Minimal rank factorizations of polynomial matrices

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Abstract

We investigate rank revealing factorizations of rank deficient $m \times n$ polynomial matrices $P(\lambda)$ into products of three, $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$, or two, $P(\lambda) = L(\lambda)R(\lambda)$, polynomial matrices. Among all possible factorizations of these types, we focus on those for which $L(\lambda)$ and/or $R(\lambda)$ is a minimal basis, since they allow us to relate easily the degree of $P(\lambda)$ with some degree properties of the factors. We call these factorizations *minimal rank factorizations*. Motivated by the well-known fact that, generically, rank deficient polynomial matrices over the complex field do not have eigenvalues, we pay particular attention to the properties of the minimal rank factorizations of polynomial matrices without eigenvalues. We carefully analyze the degree properties of generic minimal rank factorizations in the set of complex $m \times n$ polynomial matrices with normal rank at most r and degree at most d, and we prove that they are of the form $L(\lambda)R(\lambda)$, where the degrees of the r columns of $L(\lambda)$ differ at most by one, the degrees of the r rows of $R(\lambda)$ differ at most by one, and, for each $i = 1, \ldots, r$, the sum of the degrees of the ith column of $L(\lambda)$ and of the ith row of $R(\lambda)$ is equal to d. Finally, we show how these sets of polynomial matrices with generic factorizations are related to the sets of polynomial matrices with generic eigenstructures.

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1. Introduction

Given an $m \times n$ matrix A with complex entries and rank $r < \min\{m, n\}$, it is often useful to express A as the product of three factors $A = LER$ of sizes $m \times r$, $r \times r$ and $r \times n$, respectively, or as the product of two factors $A = LR$ of sizes $m \times r$ and $r \times n$, respectively.

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Such factorizations are sometimes called rank-revealing factorizations, or rank factorizations for short, since the sizes of the factors reveal the rank of the matrix. The singular value decomposition is probably the best known example of a rank-revealing factorization, though several other rank-revealing factorizations exist and are used in practice. Rank-revealing factorizations have many applications. Among them, data compression plays an important role [13]. It is well-known that a rank-revealing factorization $A = LR$ is equivalent to expressing A as a sum of r rank-1 matrices $A = v_1 u_1^T + \cdots + v_r u_r^T$, where $v_1, \ldots, v_r$ are the columns of L and $u_1^T, \ldots, u_r^T$ are the rows of R.

The main goal of this paper is to investigate rank-revealing factorizations of $m \times n$ polynomial matrices $P(\lambda)$, of normal rank[3] $r < \min\{m, n\}$ and degree d, into products of three, $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$, or two, $P(\lambda) = L(\lambda)R(\lambda)$, *polynomial matrices*. We will see that this problem is very different from the corresponding one for constant matrices and that it requires the use of completely different tools. These differences come essentially from two facts. First, from the constraint that the factors must be also polynomial matrices and, second, from the notion of degree, and the non-trivial question of how the degree of $P(\lambda)$ is related to the degrees (of the entries) of the factors.

This degree-problem motivates us to focus on rank-revealing factorizations of polynomial matrices, $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ or $P(\lambda) = L(\lambda)R(\lambda)$, where at least one of the leftmost or rightmost factors, $L(\lambda)$ or $R(\lambda)$, is a minimal basis [11]. We will prove that these factorizations allow us to relate the degree of $P(\lambda)$ with certain matching properties of the degrees of the entries of the factors. We call these factorizations *minimal rank factorizations*.

Another important ingredient of our work is the well-known fact, proved in [9], that *generic* $m \times n$ polynomial matrices with normal rank $r < \min\{m, n\}$ and degree at most d, over the complex field, do not have eigenvalues and have minimal indices with very particular properties. More precisely, the $m - r$ left minimal indices differ at most by one and the same happens with the $n - r$ right minimal indices (see Theorem 2.11 below). This result motivates us, in the first place, to study in more detail the minimal rank factorizations of polynomial matrices without eigenvalues and, in the second place, to look for some generic properties of rank-revealing factorizations of complex polynomial matrices. In this line, we prove that, generically, a factorization of $P(\lambda) = L(\lambda)R(\lambda)$ into two polynomial matrices of sizes $m \times r$ and $r \times n$ implies that the degrees of the r columns of $L(\lambda)$ differ at most by one, the degrees of the r rows of $R(\lambda)$ differ at most by one, and, for each $i = 1, \ldots, r$, the sum of the degrees of the ith column of $L(\lambda)$ and of the ith row of $R(\lambda)$ is equal to d. In this context, we will also study how the orbits of the polynomial matrices with the generic eigenstructures identified in [9], are related to the polynomial matrices with the generic factorizations that we identify in this work.

