadratification. It turns out that this is an easy consequence of the solution of what we call the T-even QRP problem. Namely, given a list $\widehat{\mathcal{L}}$ of elementary divisors and minimal indices, we determine if there exists a T-even quadratic matrix polynomial Q such that $\mathcal{L}(Q) = \widehat{\mathcal{L}}$ and, in the affirmative case, show how to construct such a Q in a simple and transparent way.

The key tools for solving the T-even QRP are two special Möbius transformations of matrix polynomials. More specifically, the *Cayley transformations* $\mathbf{c}_{+1}, \mathbf{c}_{-1} : \mathbb{F}_\infty \to \mathbb{F}_\infty$ (where $\mathbb{F}_\infty := \mathbb{F} \cup \{\infty\}$) are defined by:

$$\mathbf{c}_{+1}(\mu) = \frac{1+\mu}{1-\mu}, \qquad \mathbf{c}_{-1}(\mu) = \frac{\mu-1}{\mu+1},$$

where $\mathbf{c}_{+1}(\infty) = -1$, $\mathbf{c}_{-1}(\infty) = 1$, $\mathbf{c}_{+1}(1) = \infty$, and $\mathbf{c}_{-1}(-1) = \infty$. It is straightforward to see that $\mathbf{c}_{-1} = (\mathbf{c}_{+1})^{-1}$. Both $\mathbf{c}_{+1}$ and $\mathbf{c}_{-1}$ are rational transformations of $\mathbb{F}_\infty$, and they induce Möbius transformations on the space of all grade k matrix polynomials P given by [37, Ex. 3.10]:

$$\mathcal{C}_{+1}(P)(\mu) := (1-\mu)^k P\left(\frac{1+\mu}{1-\mu}\right), \qquad \mathcal{C}_{-1}(P)(\mu) := (\mu+1)^k P\left(\frac{\mu-1}{\mu+1}\right). \tag{7.1}$$

In particular, we have that P is a T-palindromic quadratic matrix polynomial if and only if $\mathcal{C}_{+1}(P)$ (or $\mathcal{C}_{-1}(P)$) is a T-even quadratic matrix polynomial [37, Table 9.1]. Equivalently, $\mathcal{C}_{+1}$ and $\mathcal{C}_{-1}$ each give a one-to-one correspondence between the following spaces of structured matrix polynomials of the same size:

$$\{T\text{-palindromic quadratic matrix polynomials}\} \longleftrightarrow \{T\text{-even quadratic matrix polynomials}\}.$$

Exactly these bijections, together with corresponding ones for lists of elementary divisors and minimal indices, will allow us to easily solve the T-even QRP by leveraging the solution of the T-palindromic QRP.

Definition 7.2. *Let $\mathcal{L}$ be a list of elementary divisors and minimal indices. Then $\kappa_{+1}(\mathcal{L})$ and $\kappa_{-1}(\mathcal{L})$ are new lists of elementary divisors and minimal indices obtained from $\mathcal{L}$ in the following way:*

(1) (a) *For $a \neq 1$, the finite elementary divisors of the form $(\lambda - a)^\beta$ in $\mathcal{L}$ are replaced by the elementary divisors of the form $(\lambda - \mathbf{c}_{+1}(a))^\beta$ in $\kappa_{+1}(\mathcal{L})$.*

(b) *Elementary divisors of the form $(\lambda - 1)^\beta$ in $\mathcal{L}$ are replaced by the infinite elementary divisors ω^β in $\kappa_{+1}(\mathcal{L})$.*

(c) *Infinite elementary divisors of the form ω^β in $\mathcal{L}$ are replaced by the finite elementary divisors $(\lambda + 1)^\beta$ in $\kappa_{+1}(\mathcal{L})$.*

(2) (a) *For $a \neq -1$, the finite elementary divisors of the form $(\lambda - a)^\beta$ in $\mathcal{L}$ are replaced by the elementary divisors of the form $(\lambda - \mathbf{c}_{-1}(a))^\beta$ in $\kappa_{-1}(\mathcal{L})$.*

