# On minimal bases and indices of rational matrices and their linearizations

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*Dedicated to Paul Van Dooren on the occasion of his 70th birthday*

**Abstract**

This paper presents a complete theory of the relationship between the minimal bases and indices of rational matrices and those of their strong linearizations. Such theory is based on establishing first the relationships between the minimal bases and indices of rational matrices and those of their polynomial system matrices under the classical minimality condition and certain additional conditions of properness. These first results extend under different assumptions pioneer results obtained by Verghese, Van Dooren and Kailath in 1979-80, which have not been sufficiently recognized in the modern literature in our opinion. Next, it is shown that the definitions of linearizations and strong linearizations do not guarantee any relationship between the minimal bases and indices of the linearizations and the rational matrices in general, since only the total sums of the minimal indices are related to each other in the strong case. In contrast, if the specific families of strong block minimal bases linearizations and $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations are considered, then simple relationships between the minimal bases and indices of the linearizations and the rational matrices are obtained. The relevant influence of the work of Paul Van Dooren and coworkers on these topics is emphasized throughout this paper.

**Keywords:** linearizations, minimal bases, minimal indices, polynomial system matrices, rational matrices, strong block minimal bases linearizations

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## 1 Introduction

Minimal bases and indices of rational vector spaces have been studied from different perspectives since the pioneer work of Forney [16] as a consequence of their important applications in linear systems theory and control. It was in [22] that Verghese, Van Dooren and Kailath related the minimal bases and indices of the null spaces of a rational matrix $G(\lambda)$ with those of its Rosenbrock's polynomial system matrices, that is, polynomial matrices of the form

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}, \tag{1}$$

with $A(\lambda)$ nonsingular, and with transfer function matrix $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$. For brevity, it is customary to refer to these objects simply as the minimal bases and indices of $G(\lambda)$ and $P(\lambda)$, omitting the expression "of the null spaces". Actually, the authors of [22] studied the case when the polynomial system matrices are in generalized state-space form: $A(\lambda) = \lambda E - A$ and $B(\lambda) = B$, $C(\lambda) = C$ and $D(\lambda) = D$ constant matrices. This type of polynomial system matrices was defined to be strongly irreducible if $\left[\begin{smallmatrix} \lambda E-A & B \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \lambda E-A \\ -C \end{smallmatrix}\right]$ have no zeros, finite or infinite (the infinite zeros of a rational matrix $R(\lambda)$ were defined to be the zeros at 0 of $R(1/\lambda)$, see (2) in Section 2 below). Then they showed, among other things, how the minimal bases and indices of $P(\lambda)$ of (1) and its transfer function matrix $G(\lambda)$ are related when $P(\lambda)$ is in generalized state-space form and is strongly irreducible. A year later Verghese in [21] extended the results of [22] to general strongly irreducible polynomial system matrices. However, the definition of strong irreducibility in this case is more complicated since it involves checking not only the finite zeros of $\left[\begin{smallmatrix} A(\lambda) & B(\lambda) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A(\lambda) \\ -C(\lambda) \end{smallmatrix}\right]$ but also the infinite zeros of two extensions of $P(\lambda)$.

The goal of this paper is to develop a complete theory about how the minimal bases and indices of general, that is, possibly rectangular, rational matrices and those of their linearizations are related. Linearizations [4] are matrix pencils that contain the complete pole and zero structures of the rational matrices and have received considerable attention in the literature because they can be used for computing such structures. Since linearizations are instances of polynomial system matrices, the aforementioned results by Verghese, Van Dooren and Kailath are relevant. However our definition of linearization uses the notions of coprimeness and properness (see Sections 2, 4 and 5). These are natural properties because coprimeness will allow us to relate the finite zero and pole structures of a polynomial system matrix and its transfer function matrix (see Section 2), while properness is the property that allows to relate their zero and pole structures at infinity (see [4, Lemma 2.4 and Corollary 2.5]). In addition they have the advantage over the strong irreducibility conditions in [21] of being directly checked on the blocks $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ of the polynomial system matrix itself. So, as a first step we will recover Verghese's results on the relationship between the minimal bases and indices of rational matrices and its general polynomial system matrices when these satisfy the required coprimeness and properness conditions. This is the goal of Section 3.

On the other hand, recent works like [7, 8] have dealt with the problem of recovering minimal bases and indices of rational matrices only from certain classes of linearizations and only in the case of square rational matrices. Specifically, the recovery of minimal bases and indices for square rational matrices from its Fiedler-like linearizations is studied in [7]. And in [8], minimal bases and indices of a rational matrix were obtained out of the minimal bases and indices of its $\mathbb{M}_1$ or $\mathbb{M}_2$-strong linearizations, which were introduced in [13]. In this paper, we will explore this recovery problem for any rational matrix and any of its linearizations. We will start with very general linearizations. In this case, our conclusions are that we can recover polynomial bases, but not minimal ones, and that the dimension of the left (right, respectively) nullspace of any rational matrix and that of its linearizations coincide. Also, using Van Dooren's index sum theorem [22], we can obtain the sum of the right and left minimal indices of any rational matrix from any of its strong linearizations. However, as far as the minimal indices themselves are concerned, we will show that the minimal indices of a rational matrix may differ arbitrarily from those of its strong linearizations. Nevertheless, if we specialize in the very general family of strong linearizations called strong block minimal bases linearizations introduced in [4], we can recover the minimal bases and indices of any rational matrix from those of its strong block minimal bases linearizations in a very simple way. And conversely, we can recover the minimal bases and indices of strong block minimal bases linearizations from those of the rational matrices associated with them. It is worth-mentioning that the Fiedler-like linearizations of [7] are particular cases of strong block minimal bases linearizations modulo permutations and that the $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations are very simply connected to strong block minimal bases linearizations. Using this fact, we show that the recovery results for strong block minimal bases linearizations straightforwardly apply for the $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations.

We emphasize that, apart from the new recovery results concerning strong block minimal bases linearizations, the approach described in the previous paragraph constitutes a unified treatment of the relationships between minimal bases and indices of rational matrices and those of most classes of linearizations available so far in the literature. Moreover, this approach has the advantage of establishing the connection of these relationships with the classical work of Verghese, Van Dooren and Kailath [22, 21] from the very beginning.

This paper is organized as follows: Section 2 contains the notation and some preliminary results which are used along the paper. In Section 3 the relationship between the minimal bases and indices of a rational matrix and its polynomial system matrices satisfying some conditions of coprimeness and properness is given. Section 4 relates polynomial bases of rational matrices and their linearizations in general. Section 5 shows that the minimal indices of a rational matrix and of its strong linearizations may differ arbitrarily, but that there is a connection between the sums of their left and right minimal indices. Section 6 is devoted to obtain minimal bases and indices of any rational matrix from its strong block minimal bases linearizations and vice versa. The same goal is pursued in Section 7 for $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations. Finally, some conclusions are discussed in Section 8.

## 2 Preliminaries

Most of the results included in this section are classic and can be found in standard references as [17, 19, 20], together with much more information on rational matrices.

Let $\mathbb{F}$ be an arbitrary field and $\overline{\mathbb{F}}$ its algebraic closure. Let $\mathbb{F}[\lambda]$ be the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ the field of fractions of $\mathbb{F}[\lambda]$, i.e., the field of rational functions over $\mathbb{F}$. The element of $\mathbb{F}(\lambda)$ with the degree of the numerator at most the degree of the denominator is called proper rational functions. The set of proper rational functions over $\mathbb{F}$ form a ring denoted $\mathbb{F}_{pr}(\lambda)$. If the degree of the denominator of a rational function is strictly larger than the degree of its numerator then the rational function is called strictly proper.

Vectors with entries in $\mathbb{F}[\lambda]$ are called vector polynomials. $\mathbb{F}(\lambda)^p$ stands for the vector space of $p$-tuples of rational functions. We denote by $\mathbb{F}[\lambda]^{p\times m}$ (resp., $\mathbb{F}(\lambda)^{p\times m}$, $\mathbb{F}_{pr}(\lambda)^{p\times m}$) the set of $p \times m$ matrices with entries in $\mathbb{F}[\lambda]$ (resp., $\mathbb{F}(\lambda)$, $\mathbb{F}_{pr}(\lambda)$). Matrices in $\mathbb{F}[\lambda]^{p\times m}$ are called polynomial matrices or matrix polynomials indistinctly. The degree of a polynomial matrix is the highest degree of all its entries. The square polynomial matrices whose inverses are polynomial matrices are called unimodular matrices. Matrices in $\mathbb{F}(\lambda)^{p\times m}$ are known as rational matrices and matrices with entries in $\mathbb{F}_{pr}(\lambda)$ are termed as proper rational matrices. In particular, if the entries are all strictly proper then they are called strictly proper rational matrices. Invertible matrices in $\mathbb{F}_{pr}(\lambda)^{p\times p}$, that is, square proper rational matrices whose inverses are also proper, are called biproper. Equivalently, biproper matrices are square proper rational matrices whose determinants are biproprer rational functions.

We introduce now the spectral structure (both finite and infinite) of rational matrices. Recall that two rational matrices $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are unimodularly equivalent if there exist unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that $G_2(\lambda) = U_1(\lambda)G_1(\lambda)U_2(\lambda)$. Any rational matrix is unimodularly equivalent to its finite Smith–McMillan form (see, for example, [19, Chapter 3, Section 4] or [17, Section 6.5.2]). That is to say, if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ then there are unimodular matrices $U_1(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ and $U_2(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that

$$U_1(\lambda)G(\lambda)U_2(\lambda) = \begin{bmatrix} \operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}\right) & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

where $r = \operatorname{rank} G(\lambda)$, $\epsilon_1(\lambda), \dots, \epsilon_r(\lambda), \psi_1(\lambda), \dots, \psi_r(\lambda)$ are nonzero monic (leading coefficient equal to 1) polynomials, $\epsilon_i(\lambda), \psi_i(\lambda)$ are pairwise coprime for all $i = 1, \dots, r$, and $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ while $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$, where $\mid$ stands for divisibility. The finite zeros of $G(\lambda)$ are the roots in $\overline{\mathbb{F}}$ of $\epsilon_r(\lambda)$ and its finite poles are the roots in $\overline{\mathbb{F}}$ of $\psi_1(\lambda)$. If $\lambda_0 \in \overline{\mathbb{F}}$ is a zero of $G(\lambda)$ then, for $i = 1, \dots, r$, we can write $\epsilon_i(\lambda) = (\lambda - \lambda_0)^{m_i}\widehat{\epsilon_i}(\lambda)$ with $\widehat{\epsilon_i}(\lambda_0) \neq 0$ and $m_i \geq 0$. The nonzero elements in $(m_1, \dots, m_r)$ are called the partial multiplicities of $\lambda_0$ as a zero of $G(\lambda)$. In the same way, if $\lambda_0 \in \overline{\mathbb{F}}$ is a pole of $G(\lambda)$ then, for $i = 1, \dots, r$, we can write $\psi_i(\lambda) = (\lambda - \lambda_0)^{n_i}\widehat{\psi_i}(\lambda)$ with $\widehat{\psi_i}(\lambda_0) \neq 0$ and $n_i \geq 0$. The nonzero elements in $(n_1, \dots, n_r)$ are called the partial multiplicities of $\lambda_0$ as a pole of $G(\lambda)$. We understand by finite zero structure of $G(\lambda)$ its finite zeros together with their respective partial multiplicities. Analogously, the finite pole structure of $G(\lambda)$ consists of its finite poles each with its partial multiplicities.

Rational matrices may have structure at infinity as well. Recall (see, for example, [20]) that two rational matrices of the same size $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are equivalent at infinity if there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that $G_2(\lambda) = B_1(\lambda)G_1(\lambda)B_2(\lambda)$. Any rational matrix is equivalent at infinity to its Smith–McMillan form at infinity. That is to say, if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ then there are biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that

$$B_1(\lambda)G(\lambda)B_2(\lambda) = \begin{bmatrix} \operatorname{Diag}\left((\frac{1}{\lambda})^{q_1}, \dots, (\frac{1}{\lambda})^{q_r}\right) & 0 \\ 0 & 0 \end{bmatrix}$$

where $r = \operatorname{rank} G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ are integers. These are called the invariant orders at infinity of $G(\lambda)$. They determine the zeros and poles at infinity of $G(\lambda)$, also called infinite zeros and poles. Namely, if $q_1 \leq \cdots \leq q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \leq \cdots \leq q_r$ are the invariant orders at infinity of $G(\lambda)$ then $G(\lambda)$ has $r - u + 1$ zeros at infinity each one of order $q_u, \dots, q_r$ and $k$ poles at infinity each one of order $-q_k, \dots, -q_1$. Notice that proper rational matrices have all its invariant orders at infinity nonnegative, that is, they do not have poles at infinity. Moreover, all the invariant orders at infinity of strictly proper rational matrices are positive.

Note that any rational matrix can be decomposed uniquely as $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ with $D(\lambda)$ a polynomial matrix and $G_{sp}(\lambda)$ a strictly proper matrix. When $G(\lambda)$ is not strictly proper, that is, when $D(\lambda) \neq 0$, the first invariant order at infinity of $G(\lambda)$, $q_1$, turns out to be minus the degree of the polynomial part of $G(\lambda)$, i.e., $q_1 = -\deg(D(\lambda))$ (see [4, Section 2]), where $\deg(\cdot)$ stands for "degree of".

Furthermore, any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be written as $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ where $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ is nonsingular, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$. The polynomial matrix formed with these matrices

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{3}$$

is called a polynomial system matrix of (or giving rise to) $G(\lambda)$ (see [19]). The rational matrix $G(\lambda)$ is called the transfer function matrix of $P(\lambda)$ and $\deg(\det A(\lambda))$ is known as the order of $P(\lambda)$. We allow $n$ to be equal to 0 in the definition of polynomial system matrix. In this case we say that $P(\lambda) = D(\lambda)$ is a polynomial system matrix giving rise to $G(\lambda) = D(\lambda)$, that is, $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are empty matrices. Besides, when $A(\lambda)$ is a monic linear matrix polynomial, say $A(\lambda) = \lambda I_n - A$, and $B(\lambda) = B$ and $C(\lambda) = C$ are constant matrices, $P(\lambda)$ is said to be a polynomial system matrix of $G(\lambda)$ in state-space form.