We are not aware of other similar results available in the literature, dealing with rank-revealing factorizations of polynomial matrices of degree larger than one. However, there exist factorizations of this type in the case of degree at most one, that is, in the case of matrix pencils. In fact, rank-revealing factorizations expressed as the sum of matrix pencils with rank one exist for unstructured pencils [2, 3, 4] and also for matrix pencils with symmetry structures [8]. Rank-revealing factorizations of matrix pencils have played a fundamental role in the study of the generic effect of low rank perturbations on the

[3]The concepts mentioned in this introduction are revised in Section 2.

eigenstructure of a given matrix pencil. Therefore, we hope that the results developed in this paper will have applications in the study of the generic effect of low rank perturbations on the eigenstructure of a given *polynomial matrix* of degree larger than one, which is a problem that remains open in the literature.

We emphasize that the rank-revealing factorizations of matrix pencils mentioned above have been obtained by using the Kronecker canonical form of pencils under strict equivalence [12], or structured versions of this form. Since a canonical form of this type does not exist for polynomial matrices of degree larger than one, the problem for polynomial matrices is harder than for matrix pencils and requires different tools.

The paper is organized as follows. Section 2 includes some known concepts and results that are important for obtaining the main results of this paper. Rank-revealing factorizations and minimal rank factorizations of polynomial matrices are introduced in Section 3, where their properties are also studied. Section 4 establishes the generic properties of rank-revealing factorizations and minimal rank factorizations. Finally, Section 5 presents some conclusions and possible lines of future research.

2. Preliminaries

This section summarizes the notation and some of the results previously published in the literature, that will be used in the paper. Many of the results in this paper are valid over an arbitrary field $\mathbb{F}$ while others are only valid over the field $\mathbb{C}$ of complex numbers. This will be clearly indicated in the text by using either $\mathbb{F}$ or $\mathbb{C}$. $\mathbb{F}[\lambda]$ stands for the ring of polynomials in the variable λ with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ stands for the field of fractions of $\mathbb{F}[\lambda]$, i.e., rational functions in the variable λ with coefficients in $\mathbb{F}$. A polynomial vector is a vector with entries in $\mathbb{F}[\lambda]$. $\mathbb{F}[\lambda]^{m\times n}$ and $\mathbb{F}(\lambda)^{m\times n}$ denote the sets of $m\times n$ polynomial matrices and of $m\times n$ rational matrices, respectively, over $\mathbb{F}$. The degree of a polynomial vector, $q(\lambda)$, or of a polynomial matrix, $P(\lambda)$, is the highest degree of all of its entries and is denoted by $\deg(q)$ or $\deg(P)$. The degree of the zero polynomial is defined to be $-\infty$. The set of $m\times n$ polynomial matrices of degree *at most* d is denoted by $\mathbb{F}[\lambda]_d^{m\times n}$. Given a list $\underline{\mathbf{d}} = (d_1, d_2, \ldots, d_m)$ of nonnegative integers, $\mathbb{F}[\lambda]_{\underline{\mathbf{d}}}^{m\times n}$ denotes the set of $m\times n$ polynomial matrices whose ith row has degree *at most* d_i for $i = 1,\ldots,m$. We also use $\overline{\mathbb{F}}$ for the algebraic closure of $\mathbb{F}$, I_n for the $n\times n$ identity matrix, and $0_{m\times n}$ for the $m\times n$ zero matrix, where the sizes are omitted when they are clear from the context. We need to use very often the ith row or the jth column of a polynomial matrix $P(\lambda)$ and we adopt the following compact notations for them: $P_{i*}(\lambda)$, or simply P_{i*}, denotes the ith row of $P(\lambda)$ and $P_{*j}(\lambda)$, or simply P_{*j}, denotes the jth column of $P(\lambda)$.

The normal rank of a polynomial or rational matrix $P(\lambda)$, denoted as $\operatorname{rank}(P)$, is the rank of $P(\lambda)$ considered as a matrix over the field $\mathbb{F}(\lambda)$, or the size of the largest non-identically zero minor of $P(\lambda)$. The reader can find more information on polynomial and rational matrices in the books [12, 14].

The set of $m\times n$ polynomial matrices with degree *at most* d and normal rank *at most* r is denoted by $\mathbb{F}[\lambda]_{d,r}^{m\times n}$. In the case $\mathbb{F} = \mathbb{C}$ and $r < \min\{m,n\}$, new results about factorizations of the elements of this set will be presented in Section 4. In order to avoid trivialities, every time that the symbol $\mathbb{F}[\lambda]_{d,r}^{m\times n}$ is written it should be understood that the integers d and r satisfy $d \geq 1$ and $r \geq 1$.