(b) *Elementary divisors of the form $(\lambda + 1)^\beta$ in $\mathcal{L}$ are replaced by the infinite elementary divisors ω^β in $\kappa_{-1}(\mathcal{L})$.*

(c) *Infinite elementary divisors of the form ω^β in $\mathcal{L}$ are replaced by the finite elementary divisors $(\lambda - 1)^\beta$ in $\kappa_{-1}(\mathcal{L})$.*

(3) *The left and right minimal indices in $\kappa_{+1}(\mathcal{L})$ and in $\kappa_{-1}(\mathcal{L})$ are each identical to the ones in $\mathcal{L}$.*

The importance of Definition 7.2 stems from the fact that if $\mathcal{L}$ represents the structural data of some polynomial Q, then $\kappa_{+1}(\mathcal{L})$ (resp., $\kappa_{-1}(\mathcal{L})$) will be the structural data list of $\mathcal{C}_{-1}(Q)$ (resp., $\mathcal{C}_{+1}(Q)$) [37, Thms. 5.3, 7.5].

We now introduce notions for the T-even QRP that are analogous to the ones from Section 3 for the T-palindromic QRP.

Definition 7.3. (e-quad Realizability). *A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be* e-quad realizable over the field $\mathbb{F}$ *if there exists some T-even quadratic matrix polynomial over $\mathbb{F}$ whose elementary divisors and minimal indices are exactly those in $\mathcal{L}$.*

Definition 7.4. (e-quad Symmetry). *A list $\mathcal{L}$ of elementary divisors and minimal indices over an algebraically closed field $\mathbb{F}$ is said to have* e-quad symmetry *if the following conditions are satisfied:*

(1) (a) *for any $a \neq 0, \infty$, and $\beta \geq 1$, the elementary divisors $(\lambda - a)^{\beta}$ and $(\lambda + a)^{\beta}$ appear in $\mathcal{L}$ with the same multiplicity (i.e., they appear exactly the same number of times, perhaps zero),*

(b) *any odd degree elementary divisor in $\mathcal{L}$ associated with the eigenvalues $a = 0, \infty$ has even multiplicity.*

(2) *the ordered sublist of left minimal indices is identical to the ordered sublist of right minimal indices.*

Definition 7.5. (e-quad Admissibility). *A list $\mathcal{L}$ of elementary divisors and minimal indices is said to be* e-quad admissible *if the following conditions are satisfied:*

(a) $\gamma \leq \frac{1}{2}(\delta + \mu)$,

(b) *$\mathcal{L}$ has e-quad symmetry.*

As in the case of p-quad admissibility (see Definition 3.7), condition (b) in Definition 7.5 implies condition (c) in Remark 3.8.

Finally, we are ready to prove the main result of this section, which can be viewed as the counterpart for T-even matrix polynomials of Theorem 6.1 for T-palindromic matrix polynomials.

Theorem 7.6. (e-quad Realization Theorem). *A list of elementary divisors and minimal indices $\mathcal{L}$ is e-quad realizable if and only if $\mathcal{L}$ is e-quad admissible.*

Proof. Assume that $\mathcal{L}$ is an e-quad realizable list. Then $\mathcal{L}$ satisfies (a) in Definition 7.5 as an immediate consequence of Theorem 3.4 (see the proof of Theorem 6.1). It also satisfies condition (b) in Definition 7.5 due to [34, Thm. 4.2]. Hence, $\mathcal{L}$ is e-quad admissible.

Conversely, let $\mathcal{L}$ be an e-quad admissible list of elementary divisors and minimal indices, so that $\mathcal{L}$ has e-quad symmetry. First, note that the list $\kappa_{+1}(\mathcal{L})$ has p-quad symmetry, i.e., $\kappa_{+1}(\mathcal{L})$ satisfies condition (b) in Definition 3.7. This can be seen by observing that the role played by the eigenvalues $a = 0, \infty$ in the list $\mathcal{L}$ is now played by $\mathbf{c}_{+1}(0) = 1$ and $\mathbf{c}_{+1}(\infty) = -1$ in the list $\kappa_{+1}(\mathcal{L})$. Second, the partial multiplicity sequence associated with $\mathbf{c}_{+1}(\lambda_0)$ in $\kappa_{+1}(\mathcal{L})$ coincides with the partial multiplicity sequence associated with λ_0 in $\mathcal{L}$ [37, Thm. 5.3], hence

$$\delta(\kappa_{+1}(\mathcal{L})) = \delta(\mathcal{L}) \quad \text{and} \quad \gamma(\kappa_{+1}(\mathcal{L})) = \gamma(\mathcal{L})\,. \tag{7.2}$$