Different polynomial system matrices may exist with different orders giving rise to the same transfer function matrix. A polynomial system matrix of $G(\lambda)$ is said to have least order, or to be minimal, if its order is the smallest integer for which matrix polynomials $A(\lambda)$ (nonsingular), $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ giving rise to $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$ exist ([19, Chapter 3, Section 5.1] or [20, Section 1.10]). In consequence, associated with any rational matrix $G(\lambda)$ there is a unique least order, which is the order of any minimal polynomial system matrix giving rise to $G(\lambda)$, and is denoted by $\nu(G(\lambda))$. Interested readers can find in [19, Chapter 3, Section 5.1] three algorithms to compute $\nu(G(\lambda))$ without going to the length of finding a least order polynomial system matrix giving rise to $G(\lambda)$.

One of the many characterizations of when a polynomial system matrix has least order is given in terms of coprimeness. Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ with $p + q \geq m$ are called right coprime if their only right common divisors are unimodular matrices. That is to say, if there exist $\widehat{A}(\lambda) \in \mathbb{F}[\lambda]^{p\times m}, \widehat{C}(\lambda) \in \mathbb{F}[\lambda]^{q\times m}, X(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that $A(\lambda) = \widehat{A}(\lambda)X(\lambda)$ and $C(\lambda) = \widehat{C}(\lambda)X(\lambda)$, then $X(\lambda)$ is unimodular. Let us recall some equivalent conditions that characterize when two polynomial matrices are right coprime (see, for example, [19, Chapter 2, Section 6], [20, Chapter 1], [3]):

**Proposition 2.1** *Let $A(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{q\times m}$ with $p + q \geq m$. The following conditions are equivalent:*

(i) *$A(\lambda)$ and $C(\lambda)$ are right coprime.*

(ii) *There exist matrices $X(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{m\times q}$ such that $X(\lambda)A(\lambda) + Y(\lambda)C(\lambda) = I_m$.*

(iii) $\operatorname{rank}\begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = m$ *for all* $\lambda_0 \in \overline{\mathbb{F}}$.

On the other hand, $A(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{m\times q}$, $p+q \geq m$, are left coprime if their transposes $A(\lambda)^T$ and $C(\lambda)^T$ are right coprime.

It turns out that the polynomial system matrix in (3) has least order if and only if $A(\lambda)$ and $B(\lambda)$ are left coprime and $A(\lambda)$ and $C(\lambda)$ are right coprime ([19, Chapter 3]).

A celebrated result by Rosenbrock [19, Chapter 3, Theorem 4.1] relates the finite structure (zero and pole structure) of a rational matrix with the finite structure of its minimal polynomial system matrices. Namely, when the polynomial system matrix in (3) giving rise to $G(\lambda)$ has least order, the finite zero structure of $G(\lambda)$ is the finite zero structure of $P(\lambda)$ and the finite pole structure of $G(\lambda)$ is the finite zero structure of $A(\lambda)$. A consequence of this fact is that the least order of $G(\lambda)$, $\nu(G(\lambda))$, which is the degree of the determinant of $A(\lambda)$, is equal to the sum of the partial multiplicities of the finite poles of $G(\lambda)$. In other words, $\nu(G(\lambda))$ is the sum of the degrees of the denominators in the finite Smith–McMillan form of $G(\lambda)$.

Let us introduce now the singular structure of a rational matrix. Denote by $\mathcal{N}_\ell(G(\lambda))$ and $\mathcal{N}_r(G(\lambda))$ the left and right null-spaces over $\mathbb{F}(\lambda)$ of $G(\lambda)$, respectively, i.e., if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$,

$$\begin{aligned} \mathcal{N}_\ell(G(\lambda)) &= \{x(\lambda) \in \mathbb{F}(\lambda)^p : x(\lambda)^T G(\lambda) = 0\}, \\ \mathcal{N}_r(G(\lambda)) &= \{x(\lambda) \in \mathbb{F}(\lambda)^m : G(\lambda)x(\lambda) = 0\}. \end{aligned}$$

These sets are vector subspaces of $\mathbb{F}(\lambda)^p$ and $\mathbb{F}(\lambda)^m$, respectively. For any subspace of $\mathbb{F}(\lambda)^p$, it is always possible to find a basis consisting of vector polynomials; simply take an arbitrary basis and multiply each vector by the least common multiple of the denominators of its entries. The order of a polynomial basis is defined as the sum of the degrees of its vectors (see [16]). If $\mathcal{V}$ is a subspace of $\mathbb{F}(\lambda)^p$, a minimal basis of $\mathcal{V}$ is a polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$. The fundamental result in this setting is that the non-decreasing ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same (see [16]). These degrees are called the minimal indices of $\mathcal{V}$.

We refer to a polynomial matrix $N(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$ itself as a right polynomial basis of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ if the columns of $N(\lambda)$ form a basis of $\mathcal{N}_r(G(\lambda))$. If the columns of $N(\lambda)$ form a minimal basis of $\mathcal{N}_r(G(\lambda))$ then $N(\lambda)$ is referred to as a right minimal basis of $G(\lambda)$. Notice that $l = \dim \mathcal{N}_r(G(\lambda)) \leq m$. Moreover, $l = m$ if and only if $G(\lambda) = 0$.

Analogously, a polynomial matrix $N(\lambda) \in \mathbb{F}[\lambda]^{p\times q}$ is a left polynomial (resp., minimal) basis of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ if the columns of $N(\lambda)$ form a polynomial (resp., minimal) basis of $\mathcal{N}_\ell(G(\lambda))$. As above, $q = \dim \mathcal{N}_\ell(G(\lambda)) \leq p$, and $q = p$ if and only if $G(\lambda) = 0$.

The right (resp., left) minimal indices of a rational matrix $G(\lambda)$ are the minimal indices of $\mathcal{N}_r(G(\lambda))$ (resp., $\mathcal{N}_\ell(G(\lambda))$). If $N(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$ then the right (resp., left) minimal indices of $G(\lambda)$ are the degrees of the columns of $N(\lambda)$ when ordered non-decreasingly.

One of the most usual characterizations of minimal bases is a slightly modified version of the Main Theorem given in [16], which can be also found in [10, Theorem 2.14]. Before presenting this theorem let us recall what a column proper or column reduced matrix is. Let $N(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$. We denote by $\deg(\mathrm{col}_j(N(\lambda)))$ the degree of the $j$-th column of $N(\lambda)$, that is, the degree of the highest degree entry in column $j$. Put $d_j = \deg(\mathrm{col}_j(N(\lambda)))$. The matrix $N(\lambda)$ can always be written (see [17, Section 6.3.2]) as

$$N(\lambda) = N_h \,\mathrm{Diag}(\lambda^{d_1}, \dots, \lambda^{d_l}) + L(\lambda) \tag{4}$$

where $N_h$ is the highest column degree coefficient matrix of $N(\lambda)$, and $L(\lambda)$ is a polynomial matrix collecting the remaining terms, which has lower column degrees than the corresponding ones of $N(\lambda)$. The polynomial matrix $N(\lambda)$ is called column proper or column reduced if $\mathrm{rank}\, N_h = l$.

**Theorem 2.2** *([10, Theorem 2.14]) The columns of a matrix polynomial $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if $N(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $N(\lambda)$ is column reduced.*

# 3 Minimal bases and indices of polynomial system matrices

The goal of this section is to study the relationship between the minimal bases and indices of a rational matrix and the minimal bases and indices of any polynomial system matrix giving rise to that rational matrix. We will see that the right (resp., left) minimal indices of a rational matrix are those of its polynomial system matrices when their blocks satisfy some conditions of coprimeness and properness, and under these conditions, we are able to obtain minimal bases of a rational matrix from minimal bases of their polynomial system matrices and vice versa. The results in this section are summarized in Corollary 3.9, which is obtained as a consequence of a number of results that are also interesting by themselves.

As outlined in the Introduction, the properness and coprimeness conditions used in this section allow us to recover the invariant orders at infinity and the zero and pole finite structure, respectively, of any rational matrix from those of its polynomial system matrices (see [4, Lemma 2.4 and Corollary 2.5] and the comments following Proposition 2.1, respectively).

The proofs of the following two Lemmas follow the same pattern as the first part of the proofs of Theorem 2 in [22] and Result 2 in [21] and they are omitted.

**Lemma 3.1** *Let $P(\lambda)$ of (3) be a polynomial system matrix of a rational matrix $G(\lambda)$. Then* $\operatorname{rank} P(\lambda) = n + \operatorname{rank} G(\lambda)$, $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(P(\lambda))$ *and* $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(P(\lambda))$.

**Lemma 3.2** *Under the assumptions of Lemma 3.1, if $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right polynomial basis of $P(\lambda)$ then $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$.*

Lemma 3.1 means that any polynomial system matrix and its transfer function have the same number of right minimal indices and the same number of left minimal indices. In turns, Lemma 3.2 shows how to obtain a right polynomial basis of a rational matrix from a right polynomial basis of any of its polynomial system matrices. This result can be extended to right minimal bases. This is done in [22, Theorem 2] and [21, Result 2] using the condition that $P(\lambda)$ is strongly irreducible. As mentioned in the Introduction, this condition requires that $\left[\begin{smallmatrix} A(\lambda) & B(\lambda) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A(\lambda) \\ -C(\lambda) \end{smallmatrix}\right]$ have no finite zeros and that $\left[\begin{smallmatrix} A(\lambda) & B(\lambda) & 0 \\ -C(\lambda) & D(\lambda) & I_p \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \\ 0 & I_m \end{smallmatrix}\right]$ have no infinite zeros (see [21]). The former is equivalent to $A(\lambda)$ and $B(\lambda)$ be left coprime and $A(\lambda)$ and $C(\lambda)$ be right coprime (see Proposition 2.1). And the latter is used to relate the invariant order at infinity of $P(\lambda)$ and $G(\lambda)$. We substitute the condition on the infinite zeros by the easier to check condition that $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ are proper rational functions. We will use some technical lemmas.

**Lemma 3.3** *Let $N_1(\lambda) \in \mathbb{F}[\lambda]^{n\times l}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{m\times l}$.*

(i) *If $N_1(\lambda) = R(\lambda)N_2(\lambda)$ with $R(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times m}$ then* $\deg(\operatorname{col}_j(N_1(\lambda))) \leq \deg(\operatorname{col}_j(N_2(\lambda)))$ *for $j = 1, \dots, l$.*

(ii) *If $N_1(\lambda) = R(\lambda)N_2(\lambda)$ with $R(\lambda)$ strictly proper then* $\deg(\operatorname{col}_j(N_1(\lambda))) < \deg(\operatorname{col}_j(N_2(\lambda)))$ *for $j = 1, \dots, l$.*

**Proof.-** Let $\frac{p(\lambda)}{q(\lambda)}$ be a proper rational function and let $n(\lambda)$ be a polynomial. Notice that

$$\deg(p(\lambda)) + \deg(n(\lambda)) - \deg(q(\lambda)) \leq \deg(n(\lambda)). \tag{5}$$

Let $n_{ij}^{(1)}(\lambda)$ be an arbitrary element of the $j$-th column of $N_1(\lambda)$. As $N_1(\lambda) = R(\lambda)N_2(\lambda)$, we can write $n_{ij}^{(1)}(\lambda) = \sum_{k=1}^{m} \frac{p_{ik}(\lambda)}{q_{ik}(\lambda)} n_{kj}^{(2)}(\lambda)$ where $\frac{p_{ik}(\lambda)}{q_{ik}(\lambda)}$ is the element in position $(i,k)$ of $R(\lambda)$ and $n_{kj}^{(2)}(\lambda)$ is the element in position

$(k,j)$ of $N_2(\lambda)$. It follows from (5) that for each element of the $j$-th column of $N_1(\lambda)$

$$\begin{aligned}&\deg\left(n_{ij}^{(1)}(\lambda)\right) = \deg\left(\textstyle\sum_{k=1}^{m} \frac{p_{ik}(\lambda)}{q_{ik}(\lambda)} n_{kj}^{(2)}(\lambda)\right)\\ &\leq \max_k\{\deg(p_{ik}(\lambda)) + \deg(n_{kj}^{(2)}(\lambda)) - \deg(q_{ik}(\lambda))\}\\ &\leq \max_k\{\deg(n_{kj}^{(2)}(\lambda))\} = \deg(\mathrm{col}_j(N_2(\lambda)), \qquad \text{for all } i = 1,\dots,n.\end{aligned}$$

Then $\deg(\mathrm{col}_j(N_1(\lambda))) = \max_i\{\deg(n_{ij}^{(1)}(\lambda))\} \leq \deg(\mathrm{col}_j(N_2(\lambda)))$ and (i) follows. If $R(\lambda)$ is strictly proper the previous inequality is strict. ■

The following corollary is an immediate consequence of Lemma 3.3.

**Corollary 3.4** *With the same assumptions and notation of Lemma 3.2, if $A(\lambda)^{-1}B(\lambda)$ is proper then $\deg(\mathrm{col}_j(H_1(\lambda))) \leq \deg(\mathrm{col}_j(H_2(\lambda)))$ for all $j$. The inequality is strict if $A(\lambda)^{-1}B(\lambda)$ is strictly proper.*

The following lemma relates the minimal bases of a rational matrix and its transpose as well as their minimal indices. It also states that the transpose of a polynomial system matrix gives rise to the transpose of its transfer function. It can be proved straightforwardly and, therefore, the proof is omitted.

**Lemma 3.5** (a) *For any rational matrix $G(\lambda)$, $\mathcal{N}_\ell(G(\lambda)) = \mathcal{N}_r(G(\lambda)^T)$ and $\mathcal{N}_r(G(\lambda)) = \mathcal{N}_\ell(G(\lambda)^T)$. Moreover, $H(\lambda)$ is a left minimal basis of $G(\lambda)$ if and only if it is a right minimal basis of $G(\lambda)^T$. Also the left minimal indices of $G(\lambda)$ and the right minimal indices of $G(\lambda)^T$ coincide.*

(b) *If $P(\lambda)$ is a (minimal) polynomial system matrix giving rise to $G(\lambda)$ then $P(\lambda)^T$ is a (minimal) polynomial system matrix giving rise to $G(\lambda)^T$.*

As announced, the following result shows how to obtain a minimal basis of a rational matrix from a minimal basis of any of its polynomial system matrices and relates their minimal indices under certain assumptions.