The well-known Smith form of a polynomial matrix plays a very important role in this work and the corresponding result is presented in Theorem 2.1 [12]. It requires the use of unimodular polynomial matrices, that is, square polynomial matrices with constant nonzero determinant.

Theorem 2.1. (Smith form) *Let* $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\mathrm{rank}(P) = r$. *Then there exist unimodular matrices* $U(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$, $V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ *and a diagonal matrix* $S(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *such that*

$$P(\lambda) = U(\lambda)S(\lambda)V(\lambda), \quad S(\lambda) := \left[\begin{array}{cccc|c} e_1(\lambda) & 0 & \dots & 0 & \\ 0 & e_2(\lambda) & \ddots & \vdots & \\ \vdots & \ddots & \ddots & 0 & 0_{r\times(n-r)} \\ 0 & \dots & 0 & e_r(\lambda) & \\ \hline \multicolumn{4}{c|}{0_{(m-r)\times r}} & 0_{(m-r)\times(n-r)} \end{array}\right], \tag{1}$$

where each polynomial $e_j(\lambda) \in \mathbb{F}[\lambda]$ *is monic and divides* $e_{j+1}(\lambda)$ *for* $j = 1, \dots, r-1$. *Moreover the diagonal polynomial matrix* $S(\lambda)$ *is unique.*

The unique matrix $S(\lambda)$ in (1) is the Smith form of $P(\lambda)$ and the expression $P(\lambda) = U(\lambda)S(\lambda)V(\lambda)$ is called a Smith factorization of $P(\lambda)$. Smith factorizations are not unique. The polynomials $e_j(\lambda)$ are called the invariant polynomials of $P(\lambda)$ and those that are equal to 1 are called trivial invariant polynomials. For any $\alpha \in \overline{\mathbb{F}}$, the invariant polynomials can be uniquely factorized as $e_j(\lambda) = (\lambda-\alpha)^{\sigma_j} p_j(\lambda)$, with $p_j(\lambda) \in \overline{\mathbb{F}}[\lambda]$, $p_j(\alpha) \neq 0$ and $\sigma_j \in \mathbb{N} = \{0, 1, 2, \dots\}$, for $j = 1, \dots, r$. The sequence $\sigma_1 \leq \cdots \leq \sigma_r$ is called the partial multiplicity sequence of $P(\lambda)$ at α. A root $\beta \in \overline{\mathbb{F}}$ of any of the invariant polynomials $e_j(\lambda)$ of $P(\lambda)$ is called a finite eigenvalue of $P(\lambda)$. Equivalently, $\beta \in \overline{\mathbb{F}}$ is a finite eigenvalue of $P(\lambda)$ if and only if the partial multiplicity sequence of $P(\lambda)$ at β contains at least one nonzero term.

The partial multiplicity sequence at ∞ of $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ is defined to be the partial multiplicity sequence at 0 of $\lambda^d P(1/\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ and it is said that $P(\lambda)$ has an eigenvalue at ∞ if its partial multiplicity sequence at ∞ contains at least one nonzero term, or, equivalently, if zero is an eigenvalue of $\lambda^d P(1/\lambda)$. It is easy to prove that the first term of the partial multiplicity sequence at ∞ and the degree of the polynomial are related as follows.

Lemma 2.2. [1, Lemma 2.6] *Let* $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ *with* $\mathrm{rank}(P) = r$ *and partial multiplicity sequence at* ∞ *equal to* $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_r$. *Then* $\gamma_1 = d - \deg(P)$.

Next, we refresh the concept of minimal bases of a rational subspace [11]. Let us consider the vector space $\mathbb{F}(\lambda)^n$ over the field $\mathbb{F}(\lambda)$. A subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$ is called a rational subspace. It is very easy to see that every rational subspace $\mathcal{V}$ has bases consisting entirely of polynomial vectors. Following Forney [11], we say that a minimal basis of $\mathcal{V}$ is a basis of $\mathcal{V}$ consisting of polynomial vectors whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of polynomial vectors. A key property [11] is that the ordered list of degrees of the polynomial vectors in any minimal basis of $\mathcal{V}$ is always the same. These degrees are called the minimal indices of $\mathcal{V}$. The minimal bases of any rational subspace can be characterized in different important ways [11, p. 495] (see also [14]). Among them, we emphasize the characterization in Theorem 2.4, which requires to use Definition 2.3.