Relation (7.2), together with the fact that $\mu(\kappa_{+1}(\mathcal{L})) = \mu(\mathcal{L})$ (see Definition 7.2(vii)), imply that the list $\kappa_{+1}(\mathcal{L})$ also satisfies condition (a) in Definition 3.7. Thus, $\kappa_{+1}(\mathcal{L})$ is p-quad admissible. Now from Theorem 6.1 we know that $\kappa_{+1}(\mathcal{L})$ is p-quad realizable by a T-palindromic quadratic matrix polynomial Q. Further, $\mathcal{C}_{+1}(Q)$ is a T-even quadratic matrix polynomial [37, Thm. 9.7], whose list of elementary divisors and minimal indices is precisely $\kappa_{-1}\big(\kappa_{+1}(\mathcal{L})\big) = \mathcal{L}$ [37, Thms. 5.3, 7.5]. Therefore, $\mathcal{L}$ is e-quad realizable by $\mathcal{C}_{+1}(Q)$, and this concludes the proof. □

We can also state analogs of Corollaries 6.3 and 6.4 for T-even matrix polynomials.

Corollary 7.7. *Any T-even matrix polynomial of even (nonzero) grade has a T-even strong quadratification.*

Proof. Let P be a T-even matrix polynomial of even grade k. Then $\mathcal{C}_{+1}(P)$ is a T-palindromic matrix polynomial of grade k [37, Thm. 9.7]. By Corollary 6.3, we also know that $\mathcal{C}_{+1}(P)$ has a T-palindromic strong quadratification Q. On the other hand, $\mathcal{C}_{-1}(Q)$ is a T-even quadratic matrix polynomial [37, Thm. 9.7], and a strong quadratification of $\mathcal{C}_{-1}(\mathcal{C}_{+1}(P)) = 2^k P$ [37, Cor. 8.6], [33, Prop. 2.5]. Consequently, $\mathcal{C}_{-1}(Q)$ is a T-even strong quadratification of P. □

Corollary 7.8. *Let P be a T-even matrix polynomial with odd grade $\ell \geq 3$. Then the following statements are equivalent:*

(i) *There is a T-even strong quadratification of P.*

(ii) *Any odd degree elementary divisor of P associated with $a = 0$ has even multiplicity.*

Proof. (i) $\Rightarrow$ (ii): Let us assume that Q is a T-even strong quadratification of P. Then $\mathcal{C}_{+1}(Q)$ is a T-palindromic strong quadratification of $\mathcal{C}_{+1}(P)$ [37, Cor. 8.6, Thm. 9.7]. By Corollary 6.4, any odd degree elementary divisor of $\mathcal{C}_{+1}(P)$ associated with -1 has even multiplicity, and consequently, any odd degree elementary divisor of P associated with $a := \kappa_{+1}(-1) = 0$ has even multiplicity [37, Thm. 5.3].

(ii) $\Rightarrow$ (i): Assume that any odd degree elementary divisor of P associated with $a = 0$ has even multiplicity. Then any odd degree elementary divisor of $\mathcal{C}_{+1}(P)$ associated with $\kappa_{-1}(0) = -1$ has even multiplicity [37, Thm. 5.3]. Now Corollary 6.4 implies that $\mathcal{C}_{+1}(P)$ has a T-palindromic strong quadratification Q, and consequently, $\mathcal{C}_{-1}(Q)$ is a T-even strong quadratification of $\mathcal{C}_{-1}(\mathcal{C}_{+1}(P)) = 2^k P$ [37, Cor. 8.6, Thm. 9.7] [33, Prop. 2.5]. But then $\mathcal{C}_{-1}(Q)$ is a T-even strong quadratification of P as well. □

7.1. Canonical T-even Lists and Blocks

In this last section we briefly discuss the solution of the T-even QRP, i.e., we show how to explicitly construct a quasi-canonical T-even quadratic realization for any T-even admissible list $\mathcal{L}$. Since the construction procedure is very similar to the one used in the solution of the T-palindromic QRP, we give only an outline.