**Theorem 3.6** *Let $P(\lambda)$ of* (3) *be a polynomial system matrix of a rational matrix $G(\lambda)$.*

(a) *If $A(\lambda)$ and $C(\lambda)$ are right coprime, $A(\lambda)^{-1}B(\lambda)$ is proper and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+m)\times l}$ is a right minimal basis of $P(\lambda)$, then $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. Moreover, the right minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

(b) *If $A(\lambda)$ and $B(\lambda)$ are left coprime, $C(\lambda)A(\lambda)^{-1}$ is proper and $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times q}$ is a left minimal basis of $P(\lambda)$, then $H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$. Moreover, the left minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

**Proof.**- We prove part (a). By Lemma 3.2, $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. We show that $H_2(\lambda)$ is a minimal basis of $G(\lambda)$ by applying Theorem 2.2. Let us prove first that $H_2(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$. If this were not true, there would exist $\lambda_1 \in \overline{\mathbb{F}}$ and a vector, $v \neq 0$, such that $H_2(\lambda_1)v = 0$. But since $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right minimal basis for $P(\lambda)$, $\left[\begin{smallmatrix} H_1(\lambda_1) \\ H_2(\lambda_1) \end{smallmatrix}\right]v = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$ with $w \neq 0$ and

$$P(\lambda_1)\begin{bmatrix} H_1(\lambda_1) \\ H_2(\lambda_1) \end{bmatrix} v = \begin{bmatrix} A(\lambda_1) & B(\lambda_1) \\ -C(\lambda_1) & D(\lambda_1) \end{bmatrix}\begin{bmatrix} w \\ 0 \end{bmatrix} = \begin{bmatrix} A(\lambda_1) \\ -C(\lambda_1) \end{bmatrix} w = 0.$$

This would be a contradiction because $A(\lambda)$ and $C(\lambda)$ are right coprime, i.e., $\left[\begin{smallmatrix} A(\lambda_1) \\ -C(\lambda_1) \end{smallmatrix}\right]$ has full column rank (see Proposition 2.1).

Next, let us see that $H_2(\lambda)$ is column reduced. By hypothesis and Theorem 2.2, we know that $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is column reduced. Our goal is to express the highest column degree coefficient matrix of $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ in terms of the highest column degree coefficient matrix of $H_2(\lambda)$, which is denoted by $H_{2h}$. For this purpose, note that the assumption that $A(\lambda)^{-1}B(\lambda)$ is proper implies that $-A(\lambda)^{-1}B(\lambda) = J + R(\lambda)$, where $J$ is a constant matrix and $R(\lambda)$ is strictly proper. Thus, $H_1(\lambda) = JH_2(\lambda) + R(\lambda)H_2(\lambda)$ and

$$\operatorname{col}_j\left(\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix}\right) = \begin{bmatrix} J\operatorname{col}_j(H_2(\lambda)) + R(\lambda)\operatorname{col}_j(H_2(\lambda)) \\ \operatorname{col}_j(H_2(\lambda)) \end{bmatrix}. \tag{6}$$

Bear in mind that $\operatorname{col}_j(H_2(\lambda)) \neq 0$ since $H_2(\lambda)$ is a right polynomial basis of $G(\lambda)$. Moreover, $R(\lambda)\operatorname{col}_j(H_2(\lambda))$ is a vector polynomial, because $\operatorname{col}_j(H_1(\lambda))$ and $J\operatorname{col}_j(H_2(\lambda))$ are both vector polynomials. Then, Lemma 3.3 (ii) guarantees that $\deg(R(\lambda)\operatorname{col}_j(H_2(\lambda))) < \deg(\operatorname{col}_j(H_2(\lambda)))$. Therefore, the highest degree coefficient of (6) is $\left[\begin{smallmatrix} J\operatorname{col}_j(H_{2h}) \\ \operatorname{col}_j(H_{2h}) \end{smallmatrix}\right]$, the degree $d_j$ of (6) is $d_j = \deg(\operatorname{col}_j(H_2(\lambda)))$ and the highest column degree coefficient matrix of $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} JH_{2h} \\ H_{2h} \end{smallmatrix}\right]$. This latter matrix has full column rank, which implies that $H_{2h}$ has also full column rank, since otherwise there would exist a nonzero constant vector $v$ such that $H_{2h}v = 0$ and $\left[\begin{smallmatrix} JH_{2h} \\ H_{2h} \end{smallmatrix}\right]v = 0$, which is a contradiction. This proves that $H_2(\lambda)$ is column reduced and, so, a right minimal basis of $G(\lambda)$. Since the degrees of the corresponding columns of $H_2(\lambda)$ and $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ coincide, the right minimal indices of $G(\lambda)$ and $P(\lambda)$ are the same.

Part (b) is a consequence of part (a) and Lemma 3.5. ■

We have shown so far how to obtain a minimal basis of the transfer function matrix of a polynomial system matrix out of a minimal basis of the latter, which is the most interesting scenario in applications. For completeness, we consider now the reciprocal problem. In this respect, Lemma 3.2 motivates the following result.

**Lemma 3.7** *Let $P(\lambda)$ of* (3) *be a polynomial system matrix of a rational matrix $G(\lambda)$ where $A(\lambda)$ and $C(\lambda)$ are right coprime. Let $H_2(\lambda)$ be a right polynomial basis of $G(\lambda)$ and let $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$. Then $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right polynomial basis of $P(\lambda)$.*

**Proof.**- Note that

$$P(\lambda)\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}\begin{bmatrix} -A(\lambda)^{-1}B(\lambda)H_2(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} 0 \\ G(\lambda)H_2(\lambda) \end{bmatrix} = 0.$$

Let us see first that $H_1(\lambda)$ is polynomial. As $A(\lambda)$ and $C(\lambda)$ are right coprime, by Bezout's identity (see Proposition 2.1), there exist polynomial matrices $X(\lambda)$ and $Y(\lambda)$ of appropriate sizes such that

$$\begin{bmatrix} X(\lambda) & -Y(\lambda) \end{bmatrix}\begin{bmatrix} A(\lambda) \\ -C(\lambda) \end{bmatrix} = I_n.$$

Put $H(\lambda) = X(\lambda)B(\lambda) - Y(\lambda)D(\lambda)$. Then,

$$\begin{bmatrix} X(\lambda) & -Y(\lambda) \end{bmatrix}\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = \begin{bmatrix} I_n & H(\lambda) \end{bmatrix}\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \end{bmatrix} = 0.$$

Hence $H_1(\lambda) = -H(\lambda)H_2(\lambda)$ is a matrix polynomial. Moreover, $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right polynomial basis of $P(\lambda)$, because its columns belong to $\mathcal{N}_r(P(\lambda))$, its columns are linearly independent, since $H_2(\lambda)$ is a basis of $\mathcal{N}_r(G(\lambda))$, and $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(P(\lambda))$. ■

We can prove now the reciprocal of Theorem 3.6, which shows that, under certain assumptions, minimal bases of polynomial system matrices can be obtained from minimal bases of their transfer functions.

**Theorem 3.8** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and let $P(\lambda)$ of* (3) *be a polynomial system matrix of $G(\lambda)$.*

(a) *If $A(\lambda)$ and $C(\lambda)$ are right coprime, $A(\lambda)^{-1}B(\lambda)$ is proper, $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$ then $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right minimal basis of $P(\lambda)$. Moreover, the right minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

(b) *If $A(\lambda)$ and $B(\lambda)$ are left coprime, $C(\lambda)A(\lambda)^{-1}$ is proper, $H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$ then $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a left minimal basis of $P(\lambda)$. Moreover, the left minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

**Proof.**- We prove part (a). By Lemma 3.7 and Theorem 2.2, we just need to prove that $\left[\begin{smallmatrix} H_1(\lambda_0) \\ H_2(\lambda_0) \end{smallmatrix}\right]$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$ and $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is column reduced. As $H_2(\lambda)$ is a right minimal basis of $G(\lambda)$, $H_2(\lambda_0)$ has full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$, which implies that the matrix $\left[\begin{smallmatrix} H_1(\lambda_0) \\ H_2(\lambda_0) \end{smallmatrix}\right]$ has full column rank as well. Moreover, $H_2(\lambda)$ is column reduced. Write $H_2(\lambda) = H_{2h}\,\mathrm{Diag}(\lambda^{d_1},\ldots,\lambda^{d_l}) + L_2(\lambda)$ with $H_{2h}$ of full column rank, $d_1,\ldots,d_l$ the right minimal indices of $G(\lambda)$ and the degree of the $j$-th column of $L_2(\lambda)$ less than $d_j$ for each $j$. Since $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$, with $A(\lambda)^{-1}B(\lambda)$ proper, it follows from Corollary 3.4 that each column of $H_1(\lambda)$ has degree less than or equal to the same column of $H_2(\lambda)$. Therefore, there is a matrix $H_{1h}$ such that the highest column degree coefficient matrix of $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} H_{1h} \\ H_{2h} \end{smallmatrix}\right]$, a full column rank matrix. Moreover, its column degrees are those of $H_2(\lambda)$.

Part (b) follows from (a) and Lemma 3.5. ∎

Theorems 3.6 and 3.8 together provide our next result.

**Corollary 3.9** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix and let $P(\lambda)$ of* (3) *be a minimal polynomial system matrix of $G(\lambda)$. If both $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ are proper matrices then $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right (resp., left) minimal basis of $P(\lambda)$ if and only if $H_2(\lambda)$ is a right (resp., left) minimal basis of $G(\lambda)$ and $H_1(\lambda) = -A(\lambda)^{-1}B(\lambda)H_2(\lambda)$ (resp., $H_1(\lambda) = (C(\lambda)A(\lambda)^{-1})^T H_2(\lambda)$). Moreover, the right (resp., left) minimal indices of $P(\lambda)$ and $G(\lambda)$ are the same.*

## 4 Polynomial bases of linearizations of rational matrices

The aim of this section is to study the relationship between the polynomial bases of a rational matrix and the polynomial bases of its linearizations. It is not possible to extend this relationship to minimal bases because it was already proved in [9, Theorem 4.10 (b)] that the minimal bases and indices of a polynomial matrix can not be obtained from the minimal bases and indices of its linearizations in general, and polynomial matrices are particular cases of rational matrices.

A linear pencil

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \tag{7}$$

is said to be a linearization of a rational matrix $G(\lambda)$ (see [4, Definition 3.2]) if it is a minimal polynomial system matrix of a rational matrix $\widehat{G}(\lambda)$ such that, for some nonnegative integers $s_1, s_2$, $\mathrm{Diag}(\widehat{G}(\lambda), I_{s_2})$ and $\mathrm{Diag}(G(\lambda), I_{s_1})$ are unimodularly equivalent. We can assume without loss of generality that $s_1 = s$ and $s_2 = 0$. This assumption will be adopted in the rest of the paper every time we deal with linearizations.

A first consequence of this definition is that, by the rank-nullity theorem, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) = \dim \mathcal{N}_\ell(G(\lambda))$. Therefore, $G(\lambda)$ and $\widehat{G}(\lambda)$ have the same number of right minimal indices and the same number of left minimal indices. Furthermore, by Lemma 3.1, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$. Thus, a rational matrix and any of its linearizations have the same number of right minimal indices and the same number of left minimal indices.

Proposition 4.1 relates right polynomial bases of $G(\lambda)$ and $\widehat{G}(\lambda)$. An analogous result holds for left polynomial bases of $G(\lambda)$ and $\widehat{G}(\lambda)$ as a consequence of Lemma 3.5. Such "left" result is omitted for brevity.

**Proposition 4.1** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\widehat{G}(\lambda) \in \mathbb{F}(\lambda)^{(p+s)\times(m+s)}$, $s \geq 0$. Let $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ be unimodular matrices such that $U(\lambda)\widehat{G}(\lambda)V(\lambda) = \mathrm{Diag}(G(\lambda), I_s)$.*

(a) *If $H(\lambda)$ is a right polynomial basis of $G(\lambda)$ then $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ is a right polynomial basis of $\widehat{G}(\lambda)$.*

(b) *If $\widehat{H}(\lambda)$ is a right polynomial basis of $\widehat{G}(\lambda)$ then $V(\lambda)^{-1}\widehat{H}(\lambda) = \left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ and $H(\lambda)$ is a right polynomial basis of $G(\lambda)$.*

**Proof**.- In order to prove (a) assume that $G(\lambda)H(\lambda) = 0$. We obtain, via a direct multiplication, that

$$\widehat{G}(\lambda)V(\lambda)\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix} = U(\lambda)^{-1}\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix}\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix} = 0.$$

So, $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ is a right polynomial basis of $\widehat{G}(\lambda)$, because its columns are linearly independent and $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$.

For proving (b) assume that $\widehat{G}(\lambda)\widehat{H}(\lambda) = 0$. Therefore,

$$\begin{bmatrix} G(\lambda) & 0 \\ 0 & I_s \end{bmatrix} V(\lambda)^{-1}\widehat{H}(\lambda) = 0.$$

Write $V(\lambda)^{-1} = \left[\begin{smallmatrix} V_1(\lambda) \\ V_2(\lambda) \end{smallmatrix}\right]$, where $V_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+s)}$ and $V_2(\lambda) \in \mathbb{F}[\lambda]^{s\times(m+s)}$. Thus, $G(\lambda)V_1(\lambda)\widehat{H}(\lambda) = 0$ and $V_2(\lambda)\widehat{H}(\lambda) = 0$. Set $H(\lambda) = V_1(\lambda)\widehat{H}(\lambda)$. It follows that $V(\lambda)^{-1}\widehat{H}(\lambda) = \left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ and $G(\lambda)H(\lambda) = 0$. Thus, the columns of $H(\lambda)$ form a right polynomial basis of $G(\lambda)$. ■

**Remark 4.2** Proposition 4.1 cannot be extented to right minimal bases, i.e., if $H(\lambda)$ is a right minimal basis of $G(\lambda)$, $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ may not be a right minimal basis of $\widehat{G}(\lambda)$, and if $\widehat{H}(\lambda)$ is a right minimal basis of $\widehat{G}(\lambda)$, $V(\lambda)^{-1}\widehat{H}(\lambda)$ may not contain in its first $m$ rows a minimal basis of $G(\lambda)$. Concerning part (a) of Proposition 4.1, notice that if $H(\lambda)$ is a right minimal basis of $G(\lambda)$, then $V(\lambda_0)\left[\begin{smallmatrix} H(\lambda_0) \\ 0 \end{smallmatrix}\right]$ would have full column rank for all $\lambda_0 \in \overline{\mathbb{F}}$, but it is not possible to guarantee that $V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ is column reduced. A similar observation can be made on part (b) of Proposition 4.1. Let us illustrate these remarks with an example. Take, $s = 1$,

$$G(\lambda) = \begin{bmatrix} \frac{1}{\lambda} & -1 & (\lambda^2 - \frac{1}{\lambda}) \end{bmatrix}, \quad U(\lambda) = I_2 \quad \text{and } V(\lambda) = \begin{bmatrix} 1 & 0 & \lambda^3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$\widehat{G}(\lambda) = \begin{bmatrix} \frac{1}{\lambda} & -1 & (\lambda^2 - \frac{1}{\lambda}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} V(\lambda)^{-1} = \begin{bmatrix} \frac{1}{\lambda} & -1 & -\frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is immediate to check that the polynomial matrix

$$H(\lambda) = \begin{bmatrix} 1 & \lambda \\ \lambda^2 & 1 \\ 1 & 0 \end{bmatrix}$$

is a right minimal basis of $G(\lambda)$. However

$$V(\lambda)\begin{bmatrix} H(\lambda) \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \lambda^3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ \lambda^2 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+\lambda^3 & \lambda \\ \lambda^2 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not column reduced and, so, it is a right polynomial basis of $\widehat{G}(\lambda)$ but not minimal. Conversely, it is immediate to check that the polynomial matrix

$$\widehat{H}(\lambda) = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is a right minimal basis of $\widehat{G}(\lambda)$, but

$$V(\lambda)^{-1}\widehat{H}(\lambda) = \begin{bmatrix} 1-\lambda^3 & \lambda \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is not column reduced, and its three first rows are not a right minimal basis of $G(\lambda)$.