Definition 2.3. *Let $d_1',\dots,d_n'$ be the degrees of the columns of $N(\lambda)\in\mathbb{F}[\lambda]^{m\times n}$. The highest-column-degree coefficient matrix of $N(\lambda)$, denoted by N_{hc}, is the $m\times n$ constant matrix whose* jth *column is the vector coefficient of $\lambda^{d_j'}$ in the* jth *column of $N(\lambda)$. The polynomial matrix $N(\lambda)$ is said to be column reduced if N_{hc} has full column rank.*

Similarly, let $d_1,\dots,d_m$ be the degrees of the rows of $M(\lambda)\in\mathbb{F}[\lambda]^{m\times n}$. The highest-row-degree coefficient matrix of $M(\lambda)$, denoted by M_{hr}, is the $m\times n$ constant matrix whose jth *row is the vector coefficient of λ^{d_j} in the* jth *row of $M(\lambda)$. The polynomial matrix $M(\lambda)$ is said to be row reduced if M_{hr} has full row rank.*

Theorem 2.4. *The columns (resp., rows) of a polynomial matrix $N(\lambda)\in\mathbb{F}[\lambda]^{m\times n}$ are a minimal basis of the rational subspace they span if and only if $N(\lambda_0)$ has full column (resp., row) rank for all $\lambda_0\in\overline{\mathbb{F}}$, and $N(\lambda)$ is column (resp., row) reduced.*

Next, we define four rational subspaces associated with a polynomial matrix $P(\lambda)$.

Definition 2.5. (Rational subspaces of a polynomial matrix) *Let $P(\lambda)\in\mathbb{F}[\lambda]^{m\times n}$. Then*

(i) $\mathcal{N}_\ell(P)=\{y(\lambda)\in\mathbb{F}(\lambda)^{1\times m}:\ y(\lambda)P(\lambda)=0\}\subseteq\mathbb{F}(\lambda)^{1\times m}$ *is the left nullspace of $P(\lambda)$,*

(ii) $\mathcal{N}_r(P)=\{x(\lambda)\in\mathbb{F}(\lambda)^{n\times 1}:\ P(\lambda)x(\lambda)=0\}\subseteq\mathbb{F}(\lambda)^{n\times 1}$ *is the right nullspace of $P(\lambda)$,*

(iii) $\mathcal{R}ow(P)=\{w(\lambda)P(\lambda)\ :\ w(\lambda)\in\mathbb{F}(\lambda)^{1\times m}\}\subseteq\mathbb{F}(\lambda)^{1\times n}$ *is the row space of $P(\lambda)$,*

(iv) $\mathcal{C}ol(P)=\{P(\lambda)v(\lambda)\ :\ v(\lambda)\in\mathbb{F}(\lambda)^{n\times 1}\}\subseteq\mathbb{F}(\lambda)^{m\times 1}$ *is the column space of $P(\lambda)$.*

Observe that if $\operatorname{rank}(P)=r$, then $\dim\mathcal{N}_\ell(P)=m-r$, $\dim\mathcal{N}_r(P)=n-r$ and $\dim\mathcal{R}ow(P)=\dim\mathcal{C}ol(P)=r$, by the rank-nullity theorem [12, Vol. I, p. 64]. Thus, $\mathcal{N}_\ell(P)$ has $m-r$ minimal indices, $\mathcal{N}_r(P)$ has $n-r$ minimal indices, and $\mathcal{R}ow(P)$ and $\mathcal{C}ol(P)$ have each of them r minimal indices.

Given a polynomial matrix $P(\lambda)\in\mathbb{F}[\lambda]_d^{m\times n}$, the set formed by its invariant polynomials, by its partial multiplicity sequence at ∞, by the minimal indices of $\mathcal{N}_\ell(P)$ and by the minimal indices of $\mathcal{N}_r(P)$ is often called the complete eigenstructure of $P(\lambda)$ [7, 17]. Observe that the minimal indices of $\mathcal{R}ow(P)$ and $\mathcal{C}ol(P)$ are *not* included in the complete eigenstructure of $P(\lambda)$.

The complete eigenstructure of a polynomial matrix satisfies the well-known index sum theorem, which was published for the first time in [18, Theorem 3] for general rational matrices. See [5] for the polynomial matrix specific version.

Theorem 2.6. (Index Sum Theorem) *Let $P(\lambda)\in\mathbb{F}[\lambda]_d^{m\times n}$ be a polynomial matrix of normal rank r, with invariant polynomials of degrees $\delta_1,\dots,\delta_r$, with partial multiplicity sequence at ∞ equal to $\gamma_1,\dots,\gamma_r$, with minimal indices of $\mathcal{N}_\ell(P)$ equal to $\eta_1,\dots,\eta_{m-r}$ and with minimal indices of $\mathcal{N}_r(P)$ equal to $\varepsilon_1,\dots,\varepsilon_{n-r}$. Then,*

$$rd=\sum_{i=1}^{m-r}\eta_i+\sum_{j=1}^{n-r}\varepsilon_j+\sum_{k=1}^{r}\gamma_k+\sum_{\ell=1}^{r}\delta_\ell.$$

2.1. Dual minimal bases and related properties

Dual minimal bases were defined in [6], but they are closely linked to the dual rational subspaces introduced much earlier in [11, Section 6]. For brevity, we often say in this paper that a polynomial matrix $M(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ is a minimal basis if its rows form a minimal basis of the rational subspace they span when $n \geq m$ or if its columns form a minimal basis of the rational subspace they span when $m \geq n$.