Let $\mathcal{L}$ be a list of elementary divisors and minimal indices that is e-quad admissible. Then the list $\kappa_{+1}(\mathcal{L})$ is p-quad admissible and can be realized by a direct sum of T-palindromic quadratic canonical blocks from Tables 3–6, say $P := P_1 \oplus \cdots \oplus P_s$. From [37, Prop. 3.16(c)] we know that

$$\mathcal{C}_{+1}(P) \;=\; \mathcal{C}_{+1}(P_1 \oplus \cdots \oplus P_s) \;=\; \mathcal{C}_{+1}(P_1) \oplus \cdots \oplus \mathcal{C}_{+1}(P_s)\,. \tag{7.3}$$

Since each of the blocks $\mathcal{C}_{+1}(P_i)$ is a T-even quadratic matrix polynomial [37, Thm. 9.7], so is $\mathcal{C}_{+1}(P)$. Now [37, Thms. 5.3, 7.5] implies that the elementary divisors and minimal indices of $\mathcal{C}_{+1}(P)$ are exactly those in $\mathcal{L}$, i.e., $\mathcal{C}_{+1}(P)$ is an e-quad realization of $\mathcal{L}$. But $\mathcal{C}_{+1}(P)$ is just a direct sum of $\mathcal{C}_{+1}(P_i)$'s, where each $\mathcal{C}_{+1}(P_i)$ has the same sparsity pattern (i.e., with low "anti"-bandwidth structure) as P_i.

In summary, applying the Cayley transform $\mathcal{C}_{+1}$ to blocks from Tables 3–6 produces a complete list of T-even quadratic blocks that can be used in constructing a Kronecker-like quasi-canonical form for any T-even quadratic matrix polynomial. For instance, the T-even block corresponding to type A_2 blocks after applying $\mathcal{C}_{+1}$ is:

$$\left[\begin{array}{ccc} & & \boxed{\mathcal{Q}_m(1+\mu,-2\mu,(2\mu)^2)} \\ & \boxed{\mathcal{Q}_{n-m}(1-\mu,1+\mu,(2\mu)^2)} & * \\ \boxed{\mathcal{Q}_m(1-\mu,2\mu,(2\mu)^2)} & * & \end{array}\right], \qquad (0 < m \leq n),$$

where $* = \mathcal{C}_{+1}((\lambda-1)^2)(\mu) = (1-\mu)^2 \left(\frac{1+\mu}{1-\mu} - 1\right)^2 = (2\mu)^2$.

8. Concluding Remarks

This paper has provided complete solutions to both the T-palindromic quadratic realizability problem (QRP) and to the T-even QRP, over an arbitrary algebraically closed field of characteristic different from two. These solutions have several clear advantages over previous approaches to structured and unstructured inverse polynomial eigenvalue problems. In particular, we have shown not only how to build quadratic realizations, but also have found simple characterizations of those lists of structural data that comprise the complete spectral and singular structure of some quadratic T-palindromic (resp., T-even) matrix polynomial. An important consequence of this are characterizations of those T-palindromic (resp., T-even) matrix polynomials for which there exists a T-palindromic (resp., T-even) quadratification. While parts of these results have appeared in several recent works [2, 10, 13, 25], these issues have now been completely settled in full generality.

Our systematic approach to constructing quadratic T-palindromic (resp., T-even) realizations has the additional desirable feature of producing realizations from which the given structural data can be easily read off in a completely transparent fashion. This is in stark contrast to the related results in [2, 13]. Such transparency was achieved by using direct sums of low bandwidth T-palindromic (resp., T-even) blocks, resulting in quadratic realizations that distinctly resemble the Kronecker canonical form for general matrix pencils. The main disadvantage of our "direct-sum-of-canonical-blocks" approach to the QRP is the difficulty in extending this technique to the corresponding realizability problems for matrix polynomials of higher degree. As the degree of the desired realization increases, there is likely to be a combinatorial explosion in the number of irreducible cases to be considered. Thus an argument of this type that applies to matrix polynomials of all degrees seems out of reach and impractical.