The next result relates the polynomial bases of a rational matrix and its linearizations through the unimodular matrices that connect the rational matrix and the transfer function matrix of the linearizations.

**Theorem 4.3** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ of (7) be a linearization of $G(\lambda)$ with transfer function matrix $\widehat{G}(\lambda)$. Let $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$, $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ be unimodular matrices such that $U(\lambda)\widehat{G}(\lambda)V(\lambda) =$* Diag$(G(\lambda), I_s)$.

(a) $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ *is a right polynomial basis of $L(\lambda)$ if and only if $H_2(\lambda) = V(\lambda)\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ for some right polynomial basis $H(\lambda)$ of $G(\lambda)$ and $H_1(\lambda) = -(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)H_2(\lambda)$.*

(b) $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ *is a left polynomial basis of $L(\lambda)$ if and only if $H_2(\lambda) = U(\lambda)^T\left[\begin{smallmatrix} H(\lambda) \\ 0 \end{smallmatrix}\right]$ for some left polynomial basis $H(\lambda)$ of $G(\lambda)$ and $H_1(\lambda) = ((C_1\lambda + C_0)(A_1\lambda + A_0)^{-1})^T H_2(\lambda)$.*

**Proof.**- As $L(\lambda)$ is a linearization of $G(\lambda)$, $L(\lambda)$ is a minimal polynomial system matrix and, therefore, $A_1\lambda + A_0$ and $C_1\lambda + C_0$ are right coprime and $A_1\lambda + A_0$ and $B_1\lambda + B_0$ are left coprime. Thus we can apply Lemmas 3.2 and 3.7 and Proposition 4.1 to prove part (a). To prove part (b), use Lemma 3.5 and part (a). ∎

# 5 Minimal indices of strong linearizations of rational matrices

In this section we begin to study the relationship between the minimal indices of a rational matrix and the minimal indices of its strong linearizations. As discussed in [4, Remark 3.5], strong linearizations are particular cases of linearizations and, therefore, we know that the number of right (resp., left) minimal indices of a rational matrix and of its strong linearizations coincide. However, we will show in this section that it is not possible to obtain the right (resp., left) minimal indices of a rational matrix from those of its strong linearizations in general. Nevertheless, we will prove in Theorem 5.9 that the total sum of the right and left minimal indices of a rational matrix can be easily obtained from the total sum of the right and left minimal indices of any of its strong linearizations. We postpone to Sections 6 and 7 to identify particular classes of strong linearizations that allow us to recover the minimal indices and bases of the rational matrix from those of the linearizations.

We start by recalling the definition of strong linearization of a rational matrix.

**Definition 5.1** *([4, Definition 3.4]) Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let $q_1$ be its first invariant order at infinity and $g = \min(0, q_1)$. Let $n = \nu(G(\lambda))$. A strong linearization of $G(\lambda)$ is a linear polynomial matrix*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

*such that the following conditions hold:*

(a) *if* $n > 0$ *then* $\det(A_1\lambda + A_0) \neq 0$*, and*

(b) *if* $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$*,* $\widehat{q}_1$ *is its first invariant order at infinity and* $\widehat{g} = \min(0, \widehat{q}_1)$ *then:*

(i) *there are integers* $s_1, s_2 \geq 0$ *and unimodular matrices* $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ *and* $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ *so that* $s_1 - s_2 = q - p = r - m$ *and*

$$U_1(\lambda)\,\mathrm{Diag}(G(\lambda), I_{s_1})U_2(\lambda) = \mathrm{Diag}(\widehat{G}(\lambda), I_{s_2}), \text{ and}$$

(ii) *there are biproper matrices* $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+s_1)\times(p+s_1)}$ *and* $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+s_1)\times(m+s_1)}$ *such that*

$$B_1(\lambda)\,\mathrm{Diag}(\lambda^{g}G(\lambda), I_{s_1})B_2(\lambda) = \mathrm{Diag}(\lambda^{\widehat{g}}\widehat{G}(\lambda), I_{s_2}).$$

As in the case of linearizations, we can also assume without loss of generality that $s_1 = s$ and $s_2 = 0$ in the definition of strong linearizations. We will adopt such assumption in the rest of the paper.

**Remark 5.2** As commented in [4, Remark 3.5], the requirement $n = \nu(G(\lambda))$ in Definition 5.1 might seem very restrictive. Thus, it is worth to emphasize that such requirement may be replaced by the assumptions that $L(\lambda)$ is a minimal polynomial system matrix and $A_1$ is invertible when $n > 0$, as a consequence of the discussion in [4, Remark 3.5], which are more direct requirements. We have decided to state Definition 5.1 exactly as in [4] in order to avoid confusions.

Recall that any rational matrix can be written uniquely as $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ with $D(\lambda)$ a polynomial matrix and $G_{sp}(\lambda)$ a strictly proper matrix. Moreover, if $D(\lambda) \neq 0$ then the first invariant order at infinity of $G(\lambda)$, $q_1$, is equal to $-\deg(D(\lambda))$; otherwise, if $G(\lambda)$ is strictly proper, $q_1 > 0$. We define

$$d = -\min(0, q_1) = \begin{cases} \deg(D(\lambda)) & \text{if } D(\lambda) \neq 0 \\ 0 & \text{if } D(\lambda) = 0 \end{cases}. \tag{8}$$

Notice that $g$ in Definition 5.1 is equal to $-d$.

We show now with Example 5.4 that the minimal indices of a strong linearization of a rational matrix may be arbitrarily different than the minimal indices of the rational matrix in general. In order to develop Example 5.4, we present the following lemma first.

**Lemma 5.3** *Let*

$$K_u(\lambda) = \begin{bmatrix} 1 & \lambda & & & \\ & 1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{u\times(u+1)}$$

*for any positive integer $u$ and let $0_{u,1}$ be the $u \times 1$ zero matrix. Then,*

(i) *$K_u(\lambda)$ is unimodularly equivalent to $\begin{bmatrix} I_u & 0_{u,1} \end{bmatrix}$.*

(ii) *$\lambda^{-1}K_u(\lambda)$ is equivalent at infinity to $\begin{bmatrix} I_u & 0_{u,1} \end{bmatrix}$.*

**Proof**.- In order to prove (i), multiply $K_u(\lambda)$ on the right by the unimodular matrix

$$\begin{bmatrix} 1 & -\lambda & \lambda^2 & (-\lambda)^3 & \cdots & (-\lambda)^u \\ & 1 & -\lambda & \lambda^2 & \cdots & (-\lambda)^{u-1} \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & 1 & -\lambda & \lambda^2 \\ & & & & 1 & -\lambda \\ & & & & & 1 \end{bmatrix}.$$

To prove (ii), multiply $\lambda^{-1}K_u(\lambda)$ on the right by the biproper matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -1/\lambda \\ -1/\lambda & 1 & 0 & \cdots & 0 & (-1/\lambda)^2 \\ (-1/\lambda)^2 & -1/\lambda & 1 & \cdots & 0 & (-1/\lambda)^3 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ (-1/\lambda)^{u-1} & (-1/\lambda)^{u-2} & & \cdots & 1 & (-1/\lambda)^u \end{bmatrix}.$$

■

**Example 5.4** Let $G(\lambda) = \left[\begin{smallmatrix} \lambda+\lambda^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{F}(\lambda)^{2\times 2}$. We may consider infinitely many strong linearizations of $G(\lambda)$. Let

$$L_{\epsilon,\eta}(\lambda) = \left[\begin{array}{c|ccc} \lambda & 1 & & \\ \hline -1 & \lambda & & \\ & & K_\epsilon(\lambda) & \\ & & & K_\eta(\lambda)^T \end{array}\right] \in \mathbb{F}[\lambda]^{(1+(2+\epsilon+\eta))\times(1+(2+\epsilon+\eta))}.$$

We prove now that for each pair of positive integers $\epsilon$ and $\eta$, $L_{\epsilon,\eta}(\lambda)$ is a strong linearization of $G(\lambda)$. First, notice that $L_{\epsilon,\eta}(\lambda)$ is a minimal polynomial system matrix with transfer function matrix

$$\begin{aligned} \widehat{G}_{\epsilon,\eta}(\lambda) &= \begin{bmatrix} \lambda & & \\ & K_\epsilon(\lambda) & \\ & & K_\eta(\lambda)^T \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \lambda^{-1} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda + \lambda^{-1} & & \\ & K_\epsilon(\lambda) & \\ & & K_\eta(\lambda)^T \end{bmatrix}. \end{aligned}$$

Using Lemma 5.3, it is easy to prove that $\widehat{G}_{\epsilon,\eta}(\lambda)$ is unimodularly equivalent to

$$\begin{bmatrix} \lambda+\lambda^{-1} & & & & \\ & I_\epsilon & 0_{\epsilon,1} & & \\ & & & I_\eta & \\ & & & 0_{1,\eta} & \end{bmatrix},$$

which is unimodularly equivalent to $\left[\begin{smallmatrix} G(\lambda) & 0 \\ 0 & I_{\epsilon+\eta} \end{smallmatrix}\right]$. Thus, $L_{\epsilon,\eta}(\lambda)$ is a linearization of $G(\lambda)$. Furthermore, $G(\lambda)$ can be written as

$$G(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and $\widehat{G}_{\epsilon,\eta}(\lambda)$ can be written as

$$\widehat{G}_{\epsilon,\eta}(\lambda) = \begin{bmatrix} \lambda & & \\ & K_\epsilon(\lambda) & \\ & & K_\eta(\lambda)^T \end{bmatrix} + \begin{bmatrix} \lambda^{-1} & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Therefore, with the notation of Definition 5.1, $g = \widehat{g} = -1$. The matrix $\lambda^{-1}\widehat{G}_{\epsilon,\eta}(\lambda)$ is

$$\begin{bmatrix} 1+\lambda^{-2} & & \\ & \lambda^{-1}K_\epsilon(\lambda) & \\ & & \lambda^{-1}K_\eta(\lambda)^T \end{bmatrix},$$

which, by Lemma 5.3, is equivalent at infinity to

$$\begin{bmatrix} 1+\lambda^{-2} & & & & \\ & I_\epsilon & 0_{\epsilon,1} & & \\ & & & I_\eta & \\ & & & 0_{1,\eta} & \end{bmatrix} \text{ and to } \begin{bmatrix} \lambda^{-1}G(\lambda) & 0 \\ 0 & I_{\epsilon+\eta} \end{bmatrix}.$$

Hence, $L_{\epsilon,\eta}(\lambda)$ is a strong linearization of $G(\lambda)$. Notice that the unique*right minimal index of $G(\lambda)$ is 0 and the unique left minimal index of $G(\lambda)$ is 0 as well, while the unique right minimal index of $L_{\epsilon,\eta}(\lambda)$ is $\epsilon$ and the unique left minimal index of $L_{\epsilon,\eta}(\lambda)$ is $\eta$. Thus, strong linearizations do not preserve minimal indices.*

Denote by $\mu(G(\lambda))$ the sum of the right and left minimal indices of a rational matrix $G(\lambda)$. Our next goal is to analyze how this is related with the sum of the right and left minimal indices of any of its strong linearizations. In order to study this relationship, we will make use of Van Dooren's index sum theorem, proved for the first time in [22, Theorem 3], and that we rewrite in a way convenient for our purposes in Lemma 5.5. Interested readers are referred to the recent paper [5] for more information on this fundamental result.

**Lemma 5.5** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with finite Smith–McMillan form* $\operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{p-r,m-r}\right)$. *Let $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity. Then*

$$\mu(G(\lambda)) = \sum_{i=1}^{r} \deg(\psi_i(\lambda)) - \sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - \sum_{i=1}^{r} q_i. \tag{9}$$

**Proof**.- By the index sum theorem (see [22, Theorem 3] or [17, Theorem 6.5-11]) $\mu(G(\lambda))$ is equal to the total number of poles (finite and at infinity) of $G(\lambda)$ minus the total number of zeros (finite and at infinity) of $G(\lambda)$. The total number of finite zeros of $G(\lambda)$ is the sum of all partial multiplicities of all finite zeros of $G(\lambda)$, that is, $\sum_{i=1}^{r} \deg(\epsilon_i(\lambda))$. In the same way, the total number of finite poles of $G(\lambda)$ is the sum of all partial multiplicities of all finite poles of $G(\lambda)$, i.e., $\sum_{i=1}^{r} \deg(\psi_i(\lambda))$. Therefore, the total number of finite poles minus the total number of finite zeros is $\sum_{i=1}^{r} \deg(\psi_i(\lambda)) - \sum_{i=1}^{r} \deg(\epsilon_i(\lambda))$. On the other hand, the total number of infinite poles minus the total number of infinite zeros is $-\sum_{i=1}^{r} q_i$ since the positive $q_i$ are the orders of the infinite zeros while minus the negative $q_i$ are the orders of the infinite poles. Thus, equation (9) is obtained. ■

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix, let $d$ be defined as in (8) and let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \tag{10}$$

be a linear minimal polynomial system matrix with $A_1$ invertible if $n > 0$. We say that $L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$ if the following conditions simultaneously hold:

(i) the finite poles of $G(\lambda)$ are the finite zeros of $A_1\lambda + A_0$, with the same partial multiplicities in both matrices,

(ii) the finite zeros of $G(\lambda)$ are the finite zeros of $L(\lambda)$, with the same partial multiplicities, and

(iii) the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^{-d}G(\lambda)$ if $D_1 + C_1A_1^{-1}B_1 \neq 0$ or of $\operatorname{Diag}(\lambda^{-1}I_s, \lambda^{-d-1}G(\lambda))$ otherwise.