Definition 2.7. *Two polynomial matrices $M(\lambda) \in \mathbb{F}[\lambda]^{m\times k}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{n\times k}$ are dual minimal bases if they are minimal bases satisfying $m + n = k$ and $M(\lambda)\, N(\lambda)^T = 0$.*

Observe that the dual minimal bases in Definition 2.7 satisfy that the rows of $M(\lambda)$ form a minimal basis of $\mathcal{N}_\ell(N(\lambda)^T)$ and that the columns of $N(\lambda)^T$ form a minimal basis of $\mathcal{N}_r(M(\lambda))$. As a consequence, the minimal indices of $\mathcal{N}_r(M(\lambda))$ are the degrees of the rows of $N(\lambda)$ and the minimal indices of $\mathcal{N}_\ell(N(\lambda)^T)$ are the degrees of the rows of $M(\lambda)$.

Dual minimal bases satisfy Theorem 2.8, whose "direct part" was proved in [11, p. 503] (see other proofs in [6, Remark 2.14] and in [7, Lemma 3.6]) and whose "converse part" was proved in [6, Theorem 6.1].

Theorem 2.8. *Let $M(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+n)}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{n\times(m+n)}$ be dual minimal bases with the degrees of their rows equal to $(d_1, \dots, d_m)$ and to $(d'_1, \dots, d'_n)$, respectively. Then*

$$\sum_{i=1}^{m} d_i = \sum_{j=1}^{n} d'_j \,. \tag{2}$$

Conversely, given any two lists of nonnegative integers $(d_1, \dots, d_m)$ and $(d'_1, \dots, d'_n)$ satisfying (2), there exists a pair of dual minimal bases $M(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+n)}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{n\times(m+n)}$ such that the degrees of the rows of $M(\lambda)$ and $N(\lambda)$ are $(d_1, \dots, d_m)$ and $(d'_1, \dots, d'_n)$, respectively.

A corollary of Theorem 2.8 is the following result.

Corollary 2.9. *Let $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ be a polynomial matrix of normal rank r, with minimal indices of $\mathcal{N}_\ell(P)$ equal to $\eta_1, \dots, \eta_{m-r}$, with minimal indices of $\mathcal{N}_r(P)$ equal to $\varepsilon_1, \dots, \varepsilon_{n-r}$, with minimal indices of $\mathcal{R}ow(P)$ equal to $r_1, \dots, r_r$, and with minimal indices of $\mathcal{C}ol(P)$ equal to $c_1, \dots, c_r$. Then*

$$\sum_{i=1}^{m-r} \eta_i = \sum_{i=1}^{r} c_i \qquad \text{and} \qquad \sum_{i=1}^{n-r} \varepsilon_i = \sum_{i=1}^{r} r_i \,.$$

Proof. We only prove the first equality, since the second one follows from applying the first to $P(\lambda)^T$. Let us arrange a minimal basis of $\mathcal{N}_\ell(P)$ as the rows of a matrix $M(\lambda) \in \mathbb{F}[\lambda]^{(m-r)\times m}$ and a minimal basis of $\mathcal{C}ol(P)$ as the columns of a matrix $N(\lambda)^T \in \mathbb{F}[\lambda]^{m\times r}$. Then $M(\lambda)N(\lambda)^T = 0$, which implies that $M(\lambda)$ and $N(\lambda)$ are dual minimal bases and the first equality follows from Theorem 2.8. $\square$

Combining Theorem 2.6 and Corollary 2.9, we obtain the following dual version of the Index Sum Theorem.

Corollary 2.10. (Dual version of the Index Sum Theorem) *Let $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ be a polynomial matrix of normal rank r, with invariant polynomials of degrees $\delta_1, \ldots, \delta_r$, with partial multiplicity sequence at ∞ equal to $\gamma_1, \ldots, \gamma_r$, with minimal indices of $\mathcal{R}ow(P)$ equal to $r_1, \ldots, r_r$ and with minimal indices of $\mathcal{C}ol(P)$ equal to $c_1, \ldots, c_r$. Then,*

$$rd = \sum_{i=1}^{r} c_i + \sum_{j=1}^{r} r_j + \sum_{k=1}^{r} \gamma_k + \sum_{\ell=1}^{r} \delta_\ell.$$