On the other hand, this disadvantage has had the positive effect of stimulating research, such as [17], into developing new ways of constructing matrix polynomials that transparently reveal their structural data, in order to try to overcome this obstacle. This theme will continue to motivate future research.

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References

[1] M. Al-Ammari and F. Tisseur. Hermitian matrix polynomials with real eigenvalues of definite type. Part I: Classification. *Linear Algebra Appl.*, 436:3954–3973, 2012.

[2] L. Batzke and C. Mehl. On the inverse eigenvalue problem for T-alternating and T-palindromic matrix polynomials. *Linear Algebra Appl.*, 452:172–191, 2014.

[3] M. I. Bueno and F. De Terán. Eigenvectors and minimal bases for some families of Fiedler-like linearizations. *Lin. Multilin. Algebra*, 62(32):39–62, 2014.

[4] M. I. Bueno, F. De Terán, and F. M. Dopico. Recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler linearizations. *SIAM J. Matrix Anal. Appl.*, 31(32):463–483, 2011.

[5] M. I. Bueno and S. Furtado. Palindromic linearizations of a matrix polynomial of odd degree obtained from Fiedler pencils with repetition. *Electron. J. Linear Algebra*, (23):562–577, 2012.

[6] R. Byers, D. S. Mackey, V. Mehrmann, and H. Xu. Symplectic, BVD, and Palindromic Approaches to Discrete-Time Control Problems. In P. Petkov and N. Christov, editors, *A Collection of Papers Dedicated to the 60th Anniversary of Mihail Konstantinov*, pages 81–102, Sofia, Bulgaria, 2009. Publ. House Rodina. Also available as MIMS EPrint 2008.35, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2008.

[7] F. De Terán, F. M. Dopico, and D. S. Mackey. Linearizations of singular matrix polynomials and the recovery of minimal indices. *Electron. J. Linear Algebra*, 18:371–402, 2009.

[8] F. De Terán, F M.. Dopico, and D. S. Mackey. Fiedler companion linearizations and the recovery of minimal indices. *SIAM J. Matrix Anal. Appl.*, 31(4):2181–2204, 2010.

[9] F. De Terán, F. M. Dopico, and D. S. Mackey. Palindromic companion forms for matrix polynomials of odd degree. *J. Comput. Appl. Math.*, 236:1464–1480, 2011.

[10] F. De Terán, F. M. Dopico, and D. S. Mackey. A quasi-canonical form for quadratic matrix polynomials, Part II: The singular case. In preparation, 2013.

[11] F. De Terán, F. M. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the Index Sum Theorem. *Linear Algebra Appl.*, 459:264–333, 2014.

[12] F. De Terán, F. M. Dopico, D. S. Mackey, and P. Van Dooren. Polynomial zigzag matrices, dual minimal bases, and the realization of completely singular polynomials. *Linear Algebra Appl.*, 488:460–504, 2016.

[13] F. De Terán, F. M. Dopico, and P. Van Dooren. Matrix polynomials with completely prescribed eigenstructure. *SIAM J. Matrix Anal. Appl.*, 36:302–328, 2015.

[14] J. W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$ – robust software with error bounds and applications. Part I: theory and algorithms. *ACM Trans. Math. Software*, 19:160–174, 1993.

[15] J. W. Demmel and B. Kågström. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$ – robust software with error bounds and applications. Part II: software and applications. *ACM Trans. Math. Software*, 19:175–201, 1993.

[16] F. M. Dopico, P. W. Lawrence, J. Pérez, and P. Van Dooren. Block Kronecker linearizations of matrix polynomials and their backward errors. *Numer. Math.*, 2018. doi.org/10.1007/s00211-018-0969-z.

[17] F. M. Dopico, D. S. Mackey, and P. Van Dooren. Product realizations for matrix polynomials. In preparation, 2017.

[18] D. Dummit and R. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., New Jersey, third edition, 2004.