**Theorem 5.6** *([4, Theorem 3.10]) Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $n = \nu(G(\lambda))$. Let $L(\lambda)$ be the pencil of* (10). *Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if the following two conditions hold:*

(I) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ *(which is equivalent to* $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$*), and*

(II) *$L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$.*

The following result relates the invariant orders at infinity of a rational matrix and its a strong linearization. We remark that although Lemma 5.7 was not explicitly stated in [4], it is related to discussions in [4, pp. 1682–1683].

**Lemma 5.7** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with invariant orders at infinity $q_1 \leq \cdots \leq q_r$ and $d$ be defined as in (8). Let $L(\lambda)$ of (10) be any strong linearization of $G(\lambda)$ and $q_1^L \leq \ldots \leq q_\ell^L$ be the invariant orders at infinity of $L(\lambda)$. Then $\ell = n+s+r$ and*

(i) *If $D_1+C_1A_1^{-1}B_1 \neq 0$ then $q_i^L = -1$ for $i = 1,\ldots,n+s$, and $q_{n+s+i}^L = q_i+d-1$ for $i = 1,\ldots,r$.*

(ii) *If $n > 0$ and $D_1 + C_1A_1^{-1}B_1 = 0$ then $q_i^L = -1$ for $i = 1,\ldots,n$, $q_{n+i}^L = 0$ for $i = 1,\ldots,s$, and $q_{n+s+i}^L = q_i + d$ for $i = 1,\ldots,r$.*

(iii) *If $n = 0$ and $D_1 = 0$ then $L(\lambda) = D_0$, $q_i^L = 0$ for $i = 1,\ldots,s+r$, and $q_i = -d$ for $i = 1,\ldots,r$.*

**Proof**.- By Theorem 5.6 (I) and the rank-nullity theorem, $\ell = n+s+r$ is the rank of $L(\lambda)$. As $q_1 \leq \cdots \leq q_r$ are the invariant orders at infinity of $G(\lambda)$, there exist two biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$ and $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that

$$G(\lambda) = B_1(\lambda)\,\mathrm{Diag}\left(\left(\frac{1}{\lambda}\right)^{q_1},\ldots,\left(\frac{1}{\lambda}\right)^{q_r},0_{p-r,m-r}\right)B_2(\lambda). \qquad (11)$$

We distinguish two cases:

Suppose first that $D_1+C_1A_1^{-1}B_1 \neq 0$. By Theorem 5.6 again, the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^{-d}G(\lambda)$. Since $\lambda^{-1}L(\lambda)$ and $\lambda^{-d}G(\lambda)$ are both proper rational matrices and $\operatorname{rank} L(\lambda) - \operatorname{rank} G(\lambda) = n+s$, $\lambda^{-1}L(\lambda)$ must be equivalent at infinity to $\left[\begin{smallmatrix}\lambda^{-d}G(\lambda) & 0\\ 0 & I_{n+s}\end{smallmatrix}\right]$. Thus $L(\lambda)$ is equivalent at infinity to $\begin{bmatrix}\lambda^{-d+1}G(\lambda) & 0\\ 0 & \lambda I_{n+s}\end{bmatrix}$, that is, there exist two biproper matrices $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+n+s)\times(p+n+s)}$ and $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+n+s)\times(m+n+s)}$ such that

$$L(\lambda) = B_3(\lambda)\begin{bmatrix}\lambda^{-d+1}G(\lambda) & 0\\ 0 & \lambda I_{n+s}\end{bmatrix}B_4(\lambda) = B_3(\lambda)\lambda^{-d+1}\begin{bmatrix}G(\lambda) & 0\\ 0 & \lambda^d I_{n+s}\end{bmatrix}B_4(\lambda).$$

Put $\overline{B}_1(\lambda) = B_3(\lambda)\left[\begin{smallmatrix}B_1(\lambda) & 0\\ 0 & I_{n+s}\end{smallmatrix}\right]$ and $\overline{B}_2(\lambda) = \left[\begin{smallmatrix}B_2(\lambda) & 0\\ 0 & I_{n+s}\end{smallmatrix}\right]B_4(\lambda)$, which are biproper matrices. Using (11),

$$L(\lambda) = \overline{B}_1(\lambda)\lambda^{-d+1}\begin{bmatrix} \mathrm{Diag}\left((\frac{1}{\lambda})^{q_1}, \dots, (\frac{1}{\lambda})^{q_r}, 0\right) & 0 \\ 0 & \lambda^d I_{n+s} \end{bmatrix}\overline{B}_2(\lambda)$$
$$= \overline{B}_1(\lambda)\begin{bmatrix} \mathrm{Diag}\left((\frac{1}{\lambda})^{q_1+d-1}, \dots, (\frac{1}{\lambda})^{q_r+d-1}, 0\right) & 0 \\ 0 & (\frac{1}{\lambda})^{-1} I_{n+s} \end{bmatrix}\overline{B}_2(\lambda).$$

Notice, by (8), that $q_1 + d \geq 0$. Therefore $-1 \leq q_1 + d - 1 \leq \cdots \leq q_r + d - 1$. Thus, $q_i^L = -1$ for $i = 1, \dots, n+s$, and $q_{n+s+i}^L = q_i + d - 1$ for $i = 1, \dots, r$.

Suppose now that $D_1 + C_1 A_1^{-1} B_1 = 0$. By Theorem 5.6, the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as those of $\mathrm{Diag}(\lambda^{-1}I_s, \lambda^{-d-1}G(\lambda))$. As both matrices are proper and their rank difference is $n$, $\lambda^{-1}L(\lambda)$ must be equivalent at infinity to $\begin{bmatrix} \lambda^{-d-1}G(\lambda) & 0 & 0 \\ 0 & \lambda^{-1}I_s & 0 \\ 0 & 0 & I_n \end{bmatrix}$. Thus $L(\lambda)$ is equivalent at infinity to $\begin{bmatrix} \lambda^{-d}G(\lambda) & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & \lambda I_n \end{bmatrix}$, that is, there exist two biproper matrices $B_5(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+n+s)\times(p+n+s)}$ and $B_6(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+n+s)\times(m+n+s)}$ such that

$$L(\lambda) = B_5(\lambda)\begin{bmatrix} \lambda^{-d}G(\lambda) & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & \lambda I_n \end{bmatrix}B_6(\lambda)$$
$$= B_5(\lambda)\lambda^{-d}\begin{bmatrix} G(\lambda) & 0 & 0 \\ 0 & \lambda^d I_s & 0 \\ 0 & 0 & \lambda^{d+1} I_n \end{bmatrix}B_6(\lambda).$$

By using (11) and proceeding as in the previous case, if $n > 0$ then the invariant orders at infinity of $L(\lambda)$ are $q_i^L = -1$ for $i = 1, \dots, n$, $q_{n+i}^L = 0$ for $i = 1, \dots, s$, and $q_{n+s+i}^L = q_i + d$ for $i = 1, \dots, r$. Otherwise, if $n = 0$ then $D_1 = 0$, $L(\lambda) = D_0$ and, therefore, $q_i^L = 0$ for $i = 1, \dots, s+r$. Moreover, since $D_0 = B_5(\lambda)\begin{bmatrix} \lambda^{-d}G(\lambda) & 0 \\ 0 & I_s \end{bmatrix}B_6(\lambda)$, the invariant orders at infinity of $\lambda^{-d}G(\lambda)$ must be 0 and, in consequence, $q_i = -d$ for $i = 1, \dots, r$. ■

The following lemma gives $\mu(L(\lambda))$, the sum of the right and left minimal indices of a strong linearization $L(\lambda)$ of a rational matrix $G(\lambda)$, in terms of the spectral invariants of $G(\lambda)$.

**Lemma 5.8** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix with $\epsilon_1(\lambda), \dots, \epsilon_r(\lambda)$ as numerators in its finite Smith–McMillan form and with $q_1 \leq \cdots \leq q_r$ as invariant orders at infinity. Let $d$ be defined as in (8). Let $L(\lambda)$ of* (10) *be any strong linearization of $G(\lambda)$.*

(i) *If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then*

$$\mu(L(\lambda)) \quad = s + r(1-d) + n - \textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - \sum_{i=1}^r q_i.$$

(ii) *If* $n > 0$ *and* $D_1 + C_1 A_1^{-1} B_1 = 0$ *then*

$$\mu(L(\lambda)) \quad = -dr + n - \textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - \sum_{i=1}^r q_i.$$

(iii) *If* $n = 0$ *and* $D_1 = 0$ *then* $L(\lambda) = D_0$*,* $\mu(L(\lambda)) = 0$*, and* $\epsilon_i(\lambda) = 1$ *for* $i = 1, \ldots, r$*.*

**Proof**.- We aim to apply Lemma 5.5 to $L(\lambda)$. As seen in Lemma 5.7, $\operatorname{rank} L(\lambda) = n+s+r$. Since $L(\lambda)$ is a polynomial matrix it has no finite poles. Moreover, by Theorem 5.6, its total number of finite zeros is $\sum_{i=1}^r \deg(\epsilon_i(\lambda))$. Denote by $q_i^L$, $i = 1, \ldots, n+s+r$, the invariant orders at infinity of $L(\lambda)$. By Lemma 5.5,

$$\mu(L(\lambda)) = -\sum_{i=1}^{r} \deg(\epsilon_i(\lambda)) - \sum_{i=1}^{n+s+r} q_i^L.$$

By Lemma 5.7:

(i) If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then

$$\begin{aligned} \mu(L(\lambda)) \quad &= -\textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - (\sum_{i=1}^{n+s}(-1) + \sum_{i=1}^r (q_i + d - 1)) \\ &= -\textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) + n + s + r - dr - \sum_{i=1}^r q_i. \end{aligned}$$

(ii) If $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$ then

$$\begin{aligned} \mu(L(\lambda)) \quad &= -\textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - (\sum_{i=1}^{n}(-1) + \sum_{i=1}^r (q_i + d)) \\ &= -\textstyle\sum_{i=1}^r \deg(\epsilon_i(\lambda)) + n - dr - \sum_{i=1}^r q_i. \end{aligned}$$

(iii) If $n = 0$ and $D_1 = 0$ then $L(\lambda) = D_0$ and $\mu(L(\lambda)) = -\sum_{i=1}^r \deg(\epsilon_i(\lambda)) - 0$. But since $L(\lambda)$ is constant its total number of finite zeros is 0 and, therefore, $\epsilon_i(\lambda) = 1$ for $i = 1, \ldots, r$.

∎

Finally, the following result shows the relationship between the sum of the right and left minimal indices of a rational matrix and of its strong linearizations.

**Theorem 5.9** *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be any rational matrix of rank* $r$*. Let* $d$ *be defined as in (8). Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

*be any strong linearization of* $G(\lambda)$*. Then*

$$\mu(G(\lambda)) = \begin{cases} \mu(L(\lambda)) + dr - (r+s), & \text{if } D_1 + C_1 A_1^{-1} B_1 \neq 0 \\ \mu(L(\lambda)) + dr, & \text{if } n > 0 \text{ and } D_1 + C_1 A_1^{-1} B_1 = 0 \\ dr, & \text{if } n = 0 \text{ and } D_1 = 0 \end{cases}.$$

**Proof**.- Let $\mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{p-r,m-r}\right)$ be the finite Smith–McMillan form of $G(\lambda)$ and $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity. By definition of strong linearization, $n = \nu(G(\lambda))$. Moreover, $\nu(G(\lambda)) = \sum_{i=1}^r \deg(\psi_i(\lambda))$ and, therefore, $n = \sum_{i=1}^r \deg(\psi_i(\lambda))$. By using Lemma 5.5, $\mu(G(\lambda)) = n - \sum_{i=1}^r \deg(\epsilon_i(\lambda)) - \sum_{i=1}^r q_i$. Now, by Lemma 5.8:

(i) If $D_1 + C_1 A_1^{-1} B_1 \neq 0$ then $\mu(L(\lambda)) = s + r(1-d) + \mu(G(\lambda))$.

(ii) If $n > 0$ and $D_1 + C_1 A_1^{-1} B_1 = 0$ then $\mu(L(\lambda)) = -dr + \mu(G(\lambda))$.

(iii) If $n = 0$ and $D_1 = 0$ then $\mu(L(\lambda)) = 0$ and, by Lemmas 5.7 and 5.8, $\mu(G(\lambda)) = dr$.

■

**Example 5.10** We show that, certainly, the previous result is satisfied for the matrices in Example 5.4. It was proved that the matrices $L_{\epsilon,\eta}(\lambda)$ are strong linearizations of $G(\lambda) = \left[\begin{smallmatrix} \lambda+\lambda^{-1} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Notice that, under the same notation as above, $r = 1$, $d = 1$, $n = 1$, $s = \epsilon + \eta$, $A_1 = 1, B_1 = 0, C_1 = 0$ and $D_1 + C_1 A_1^{-1} B_1 \neq 0$. As we proved $\mu(G(\lambda)) = 0$ and $\mu(L_{\epsilon,\eta}(\lambda)) = \epsilon + \eta$. Thus, $\mu(G(\lambda)) = \mu(L_{\epsilon,\eta}(\lambda)) + dr - (r+s)$, as calimed.

# 6 Minimal bases and indices of strong block minimal bases linearizations of rational matrices

The aim of this section is to study the relationship between the minimal bases and indices of a rational matrix and the minimal bases and indices of its strong block minimal bases linearizations. This family of strong linearizations is a rather general family introduced in [4, Theorem 5.11] which includes modulo permutations other families of Fiedler-like linearizations of rational matrices [1, 2, 7], as a consequence of the results in [6] and [13, Lemma 2.7]. Strong block minimal bases linearizations of rational matrices are built on strong block minimal bases linearizations of polynomial matrices, presented previously in [12, Definition 3.1] (see [11] for an expanded version of this latter reference). In order to introduce these families of linearizations and prove the results in this section, we need to recall first a number of concepts in the next paragraphs.

A matrix polynomial $N(\lambda) \in \mathbb{F}[\lambda]^{m \times l}$ with $m < l$ is a minimal basis if the columns of $N(\lambda)^T$ form a minimal basis of the subspace they span. Moreover, two matrix polynomials $K(\lambda) \in \mathbb{F}[\lambda]^{m_1 \times l}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{m_2 \times l}$ are dual minimal bases if they are both minimal bases satisfying $m_1 + m_2 = l$ and $K(\lambda) N(\lambda)^T = 0$ (see [12, 16]).