2.2. Generic complete eigenstructures in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$

We recall in this subsection the main results of [9]. For that, we need to introduce some concepts. First, we introduce a distance in the vector space (over the field $\mathbb{C}$) $\mathbb{C}[\lambda]_d^{m\times n}$ in terms of the Frobenius matrix norm of complex matrices as follows: Given $P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathbb{C}[\lambda]_d^{m\times n}$ and $Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathbb{C}[\lambda]_d^{m\times n}$, where $P_i, Q_i \in \mathbb{C}^{m\times n}$, for $i = 0, \ldots, d$, the distance between $P(\lambda)$ and $Q(\lambda)$ is

$$\rho(P, Q) := \left(\sum_{i=0}^{d} ||P_i - Q_i||_F^2 \right)^{\frac{1}{2}}. \tag{3}$$

This makes $\mathbb{C}[\lambda]_d^{m\times n}$ a metric space and allows us to define in it limits, open and closed sets, closures of sets and any other topological concept. The closure of any subset $\mathcal{A}$ of $\mathbb{C}[\lambda]_d^{m\times n}$ will be denoted by $\overline{\mathcal{A}}$.

Given $P(\lambda) \in \mathbb{C}[\lambda]_d^{m\times n}$, we define the orbit of $P(\lambda)$, denoted by $\mathcal{O}(P)$, as the set of polynomial matrices in $\mathbb{C}[\lambda]_d^{m\times n}$ with the same complete eigenstructure as $P(\lambda)$. The closure of $\mathcal{O}(P)$ is denoted by $\overline{\mathcal{O}}(P)$. Observe that all the polynomial matrices in $\mathcal{O}(P)$ have the same rank, since the complete eigenstructure determines the rank, and the same degree, since the first term in the partial multiplicity sequence at ∞ determines the degree according to Lemma 2.2.

The main result in [9] describes $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ in terms of closures of orbits of certain polynomial matrices with very particular complete eigenstructures. It is stated in the next theorem.

Theorem 2.11. [9, Theorem 3.2] *Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r < \min\{m, n\}$. Define $rd + 1$ complete eigenstructures $\mathbf{K}_a$ of polynomial matrices in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ with r invariant polynomials all equal to one, with all the terms of the partial multiplicity sequence at ∞ equal to zero (equivalently, without finite or infinite eigenvalues), with $m - r$ minimal indices of the left null space equal to β and $\beta + 1$, and with $n - r$ minimal indices of the right null space equal to α and $\alpha + 1$, as follows:*

$$\mathbf{K}_a : \{\underbrace{\alpha + 1, \ldots, \alpha + 1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}, \underbrace{\beta + 1, \ldots, \beta + 1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}\} \tag{4}$$

for $a = 0, 1, \ldots, rd$, where $\alpha = \lfloor a/(n - r) \rfloor$, $s = a \mod (n - r)$, $\beta = \lfloor (rd - a)/(m - r) \rfloor$, and $t = (rd - a) \mod (m - r)$. Then,

(i) *There exists a polynomial matrix $K_a(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$ of degree exactly d and normal rank exactly r with the complete eigenstructure $\mathbf{K}_a$ for $a = 0, 1, \ldots, rd$;*

(ii) *For every polynomial matrix* $M(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$, *there exists an integer* a *such that* $\overline{\mathcal{O}}(K_a) \supseteq \overline{\mathcal{O}}(M)$;

(iii) $\overline{\mathcal{O}}(K_a) \bigcap \mathcal{O}(K_{a'}) = \emptyset$ *whenever* $a \neq a'$;

(iv) $\mathbb{C}[\lambda]_{d,r}^{m\times n} = \displaystyle\bigcup_{0\leq a\leq rd} \overline{\mathcal{O}}(K_a)$ *and* $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ *is a closed subset of* $\mathbb{C}[\lambda]_{d}^{m\times n}$.

Moreover, it was proved in [9, Corollary 3.3] that for each $a = 0, 1, \ldots, rd$, the orbit $\mathcal{O}(K_a)$ is an open subset of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ (in the subspace topology of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ corresponding to the distance (3)). This means that $\bigcup_{0\leq a\leq rd} \mathcal{O}(K_a)$ is an open and dense subset of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$, which justifies to term the complete eigenstructures in (4) as the generic eigenstructures of the polynomial matrices in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$. As we have explained in Section 1, one of the main objectives of this paper is to provide an alternative description of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ in terms of the union of the closures of some sets of polynomial matrices that can be factorized in certain specific ways and to relate this description with that in Theorem 2.11-(iv). This is done in Section 4.