[19] H. Faßbender, D. S. Mackey, N. Mackey, and C. Schröder. Structured polynomial eigenproblems related to time-delay systems. *Electron. Trans. Numer. Anal.*, 31:306–330, 2009.

[20] G. D. Forney. Minimal bases of rational vector spaces, with applications to multivariable linear systems. *SIAM J. Control.*, 13:493–520, 1975.

[21] G. Frobenius. Theorie der linearen Formen mit ganzen Coefficienten. *J. Reine Angew. Math. (Crelle)*, 86:146–208, 1878.

[22] F. R. Gantmacher. *The Theory of Matrices*. Chelsea, New York, 1959.

[23] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*. Academic Press, New York, 1982.

[24] N. Higham, D. S. Mackey, N. Mackey, and T. Tisseur. Symmetric linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 29(1):143–159, 2006.

[25] T.-M. Huang, W.-W. Lin, and W.-S. Su. Palindromic quadratization and structure-preserving algorithm for palindromic matrix polynomials of even degree. *Numer. Math.*, 118:713–735, 2011.

[26] I. Ipsen. Accurate eigenvalues for fast trains. *SIAM News*, 37:1–2, 2004.

[27] S. Johansson, B. Kågström, and P. Van Dooren. Stratification of full rank polynomial matrices. *Linear Algebra Appl.*, 439:1062–1090, 2013.

[28] P. Lancaster and I. Zaballa. On the inverse symmetric quadratic eigenvalue problem. *SIAM J. Matrix Anal. Appl.*, 35:254–278, 2014.

[29] D. S. Mackey. *Structured Linearizations for Matrix Polynomials*. PhD thesis, The University of Manchester, UK, 2006.

[30] D. S. Mackey. A quasi-canonical form for quadratic matrix polynomials, Part I: The regular case. In preparation, 2011.

[31] D. S. Mackey. Minimal indices and minimal bases via filtrations. MIMS EPrint 2012.82, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, 2012. Accepted for publication in Electron. J. Linear Algebra.

[32] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. *SIAM J. Matrix Anal. Appl.*, 28(4):971–1004, 2006.

[33] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: Good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28(4):1029–1051, 2006.

[34] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Jordan structures of alternating matrix polynomials. *Linear Algebra Appl.*, 432(4):867–891, 2010.

[35] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms for palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[36] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Skew-symmetric matrix polynomials and their Smith forms. *Linear Algebra Appl.*, 438:4625–4653, 2013.

[37] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Möbius transformations of matrix polynomials. *Linear Algebra Appl.*, 470:120–184, 2015.

[38] D. S. Mackey and F. Tisseur. The Hermitian quadratic realizability problem. In preparation, 2013.

[39] E. Marques de Sá. Imbedding conditions for λ-matrices. *Linear Algebra Appl.*, 24:33–50, 1979.

[40] V. Mehrmann and D. Watkins. Polynomial eigenvalue problems with Hamiltonian structure. *Electron. Trans. Numer. Anal.*, 13:106–118, 2002.

[41] V. Noferini. The behaviour of the complete eigenstructure of a polynomial matrix under a generic rational transformation. *Electron. J. Linear Algebra*, 23:607–624, 2012.

[42] V. Perović. *Spectrally equivalent matrix polynomials: non-standard representations and preservation of structure*. PhD thesis, Western Michigan University, Michigan, USA, 2015.

[43] C. Schröder. A structured Kronecker form for the palindromic eigenvalue problem. *Proc. Appl. Math. and Mech., GAMM Annual Meeting 2006 - Berlin*, 721–722(6), 2006.

[44] C. Schröder. *Palindromic and even eigenvalue problems – analysis and numerical methods.* PhD thesis, Technische Universität, Berlin, Germany, 2008.

[45] R. C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 323–371(147), 1991.

[46] S. Zaglmayr, J. Schöberl, and U. Langer. Eigenvalue problems in surface acoustic wave filter simulations. In A. Di Bucchianico, R.M.M. Mattheij, and M.A. Peletier, editors, *Progress in Industrial Mathematics at ECMI*, pg. 74–98. 2005.