Let us recall the definition of strong block minimal bases pencils associated to a polynomial matrix (see [12, Definition 3.1 and Theorem 3.3] or [4, Definition 5.2]). Let $P(\lambda) \in \mathbb{F}[\lambda]^{p \times m}$ be a polynomial matrix. A strong block minimal bases pencil associated to $P(\lambda)$ is a linear polynomial matrix with the following structure

$$\mathcal{L}(\lambda) = \begin{array}{c} \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{array}{l} \}\, p+\widehat{p} \\ \}\, \widehat{m} \end{array} \\ \quad \underbrace{\qquad\quad}_{m+\widehat{m}} \quad \underbrace{\qquad\quad}_{\widehat{p}} \end{array}, \tag{12}$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ (respectively $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ (respectively $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$) dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$) all equal, and such that

$$P(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T. \tag{13}$$

If, in addition, $\deg(P(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ then $\mathcal{L}(\lambda)$ is said to be a strong block minimal bases pencil associated to $P(\lambda)$ with sharp degree. The key property is that any strong block minimal bases pencil associated to $P(\lambda)$ is a strong linearization of $P(\lambda)$ [12, Theorem 3.3].

Let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be the unique decomposition of $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal order state-space realization of $G_{sp}(\lambda)$ with $n = \nu(G(\lambda))$. Assume[1] that $\deg(D(\lambda)) > 1$ and let (12) be a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree, with $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, such that $D(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$. Let $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$, $\widehat{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$, $\widehat{K}_2 \in \mathbb{F}^{p\times(p+\widehat{p})}$ and $\widehat{N}_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$ be matrices such that for $i = 1, 2$

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix} \tag{14}$$

are unimodular (see in [4, Lemma 5.5] the result that guaratees that all these matrices exist and are well-defined). Let $T, S \in \mathbb{F}^{n\times n}$ be any nonsingular constant matrices. By [4, Theorem 5.11] the linear polynomial matrix

$$L(\lambda) = \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CS & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \tag{15}$$

is a strong linearization of $G(\lambda)$ and is called strong block minimal bases linearization of $G(\lambda)$.

[1] If $\deg(D(\lambda)) \le 1$, then the polynomial system matrix $\left[\begin{smallmatrix} \lambda I_n - A & B \\ -C & D(\lambda) \end{smallmatrix}\right]$ with transfer function matrix $G(\lambda)$ gives directly a strong linearization of $G(\lambda)$, as discussed in [4], and the idea of strong block minimal bases linearizations is of no interest.

Furthermore, by [4, Theorem 5.7], there are matrices $X(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times m}$ $(X(\lambda) = \widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T)$, $Y(\lambda) \in \mathbb{F}[\lambda]^{p\times\widehat{m}}$ $(Y(\lambda) = N_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T)$, and $Z(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times\widehat{m}}$ $(Z(\lambda) = \widehat{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T)$ such that

$$V(\lambda) = \begin{bmatrix} N_1(\lambda)^T & \widehat{N}_1(\lambda)^T & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix} \text{ and } U(\lambda) = \begin{bmatrix} N_2(\lambda) & -Y(\lambda) \\ 0 & I_{\widehat{m}} \\ \widehat{N}_2(\lambda) & -Z(\lambda) \end{bmatrix} \tag{16}$$

are unimodular matrices and

$$U(\lambda)\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} V(\lambda) = \operatorname{Diag}(D(\lambda), I_{\widehat{m}+\widehat{p}}),$$

as can be easily checked through a direct matrix multiplication. Moreover, $U(\lambda)\left[\begin{smallmatrix} -\widehat{K}_2^T CS \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -CS \\ 0 \end{smallmatrix}\right]$ and $\begin{bmatrix} TB\widehat{K}_1 & 0 \end{bmatrix} V(\lambda) = \begin{bmatrix} TB & 0 \end{bmatrix}$. Thus,

$$\begin{bmatrix} T^{-1} & 0 \\ 0 & U(\lambda) \end{bmatrix} L(\lambda) \begin{bmatrix} S^{-1} & 0 \\ 0 & V(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda I_n - A & B & 0 \\ -C & D(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}+\widehat{p}} \end{bmatrix}.$$

Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$, i.e.,

$$\widehat{G}(\lambda) = \begin{bmatrix} M(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B\widehat{K}_1 & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}. \tag{17}$$

Taking into account the developments above, a straightforward computation yields

$$U(\lambda)\widehat{G}(\lambda)V(\lambda) = \operatorname{Diag}(G(\lambda), I_{\widehat{m}+\widehat{p}}), \tag{18}$$

which implies, among other properties, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$ and $\dim \mathcal{N}_\ell(\widehat{G}(\lambda)) = \dim \mathcal{N}_\ell(G(\lambda))$, in agreement with the properties of any (strong) linearization of $G(\lambda)$.

In order to investigate the relationship between the minimal bases and indices of a rational matrix and those of its strong block minimal bases linearizations, we prove Lemma 6.1. This lemma first establishes the relationship between vectors in the right null-space of the rational matrix and in the right null-spaces of the transfer functions of any of its strong block minimal bases linearizations. Secondly, it relates the right minimal bases of the rational matrix and those of the transfer functions of its strong block minimal bases linearizations. Lemma 6.1 is based on [12, Lemma A.1], which is a similar result corresponding to strong block minimal bases pencils of polynomial matrices.

**Lemma 6.1** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and let $\widehat{N}_2(\lambda)$ be the matrix in (14).*

(a) *If* $h(\lambda) \in \mathcal{N}_r(G(\lambda))$ *then*

$$z(\lambda) = \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h(\lambda) \in \mathcal{N}_r(\widehat{G}(\lambda)).$$

*Moreover, if* $0 \neq h(\lambda) \in \mathcal{N}_r(G(\lambda))$ *is a vector polynomial then* $z(\lambda)$ *is also a vector polynomial and*

$$\deg(z(\lambda)) = \deg(N_1(\lambda)^T h(\lambda)) = \deg(N_1(\lambda)) + \deg(h(\lambda)). \tag{19}$$

(b) *If* $\{h_1(\lambda), \ldots, h_l(\lambda)\}$ *is a right minimal basis of* $G(\lambda)$ *then*

$$\left\{ \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_1(\lambda), \ldots, \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_l(\lambda) \right\}$$

*is a right minimal basis of* $\widehat{G}(\lambda)$.

**Proof.**- By Proposition 4.1, equation (18) and using the structure of $V(\lambda)$ in (16) (recall that $X(\lambda) = \widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T$) we obtain the first part of (a). Now, we are going to prove (19) following the ideas of [12, Lemma A.1]. It can be seen as in the proof of [12, Lemma A.1] that for any vector polynomial $g(\lambda) \neq 0$

$$\deg(N_1(\lambda)^T g(\lambda)) = \deg(N_1(\lambda)) + \deg(g(\lambda)), \tag{20}$$

for any vector polynomial $y(\lambda) \neq 0$

$$\deg(K_2(\lambda)^T y(\lambda)) = \deg(K_2(\lambda)) + \deg(y(\lambda)) = 1 + \deg(y(\lambda)), \tag{21}$$

and

$$\deg(z(\lambda)) = \max\{\deg(N_1(\lambda)^T h(\lambda)), \deg(X(\lambda)h(\lambda))\}. \tag{22}$$

If $X(\lambda)h(\lambda) = 0$ then (19) follows. Otherwise, use $0 = \widehat{G}(\lambda)z(\lambda)$ and consider the expression of $\widehat{G}(\lambda)$ in (17)

$$\begin{aligned} 0 &= \begin{bmatrix} M(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B \widehat{K}_1 & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} N_1(\lambda)^T \\ -X(\lambda) \end{bmatrix} h(\lambda) \\ &= \begin{bmatrix} M(\lambda)N_1(\lambda)^T + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B - K_2(\lambda)^T X(\lambda) \\ 0 \end{bmatrix} h(\lambda). \end{aligned}$$

Therefore, $M(\lambda)N_1(\lambda)^T h(\lambda) - K_2(\lambda)^T X(\lambda)h(\lambda) = -\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)$. Since the expression on the left hand side of this equality is polynomial, the expression on the right hand side must be polynomial. Moreover, by Lemma 3.3, $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) < \deg(h(\lambda))$ since $\widehat{K}_2^T C(\lambda I_n - A)^{-1} B$ is strictly proper. Write the previous expression as

$$K_2(\lambda)^T X(\lambda)h(\lambda) = M(\lambda)N_1(\lambda)^T h(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda).$$

Notice that (21) implies that

$$1+\deg(X(\lambda)h(\lambda)) = \deg(M(\lambda)N_1(\lambda)^T h(\lambda) + \widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)).$$

Let us see now that, using the previous expression,

$$\deg(X(\lambda)h(\lambda)) \leq \deg(N_1(\lambda)^T h(\lambda)). \tag{23}$$

If $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) \leq \deg(M(\lambda)N_1(\lambda)^T h(\lambda))$ then

$$1+\deg(X(\lambda)h(\lambda)) \leq \deg(M(\lambda)N_1(\lambda)^T h(\lambda)) \leq 1+\deg(N_1(\lambda)^T h(\lambda)).$$

Otherwise, if $\deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) > \deg(M(\lambda)N_1(\lambda)^T h(\lambda))$ then

$$1+\deg(X(\lambda)h(\lambda)) = \deg(\widehat{K}_2^T C(\lambda I_n - A)^{-1} B h(\lambda)) < \deg(h(\lambda)) \text{ and}$$

$$\deg(X(\lambda)h(\lambda)) < \deg(h(\lambda)) - 1 < \deg(h(\lambda)) + \deg(N_1(\lambda)) = \deg(N_1(\lambda)^T h(\lambda)).$$

Therefore, (20), (22) and (23) prove that $\deg(z(\lambda)) = \deg(N_1(\lambda)) + \deg(h(\lambda))$.

The proof of part (b) is similar to the proof of [12, Lemma A.1] taking into account that $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(G(\lambda))$. Therefore, the details are omitted. ■

As a corollary of Lemma 6.1 we get the following result on the relationship between the minimal indices of a rational matrix and of the transfer function of any of its strong block minimal bases linearizations.

**Corollary 6.2** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$.*

(a) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)$.*

(b) *If $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$ then $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $\widehat{G}(\lambda)$.*

**Proof.**- Part (a) follows from part (b) of Lemma 6.1 and (19). Suppose now that $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$. By Lemma 3.5, $\eta_1 \leq \cdots \leq \eta_q$ are the right minimal indices of $G(\lambda)^T$. Notice that $L(\lambda)^T$ is a strong block minimal bases linearization of $G(\lambda)^T$ with transfer function matrix $\widehat{G}(\lambda)^T$. Observe that $S^T$, $A^T$, $T^T$, $B^T$, $C^T$, $\widehat{K}_1$, $\widehat{K}_2$, $M(\lambda)^T$, $K_1(\lambda)$, $K_2(\lambda)$ in $L(\lambda)^T$ play the role of $T$, $A$, $S$, $-C$, $-B$, $\widehat{K}_2$, $\widehat{K}_1$, $M(\lambda)$, $K_2(\lambda)$, $K_1(\lambda)$ in $L(\lambda)$ respectively. In particular, $K_2(\lambda)$ in $L(\lambda)^T$ plays the role of $K_1(\lambda)$ in $L(\lambda)$. Thus, by part (a), $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)^T$. By Lemma 3.5 again, $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $\widehat{G}(\lambda)$. ■

Now, we provide a recovery result for the minimal bases of a rational matrix from the minimal bases of the transfer functions of any of its strong block minimal bases linearizations, i.e., the converse of Lemma 6.1-(b).

**Lemma 6.3** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $\widehat{G}(\lambda)$ be its transfer function matrix, as in (17). Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$ and $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ be the matrices appearing in (14).*

(a) *Any right minimal basis of $\widehat{G}(\lambda)$ has the form*

$$\left\{\begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_1(\lambda), \dots, \begin{bmatrix} N_1(\lambda)^T \\ -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T \end{bmatrix} h_l(\lambda)\right\}$$

*where $\{h_1(\lambda), \dots, h_l(\lambda)\}$ is some right minimal basis of $G(\lambda)$.*

(b) *Any left minimal basis of $\widehat{G}(\lambda)$ has the form*

$$\left\{\begin{bmatrix} N_2(\lambda)^T \\ -\widehat{N}_1(\lambda)M(\lambda)^T N_2(\lambda)^T \end{bmatrix} j_1(\lambda), \dots, \begin{bmatrix} N_2(\lambda)^T \\ -\widehat{N}_1(\lambda)M(\lambda)^T N_2(\lambda)^T \end{bmatrix} j_q(\lambda)\right\}$$

*where $\{j_1(\lambda), \dots, j_q(\lambda)\}$ is some left minimal basis of $G(\lambda)$.*

**Proof.-** The proof is like the one of [11, Lemma 7.1]. Therefore, it is omitted. ∎

**Remark 6.4** Lemma 6.3 implies that a right (resp., left) minimal basis of $G(\lambda)$ can be obtained, or recovered, from any right (resp., left) minimal basis of $\widehat{G}(\lambda)$, as it is described in this remark. Let us focus for brevity only on right minimal bases, since the procedure for left minimal bases is completely analogous. Note first that the vectors $\{\widehat{h}_1(\lambda), \dots, \widehat{h}_l(\lambda)\}$ obtained by taking the top $m + \widehat{m}$ entries of the vectors of any right minimal basis of $\widehat{G}(\lambda)$ are always of the form

$$\{\widehat{h}_1(\lambda), \dots, \widehat{h}_l(\lambda)\} = \{N_1(\lambda)^T h_1(\lambda), \dots, N_1(\lambda)^T h_l(\lambda)\}, \tag{24}$$

with $\{h_1(\lambda), \dots, h_l(\lambda)\}$ a right minimal basis of $G(\lambda)$. Then, it is enough to multiply each $\widehat{h}_j(\lambda)$ by a left inverse of $N_1(\lambda)^T$ in order to get the right minimal basis $\{h_1(\lambda), \dots, h_l(\lambda)\}$ of $G(\lambda)$. Such left inverse may be, for instance, the matrix $\widehat{K}_1$ in (14). Moreover, in some cases important in applications, the matrices $N_1(\lambda)$ and $\widehat{K}_1$ are very simple and allow us to recover a right minimal basis of $G(\lambda)$ without the need of performing any matrix multiplication. This happens, for instance, if $K_1(\lambda) = L_\varepsilon(\lambda) \otimes I_m$ (and $K_2(\lambda) = L_\eta(\lambda) \otimes I_p$) in (15), where

$$L_k(\lambda) = \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)},$$

which corresponds to the well-known block Kronecker linearizations of the polynomial part of $G(\lambda)$ [12, Section 4] (see also [4, Examples 5.3 and 5.6]). In this case,

$$N_1(\lambda)^T = \begin{bmatrix} \lambda^\varepsilon \\ \vdots \\ \lambda \\ 1 \end{bmatrix} \otimes I_m \quad \text{and} \quad \widehat{K}_1 = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \otimes I_m.$$

Thus a minimal bases of $G(\lambda)$ can be obtained just by taking the last $m$ entries of the vectors $\{\widehat{h}_1(\lambda), \dots, \widehat{h}_l(\lambda)\}$ in (24).