2.3. Generic polynomial matrices in $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$

The last subsection in these preliminaries presents a result from [10] that describes the generic polynomial matrices in the vector space (over the field $\mathbb{C}$) $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$, where $r, s > 0$. Observe that if $\underline{\mathbf{d}} = (d_1, d_2, \ldots, d_r)$ and $d = \max_{1\leq i\leq r} d_i$, then $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$ is a subspace of $\mathbb{C}[\lambda]_{d}^{r\times(r+s)}$ and we can use the distance (3) in $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$. Moreover, this allows us to define naturally the partial multiplicity sequence at ∞ of any $M(\lambda) \in \mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$ as the partial multiplicity at 0 of $\lambda^d P(1/\lambda)$.

Next, we define an important subset of $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$, which is proved to be generic in Theorem 2.13.

Definition 2.12. *Let* $r, s > 0$ *be two positive integers, consider the set* $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$, *where* $\underline{\mathbf{d}} = (d_1, d_2, \ldots, d_r)$ *is a list of nonnegative integers, and define*

$$k' = \left\lceil \frac{\sum_{i=1}^{r} d_i}{s} \right\rceil \quad \text{and} \quad s\,k' = \sum_{i=1}^{r} d_i + t, \quad \text{where } 0 \leq t < s.$$

Then $\mathcal{G}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)} \subset \mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$ *is the set of polynomial matrices whose* i*th row has degree exactly* d_i, *for* $i = 1, \ldots, r$, *whose rows form a minimal basis, and such that their right nullspaces have* s *minimal indices,* t *of them equal to* $k' - 1$ *and* $s - t$ *equal to* k'.

Theorem 5.3 in [10] implies that $\mathcal{G}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$ is equal to the set of the polynomial matrices that have full-trimmed-Sylvester rank[4]. Combining this fact with [10, Theorem 6.2], we obtain the following result.

Theorem 2.13. $\mathcal{G}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$ *is an open and dense subset of* $\mathbb{C}[\lambda]_{\underline{\mathbf{d}}}^{r\times(r+s)}$.

[4] The reader can see in [10, Definition 5.1], the definition of this concept, though it is not of interest in the present paper.

3. Minimal rank factorizations of polynomial matrices

We consider in this section factorizations of a polynomial matrix $P(\lambda)$ *into products of other polynomial matrices* that reveal the normal rank and the degree of $P(\lambda)$.

Definition 3.1. *Let* $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\operatorname{rank}(P) = r > 0$. *A factorization of* $P(\lambda)$ *as* $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ *with* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}, E(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$ *and* $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ *is called a rank factorization of* $P(\lambda)$.

The name "rank factorization" in the definition above reminds us that the sizes of the factors $L(\lambda), E(\lambda)$, and $R(\lambda)$ reveal the rank of $P(\lambda)$. Standard linear algebra properties of matrices over the field $\mathbb{F}(\lambda)$, in particular, the inequality $\operatorname{rank}(L(\lambda)E(\lambda)R(\lambda)) \leq \min\{\operatorname{rank}(L), \operatorname{rank}(E), \operatorname{rank}(R)\}$, immediately imply the following simple well-known results.

Lemma 3.2. *Let* $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\operatorname{rank}(P) = r > 0$ *and* $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ *be a rank factorization of* $P(\lambda)$. *Then,*

(i) $\operatorname{rank}(L) = \operatorname{rank}(E) = \operatorname{rank}(R) = r$ *and, in particular,* $E(\lambda)$ *is nonsingular,*

(ii) $\mathcal{N}_\ell(P) = \mathcal{N}_\ell(L)$,

(iii) $\mathcal{N}_r(P) = \mathcal{N}_r(R)$,

(iv) $\mathcal{R}ow(P) = \mathcal{R}ow(R)$, *and the rows of* $R(\lambda)$ *are a polynomial basis of* $\mathcal{R}ow(P)$,

(v) $\mathcal{C}ol(P) = \mathcal{C}ol(L)$, *and the columns of* $L(\lambda)$ *are a polynomial basis of* $\mathcal{C}ol(P)$.

The following simple lemma is valid for rational matrices (and, so, for constant and polynomial matrices) and is also very easy to prove.

Lemma 3.3. *Let* $L(\lambda) \in \mathbb{F}(\lambda)^{m\times r}, E(\lambda) \in \mathbb{F}(\lambda)^{r\times r}$ *and* $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ *with* $\operatorname{rank}(L) = \operatorname{rank}(R) = r > 0$. *Then* $\operatorname{rank}(L(\lambda)E(\lambda)R(\lambda)) = \operatorname{rank}(E(\lambda))$.

We will often consider rank factorizations with $E(\lambda) = I_r$. In this case a rank factorization is expressed as the product of just two factors as $P(\lambda) = L(\lambda)R(\lambda)$. Observe that rank factorizations are mainly of interest for polynomial matrices $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ with $\operatorname{rank}(P) < \min\{m, n\}$, because if $\operatorname{rank}(P) = m$ or $\operatorname{rank}(P) = n$, then $P(\lambda) = I_m P(\lambda)$ or $P(\lambda) = P(\lambda)I_n$ is a rank factorization of $P(\lambda)$. Therefore, we will only consider the case $\operatorname{rank}(P) < \min\{m, n\}$ in the rest of the paper.