The next Theorem 6.5 is the main result in this section, together with Theorem 6.7, and one of the most relevant results in this paper. Theorem 6.5 describes the complete relationship between the minimal bases of a rational matrix and the minimal bases of its strong block minimal bases linearizations in both directions. It follows from combining results in Section 3 with results previously obtained in this section.

**Theorem 6.5** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$, $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$ and $\widehat{N}_1(\lambda)$ and $\widehat{N}_2(\lambda)$ be the matrices appearing in (14).*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ *is a right minimal basis of $L(\lambda)$ if and only if*

$$\begin{aligned} H_1(\lambda) &= -S^{-1}(\lambda I_n - A)^{-1} B H(\lambda), \\ H_2(\lambda) &= N_1(\lambda)^T H(\lambda), \\ H_3(\lambda) &= -\widehat{N}_2(\lambda) M(\lambda) N_1(\lambda)^T H(\lambda) \end{aligned}$$

*for some right minimal basis $H(\lambda)$ of $G(\lambda)$.*

(b) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ *is a left minimal basis of $L(\lambda)$ if and only if*

$$\begin{aligned} H_1(\lambda) &= (C(\lambda I_n - A)^{-1} T^{-1})^T H(\lambda), \\ H_2(\lambda) &= N_2(\lambda)^T H(\lambda), \\ H_3(\lambda) &= -\widehat{N}_1(\lambda) M(\lambda)^T N_2(\lambda)^T H(\lambda) \end{aligned}$$

*for some left minimal basis $H(\lambda)$ of $G(\lambda)$.*

**Proof.**- Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. Notice that both $(T(\lambda I_n - A)S)^{-1} \begin{bmatrix} TB\widehat{K}_1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -\widehat{K}_2^T CS \\ 0 \end{bmatrix} (T(\lambda I_n - A)S)^{-1}$ are strictly proper matrices. By Corollary 3.9, $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ is a right minimal basis of $L(\lambda)$ if and only if $\begin{bmatrix} H_2(\lambda) \\ H_3(\lambda) \end{bmatrix}$ is a right minimal basis of $\widehat{G}(\lambda)$ and $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}B\widehat{K}_1 H_2(\lambda)$. Now, by Lemma 6.3, $H_2(\lambda) = N_1(\lambda)^T H(\lambda)$ and $H_3(\lambda) = -\widehat{N}_2(\lambda)M(\lambda)N_1(\lambda)^T H(\lambda)$ for some $H(\lambda)$ right minimal basis of $G(\lambda)$.

Part (b) is proved similarly. ∎

**Remark 6.6** Theorem 6.5 implies that a right (resp., left) minimal basis of $G(\lambda)$ can be recovered from any right (resp., left) minimal basis of any of its strong block minimal bases linearizations. Such recovery procedure is completely analogous to the one described in Remark 6.4 except for the following minor variation: in the case of Theorem 6.5 the right (resp., left) minimal bases of $G(\lambda)$ have to be recovered from the entries $n+1, n+2, \ldots, n+m+\widehat{m}$ (resp., $n+1, n+2, \ldots, n+p+\widehat{p}$) of the vectors of the right (resp., left) minimal bases of its strong block minimal bases linearizations. As in Remark 6.4, the recovery is extremely simple for strong block minimal bases linearizations of $G(\lambda)$ constructed from a block Kronecker linearization of its polynomial part.

In the last result of this section, the relationship between the minimal indices of a rational matrix and those of its strong block minimal bases linearizations is established.

**Theorem 6.7** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda)$ as in (15) be a strong block minimal bases linearization of $G(\lambda)$. Let $N_1(\lambda)$ be a minimal basis dual to $K_1(\lambda)$ and $N_2(\lambda)$ be a minimal basis dual to $K_2(\lambda)$.*

(a) *If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $L(\lambda)$.*

(b) *If $\eta_1 \leq \cdots \leq \eta_q$ are the left minimal indices of $G(\lambda)$ then $\eta_1 + \deg(N_2(\lambda)) \leq \cdots \leq \eta_q + \deg(N_2(\lambda))$ are the left minimal indices of $L(\lambda)$.*

**Proof.-** Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. If $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then, by Corollary 6.2, $\varepsilon_1 + \deg(N_1(\lambda)) \leq \cdots \leq \varepsilon_l + \deg(N_1(\lambda))$ are the right minimal indices of $\widehat{G}(\lambda)$. Now, by Theorem 3.6, these are the right minimal indices of $L(\lambda)$.

A similar proof can be done in order to prove (b). ∎

# 7 Minimal bases and indices of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices

$\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of square rational matrices have been recently introduced in [13] by combining results from [4] with the $\mathbb{M}_1$ and $\mathbb{M}_2$ ansatz spaces of linearizations of a polynomial matrix developed in [15], which in turn are inspired by the pioneer $\mathbb{L}_1$ and $\mathbb{L}_2$ vector spaces of linearizations of matrix polynomials introduced in [18]. Among other properties, $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices allow us to deal very easily with rational matrices whose polynomial part is expressed in any orthogonal basis. In this section, we study the minimal bases and indices of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices. Since these families of linearizations are closely connected to strong block minimal bases linearizations, it is not surprising that the results of this section are easily obtained from combining those in Section 6 with specific properties of $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations. In order to proceed, we need to recap first some results and notations taken from [13].

The following lemma establishes a general result about the relationship between the minimal bases and indices of two rational matrices connected by a nonsingular constant matrix on the left. We will see that this simple result will allow us to obtain the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$, $\mathbb{M}_2$-strong linearizations. The reason is that an $\mathbb{M}_1$-strong linearization is a strong block minimal bases linearization premultiplied by a nonsingular constant matrix, and an $\mathbb{M}_2$-strong linearization is a strong block minimal bases linearization postmultiplied by a nonsingular constant matrix.

**Lemma 7.1** *Let $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $X \in \mathbb{F}^{p\times p}$ be nonsingular such that $G_2(\lambda) = XG_1(\lambda)$. Then, $H(\lambda)$ is a right minimal basis of $G_1(\lambda)$ if and only if $H(\lambda)$ is a right minimal basis of $G_2(\lambda)$ and $\overline{H}(\lambda)$ is a left minimal basis of $G_1(\lambda)$ if and only if $X^{-T}\overline{H}(\lambda)$ is a left minimal basis of $G_2(\lambda)$. Moreover, $G_1(\lambda)$ and $G_2(\lambda)$ have the same right minimal indices and the same left minimal indices.*

**Proof**.- Notice that $G_1(\lambda)H(\lambda) = 0$ if and only if $G_2(\lambda)H(\lambda) = 0$. Moreover, by Lemma 3.5, $\overline{H}(\lambda)$ is a left minimal basis of $G_1(\lambda)$ if and only if $\overline{H}(\lambda)$ is a right minimal basis of $G_1(\lambda)^T$. Furthermore, $G_1(\lambda)^T\overline{H}(\lambda) = 0$ if and only if $G_2(\lambda)^T X^{-T}\overline{H}(\lambda) = 0$ and, by [10, Lemma 2.16], $X^{-T}\overline{H}(\lambda)$ is a minimal basis with the same column degrees as $\overline{H}(\lambda)$. Therefore, $\overline{H}(\lambda)$ is a right minimal basis of $G_1(\lambda)^T$ if and only if $X^{-T}\overline{H}(\lambda)$ is a right minimal basis of $G_2(\lambda)^T$ and, by Lemma 3.5 again, $X^{-T}\overline{H}(\lambda)$ is a left minimal basis of $G_2(\lambda)$. ∎

The definitions of the $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations introduced in Subsections 7.1 and 7.2 are based on the matrices and vectors presented in the next paragraphs. Consider a polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$ of $\mathbb{F}[\lambda]$, viewed as an $\mathbb{F}$-vector space, with $\phi_j(\lambda)$ a polynomial of degree $j$, that satisfies the following three-term recurrence relation:

$$\alpha_j\phi_{j+1}(\lambda) = (\lambda - \beta_j)\phi_j(\lambda) - \gamma_j\phi_{j-1}(\lambda) \quad j \geq 0 \tag{25}$$

where $\alpha_j, \beta_j, \gamma_j \in \mathbb{F}$, $\alpha_j \neq 0$, $\phi_{-1}(\lambda) = 0$, and $\phi_0(\lambda) = 1$.

Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ be a rational matrix, let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal order state-space realization of $G_{sp}(\lambda)$, where $n = \nu(G(\lambda))$. Assume that $\deg(D(\lambda)) \geq 2$. Write $D(\lambda)$ in terms of the polynomial basis $\{\phi_j(\lambda)\}_{j=0}^{\infty}$, as

$$D(\lambda) = D_k\phi_k(\lambda) + D_{k-1}\phi_{k-1}(\lambda) + \cdots + D_1\phi_1(\lambda) + D_0\phi_0(\lambda) \tag{26}$$

with $D_k \neq 0$. Let

$$\Phi_k(\lambda) = [\phi_{k-1}(\lambda)\cdots\phi_1(\lambda)\ \phi_0(\lambda)]^T, \tag{27}$$

$$m_\Phi^D(\lambda) = \begin{bmatrix} \frac{(\lambda-\beta_{k-1})}{\alpha_{k-1}}D_k + D_{k-1} & D_{k-2} - \frac{\gamma_{k-1}}{\alpha_{k-1}}D_k & D_{k-3} & \cdots & D_1 & D_0 \end{bmatrix},$$

$$M_\Phi(\lambda) = \begin{bmatrix} -\alpha_{k-2} & (\lambda-\beta_{k-2}) & -\gamma_{k-2} & & & \\ & -\alpha_{k-3} & (\lambda-\beta_{k-3}) & -\gamma_{k-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\alpha_1 & (\lambda-\beta_1) & -\gamma_1 \\ & & & & -\alpha_0 & (\lambda-\beta_0) \end{bmatrix},$$

$$F_\Phi^D(\lambda) = \begin{bmatrix} m_\Phi^D(\lambda) \\ M_\Phi(\lambda)\otimes I_m \end{bmatrix}. \tag{28}$$

### 7.1 Minimal bases and indices of $\mathbb{M}_1$-strong linearizations of rational matrices

We investigate first the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$-strong linearizations.

For any nonsingular constant matrices $T, S \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$\begin{aligned} L(\lambda) &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & 0_{n\times(k-1)m} & TB \\ \hline -CS & \multicolumn{2}{c}{m_\Phi^D(\lambda)} \\ 0_{(k-1)m\times n} & \multicolumn{2}{c}{M_\Phi(\lambda)\otimes I_m} \end{array}\right] \\ &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & 0_{n\times(k-1)m} & TB \\ \hline -CS & \multicolumn{2}{c}{\multirow{2}{*}{$F_\Phi^D(\lambda)$}} \\ 0_{(k-1)m\times n} & & \end{array}\right] \end{aligned} \tag{29}$$

is a strong linearization of $G(\lambda)$ (see [13, Theorem 3.8]). Taking into account also [13, Lemmas 3.3 and 3.7] and their proofs, note that $L(\lambda)$ is a strong block minimal bases linearization of $G(\lambda)$ as in (15) with $M(\lambda) = m_\Phi^D(\lambda)$, $K_1(\lambda) = M_\Phi(\lambda)\otimes I_m$, $K_2(\lambda)$ empty, $N_1(\lambda) = (\Phi_k(\lambda)\otimes I_m)^T = \Phi_k(\lambda)^T\otimes I_m$, $N_2(\lambda) = I_m$, $\widehat{K}_1 = e_k^T \otimes I_m$, $\widehat{K}_2 = I_m$ and $\widehat{N}_2(\lambda)$ empty, where $e_k$ is the $k$th canonical vector of size $k\times 1$. Moreover, $\widehat{N}_1(\lambda)$ is of the form $\widehat{N}_1(\lambda) = Q(\lambda)^T \otimes I_m$ with $Q(\lambda) = \left[\begin{smallmatrix} M_\Phi(\lambda) \\ e_k^T \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} I_{k-1} \\ 0 \end{smallmatrix}\right]$.

Furthermore, let $v \in \mathbb{F}^k$, $J \in \mathbb{F}^{km\times(k-1)m}$ with $\begin{bmatrix} v\otimes I_m & J\end{bmatrix}$ nonsingular and let $\mathcal{L}(\lambda) = \begin{bmatrix} v\otimes I_m & J\end{bmatrix} F_\Phi^D(\lambda)$. Then, the linear polynomial matrix

$$\begin{aligned} L_1(\lambda) &= \left[\begin{array}{c|cc} I_n & \multicolumn{2}{c}{0} \\ \hline 0 & v\otimes I_m & J \end{array}\right] \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -CS & m_\Phi^D(\lambda) \\ 0_{(k-1)m\times n} & M_\Phi(\lambda)\otimes I_m \end{array}\right] \\ &= \left[\begin{array}{c|c} T(\lambda I_n - A)S & 0_{n\times(k-1)m} \quad TB \\ \hline -(v\otimes I_m)CS & \mathcal{L}(\lambda) \end{array}\right] \end{aligned} \tag{30}$$

is a strong linearization of $G(\lambda)$, which is called $\mathbb{M}_1$-strong linearization of $G(\lambda)$ (see [13, Theorem 3.9]). Put $X = \left[\begin{array}{c|cc} I_n & \multicolumn{2}{c}{0} \\ \hline 0 & v\otimes I_m & J \end{array}\right]$, which is nonsingular. Thus, $L_1(\lambda) = XL(\lambda)$.

With all these results at hand, Theorem 7.2 establishes the relationships between the minimal bases and indices of a rational matrix and its $\mathbb{M}_1$-strong linearizations.

**Theorem 7.2** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ and let $L_1(\lambda)$ as in (30) be an $\mathbb{M}_1$-strong linearization of $G(\lambda)$. Let $\Phi_k(\lambda)$ be as in (27).*

(a) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda)\end{bmatrix}$ *is a right minimal basis of $L_1(\lambda)$ if and only if $H_{k+1}(\lambda)$ is a right minimal basis of $G(\lambda)$ and*

$$\begin{aligned} H_1(\lambda) &= -S^{-1}(\lambda I_n - A)^{-1}BH_{k+1}(\lambda), \\ H_i(\lambda) &= \phi_{k-i+1}(\lambda)H_{k+1}(\lambda), \ i = 2,\dots,k. \end{aligned}$$

(b) *If $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda)\end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ then $(v^T\otimes I_m)H_2(\lambda)$ is a left minimal basis of $G(\lambda)$ and $H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T(v^T\otimes I_m)H_2(\lambda)$.*

(c) *If $H(\lambda)$ is a left minimal basis of $G(\lambda)$ then $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda)\end{bmatrix}$ is a left minimal basis of $L_1(\lambda)$ where*

$$\begin{aligned} H_1(\lambda) &= (C(\lambda I_n - A)^{-1}T^{-1})^T H(\lambda), \\ H_2(\lambda) &= \begin{bmatrix} v\otimes I_m & J\end{bmatrix}^{-T} \begin{bmatrix} H(\lambda) \\ -\widehat{N}_1(\lambda)m_\Phi^D(\lambda)^T H(\lambda)\end{bmatrix} \end{aligned}$$

*with $\widehat{N}_1(\lambda) = Q(\lambda)^T \otimes I_m$ such that $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T\end{bmatrix}^{-1}\begin{bmatrix} I_{k-1} \\ 0\end{bmatrix}$.*

(d) *If $\varepsilon_1 \le \cdots \le \varepsilon_l$ are the right minimal indices of $G(\lambda)$ then $\varepsilon_1 + k - 1 \le \cdots \le \varepsilon_l + k - 1$ are the right minimal indices of $L_1(\lambda)$.*

(e) *If $\eta_1 \le \cdots \le \eta_l$ are the left minimal indices of $G(\lambda)$ then $\eta_1 \le \cdots \le \eta_l$ are the left minimal indices of $L_1(\lambda)$.*

**Proof.-** To prove (a), by using Lemma 7.1, we get that $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ is a right minimal basis of $L_1(\lambda)$ if and only if it is a right minimal basis of $L(\lambda)$ in (29). By the fact that $L(\lambda)$ is a strong block minimal basis linearization of $G(\lambda)$ and Theorem 6.5 (with $\widehat{N}_2(\lambda)$ empty), this occurs if and only if $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1} B H(\lambda)$ and $\begin{bmatrix} H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix} = (\Phi_k(\lambda) \otimes I_m) H(\lambda)$ for some right minimal basis $H(\lambda)$ of $G(\lambda)$. But, since $\phi_0(\lambda) = 1$, $H(\lambda)$ is uniquely determined as $H(\lambda) = H_{k+1}(\lambda)$, and $H_i(\lambda) = \phi_{k-i+1}(\lambda) H_{k+1}(\lambda)$, $i = 2, \ldots, k$.

The proof of the other parts can be done similarly by using Lemma 7.1 and Theorems 6.5 or 6.7. Observe that in this case $G(\lambda)$ is square and, therefore, it has a number of left minimal indices equal to the number of right minimal indices. ∎

**Remark 7.3** Part (a) of Theorem 7.2, together with the fact that $\phi_0(\lambda) = 1$, provides a very simple recovery rule of a right minimal basis of $G(\lambda)$ from any right minimal basis of any of its $\mathbb{M}_1$-strong linearizations: simply take the last $m$ rows of the right minimal basis of the $\mathbb{M}_1$-strong linearization. Part (b) of Theorem 7.2 also provides a simple recovery rule of a left minimal basis of $G(\lambda)$ from any left minimal basis of any of its $\mathbb{M}_1$-strong linearizations, though in this case some arithmetic operations are required unless $v$ is one of the canonical vectors of $\mathbb{F}^k$.

### 7.2 Minimal bases and indices of $\mathbb{M}_2$-strong linearizations of rational matrices

We investigate now the relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_2$-strong linearizations. The developments and results are very similar to those in Subsection 7.1 and, therefore, are described briefly.

Let $Q(\lambda)$ be a $km \times lm$ pencil of the form $Q(\lambda) = \sum_{i=1}^{k} \sum_{j=1}^{l} e_i e_j^T \otimes Q_{ij}(\lambda)$ for certain $m \times m$ pencils $Q_{ij}(\lambda)$, and where $e_i$ (resp., $e_j$) is the $i$th (resp., $j$th) canonical vector in $\mathbb{F}^k$ (resp., $\mathbb{F}^l$). The $lm \times km$ pencil $Q(\lambda)^{\mathcal{B}} = \sum_{i=1}^{k} \sum_{j=1}^{l} e_j e_i^T \otimes Q_{ij}(\lambda)$ is the block-transpose of $Q(\lambda)$. Notice that the block-transpose of $F_{\Phi}^{D}(\lambda)$ in (28) is $F_{\Phi}^{D}(\lambda)^{\mathcal{B}} = [m_{\Phi}^{D}(\lambda)^{\mathcal{B}} \quad M_{\Phi}(\lambda)^T \otimes I_m]$.

For any nonsingular constant matrices $T, S \in \mathbb{F}^{n \times n}$ the linear polynomial matrix

$$
\begin{aligned}
\mathbb{L}(\lambda) &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB & 0_{n\times(k-1)m} \\ \hline 0_{(k-1)m\times n} & & \\ -CS & m_\Phi^D(\lambda)^{\mathcal{B}} & M_\Phi(\lambda)^T \otimes I_m \end{array}\right] \\
&= \left[\begin{array}{c|c} T(\lambda I_n - A)S & TB \quad 0_{n\times(k-1)m} \\ \hline 0_{(k-1)m\times n} & \\ -CS & F_\Phi^D(\lambda)^{\mathcal{B}} \end{array}\right]
\end{aligned} \tag{31}
$$

is a strong linearization of $G(\lambda)$ (see [13, Theorem 4.3]). Notice that $\mathbb{L}(\lambda)$ is a strong block minimal bases linearization of $G(\lambda)$ as in (15) with $M(\lambda) = m_\Phi^D(\lambda)^{\mathcal{B}}$, $K_1(\lambda)$ empty, $K_2(\lambda) = M_\Phi(\lambda)\otimes I_m$, $N_1(\lambda) = I_m$, $N_2(\lambda) = (\Phi_k(\lambda)\otimes I_m)^T = \Phi_k(\lambda)^T \otimes I_m$, $\widehat{K}_1 = I_m$, $\widehat{K}_2 = e_k^T \otimes I_m$ and $\widehat{N}_1(\lambda)$ empty. Moreover, $\widehat{N}_2(\lambda) = Q(\lambda)^T \otimes I_m$ such that $Q(\lambda) = \left[\begin{smallmatrix} M_\Phi(\lambda) \\ e_k^T \end{smallmatrix}\right]^{-1}\left[\begin{smallmatrix} I_{k-1} \\ 0 \end{smallmatrix}\right]$.

Furthermore, let $w \in \mathbb{F}^k$, $J \in \mathbb{F}^{km\times(k-1)m}$ with $\left[\begin{smallmatrix} w^T\otimes I_m \\ J^{\mathcal{B}} \end{smallmatrix}\right]$ nonsingular and $\mathcal{L}(\lambda) = F_\Phi^D(\lambda)^{\mathcal{B}}\left[\begin{smallmatrix} w^T\otimes I_m \\ J^{\mathcal{B}} \end{smallmatrix}\right]$. Then, the linear polynomial matrix

$$
\begin{aligned}
\mathbb{L}_2(\lambda) &= \left[\begin{array}{c|cc} T(\lambda I_n - A)S & TB & 0_{n\times(k-1)m} \\ \hline 0_{(k-1)m\times n} & & \\ -CS & m_\Phi^D(\lambda)^{\mathcal{B}} & M_\Phi(\lambda)^T \otimes I_m \end{array}\right]\left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & w^T \otimes I_m \\ 0 & J^{\mathcal{B}} \end{array}\right] \\
&= \left[\begin{array}{c|c} T(\lambda I_n - A)S & TB(w^T \otimes I_m) \\ \hline 0_{(k-1)m\times n} & \\ -CS & \mathcal{L}(\lambda) \end{array}\right]
\end{aligned} \tag{32}
$$

is a strong linearization of $G(\lambda)$, which is called $\mathbb{M}_2$-strong linearization of $G(\lambda)$ (see [13, Theorem 4.4]). Put $Y = \left[\begin{array}{c|c} I_n & 0 \\ \hline 0 & w^T \otimes I_m \\ 0 & J^{\mathcal{B}} \end{array}\right]$, which is nonsingular. Thus, $\mathbb{L}_2(\lambda) = \mathbb{L}(\lambda)Y$.

The relationship between the minimal bases and indices of a rational matrix and its $\mathbb{M}_2$-strong linearizations is given in Theorem 7.4.

**Theorem 7.4** *Let $G(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ and let $\mathbb{L}_2(\lambda)$ as in (32) be an $\mathbb{M}_2$-strong linearization of $G(\lambda)$. Let $\Phi_k(\lambda)$ be as in (27).*

(a) *If $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right minimal basis of $\mathbb{L}_2(\lambda)$ then $(w^T \otimes I_m)H_2(\lambda)$ is a right minimal basis of $G(\lambda)$ and $H_1(\lambda) = -S^{-1}(\lambda I_n - A)^{-1}B(w^T \otimes I_m)H_2(\lambda)$.*

(b) *If $H(\lambda)$ is a right minimal basis of $G(\lambda)$ then $\left[\begin{smallmatrix} H_1(\lambda) \\ H_2(\lambda) \end{smallmatrix}\right]$ is a right minimal basis of $\mathbb{L}_2(\lambda)$ where*

$$
\begin{aligned}
H_1(\lambda) &= -S^{-1}(\lambda I_n - A)^{-1}BH(\lambda), \\
H_2(\lambda) &= \begin{bmatrix} w^T \otimes I_m \\ J^{\mathcal{B}} \end{bmatrix}^{-1}\begin{bmatrix} H(\lambda) \\ -\widehat{N}_2(\lambda)m_\Phi^D(\lambda)^{\mathcal{B}}H(\lambda) \end{bmatrix}
\end{aligned}
$$

*with* $\widehat{N}_2(\lambda) = Q(\lambda)^T \otimes I_m$ *such that* $Q(\lambda) = \begin{bmatrix} M_\Phi(\lambda) \\ e_k^T \end{bmatrix}^{-1} \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} H_1(\lambda) \\ H_2(\lambda) \\ \vdots \\ H_{k+1}(\lambda) \end{bmatrix}$ *is a left minimal basis of* $\mathbb{L}_2(\lambda)$ *if and only if* $H_{k+1}(\lambda)$ *is a left minimal basis of* $G(\lambda)$ *and*

$$H_1(\lambda) = (C(\lambda I_n - A)^{-1}T^{-1})^T H_{k+1}(\lambda),$$
$$H_i(\lambda) = \phi_{k-i+1}(\lambda) H_{k+1}(\lambda), \; i = 2, \ldots, k.$$

(d) *If* $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ *are the right minimal indices of* $G(\lambda)$ *then* $\varepsilon_1 \leq \cdots \leq \varepsilon_l$ *are the right minimal indices of* $\mathbb{L}_2(\lambda)$.

(e) *If* $\eta_1 \leq \cdots \leq \eta_l$ *are the left minimal indices of* $G(\lambda)$ *then* $\eta_1 + k - 1 \leq \cdots \leq \eta_l + k - 1$ *are the left minimal indices of* $\mathbb{L}_2(\lambda)$.

**Proof**.- The proof can be done by using Lemmas 3.5(a) and 7.1, and Theorems 6.5 and 6.7, and by following the same pattern as in the proof of Theorem 7.2. ■

**Remark 7.5** Comments similar to those in Remark 7.3 can be done in order to apply Theorem 7.4 to recover minimal bases of $G(\lambda)$ from those of any of its $\mathbb{M}_2$-strong linearizations. The only difference to be emphasized is that the roles of left and right minimal bases are interchanged in Theorems 7.2 and 7.4.

# 8 Conclusions

In this paper a complete theory about the relationship between the minimal bases and indices of a rational matrix and those of its polynomial system matrices, as well as those of its strong linearizations has been developed. In order to develop such theory a number of additional results have been obtained for general (i.e., not necessarily minimal) polynomial basis of rational matrices.

The original contributions of this paper are organized into two clearly different parts. On the one hand those in Sections 3, 4 and 5, which deal with general polynomial system matrices, general linearizations and general strong linearizations of rational matrices. On the other hand those contributions in Sections 6 and 7, which deal with specific (though large) families of strong linearizations. In the case of polynomial system matrices, we have shown that, under the standard assumption of minimality and a certain additional condition of properness, the minimal indices of the polynomial system matrices and their transfer functions are exactly the same and their minimal bases are easily related to each other. These results are connected to pioneer results by Paul Van Dooren and coworkers [22, 21], who proved similar results under different assumptions. In contrast, we have shown that the minimal bases and indices of a rational matrix and those of its linearizations and strong linearizations are not related to each other in general, and that only the sum of the left and the right minimal indices are determined by each other in the case of strong linearizations. This latter result is based on combining the fundamental index sum theorem obtained by Paul Van Dooren in [22] with the properties of strong linearizations.

In the case of the specific families of strong block minimal bases linearizations and $\mathbb{M}_1$ and $\mathbb{M}_2$-strong linearizations of rational matrices, we have proved that the minimal indices and bases of the linerizations and the rational matrices are easily related to each other and that any of them can be obtained from the others and vice versa. In this context, it is worth to emphasize the important unifying role played by strong block minimal bases linearizations of rational matrices which include modulo perturbations most of the Fiedler-like linearizations developed so far in the literature, among many other linearizations. We remark again the influence of the work of Paul Van Dooren on these results since strong block minimal bases linearizations of rational matrices are based on the corresponding concept for polynomial matrices, introduced by Van Dooren and coworkers in [12].

Finally, we would like to stress that, in our opinion, the results in this paper are carefully proved in rather simple and constructive manners, forming in this way a body of well established techniques that can be used in the future for solving similar problems concerning other families of strong linearizations of rational matrices.

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