An example of a rank factorization of a polynomial matrix can be obtained from truncating the Smith factorization and from elementary properties of matrix multiplication. This is stated in the next lemma.

Lemma 3.4. *Let* $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\min\{m, n\} > \operatorname{rank}(P) = r > 0$ *and Smith factorization* $P(\lambda) = U(\lambda)S(\lambda)V(\lambda)$ *as in* (1). *Let* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ *be the polynomial matrix whose columns are the first* r *columns of* $U(\lambda)$, $E(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$ *be the diagonal polynomial matrix whose diagonal entries are the first* r *diagonal entries of* $S(\lambda)$, *and* $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ *be the polynomial matrix whose rows are the first* r *rows of* $V(\lambda)$. *Then,* $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ *is a rank factorization of* $P(\lambda)$.

Definition 3.5. *The factorization $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ in Lemma 3.4 is called a Smith rank factorization of $P(\lambda)$.*

Smith rank factorizations $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ reveal the invariant polynomials of $P(\lambda)$ in $E(\lambda)$, which is a very important information, in addition to the rank of $P(\lambda)$. However, in general, the columns of $L(\lambda)$ are not a minimal basis of $\mathcal{C}ol(P)$ and the rows of $R(\lambda)$ are not a minimal basis of $\mathcal{R}ow(P)$. Moreover, the degree properties of a Smith rank factorization of $P(\lambda)$ are not optimal in general. The next example illustrates these facts.

Example 3.6. Let

$$P(\lambda) = \begin{bmatrix} \lambda^8 & 0 & 0 \\ \lambda^6 + 1 & -\lambda^7 & -\lambda^5 \\ 1 & -\lambda^7 & -\lambda^5 \end{bmatrix}. \tag{5}$$

Then,

$$P(\lambda) = \begin{bmatrix} \lambda^8 & \lambda^2 & 1 \\ \lambda^6 + 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \\ 0 & \lambda^2 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$P(\lambda) = \begin{bmatrix} \lambda^8 & \lambda^2 \\ \lambda^6 + 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{11} \end{bmatrix} \begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \\ 0 & \lambda^2 & 1 \end{bmatrix} =: L(\lambda)E(\lambda)R(\lambda) \tag{6}$$

are, respectively, a Smith factorization and a Smith rank factorization of $P(\lambda)$. Note that according to Theorem 2.4 neither $L(\lambda)$ nor $R(\lambda)$ are minimal bases because their highest-column-degree and highest-row-degree coefficients are, respectively,

$$L_{hc} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad R_{hr} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

which do not have full column and full row rank, respectively. Observe that the degree of $P(\lambda)$ is 8 and that is not equal to the sum of the degrees of the three factors in (6), which is 26. This inequality is expected because the entries with highest degrees in each factor do not interact when the product $L(\lambda)E(\lambda)R(\lambda)$ is computed. But note also that if (6) is expanded into a sum of rank one matrices as follows

$$P(\lambda) = \begin{bmatrix} \lambda^8 \\ \lambda^6 + 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \end{bmatrix} + \begin{bmatrix} \lambda^2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda^{11} \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \end{bmatrix}, \tag{7}$$

then the degrees of both terms are equal to 15, again much larger than $\deg(P) = 8$.

In the rest of this section, we explore other rank factorizations, different from Smith rank factorizations, of a polynomial matrix $P(\lambda)$ whose factors provide minimal bases of $\mathcal{C}ol(P)$ and/or $\mathcal{R}ow(P)$ and reveal the degree of $P(\lambda)$. We emphasize that, in general, such factorizations do not reveal explicitly the invariant polynomials of $P(\lambda)$.

We will need in the sequel the two auxiliary Lemmas 3.7 and 3.8. Lemma 3.7 implies, in particular, that any rank factorization of a polynomial matrix $P(\lambda)$ with normal rank equal to one reveals the degree of $P(\lambda)$ via the sum of the degrees of the three factors. The simple proof of this lemma is omitted.

Lemma 3.7. *Let* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times 1}, E(\lambda) \in \mathbb{F}[\lambda]^{1\times 1}, R(\lambda) \in \mathbb{F}[\lambda]^{1\times n}$ *and* $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$. *Then* $\deg(P) = \deg(L) + \deg(E) + \deg(R)$.

Lemma 3.8 is a consequence of [19, Theorem 2.5.7], which introduces an algorithm for transforming any polynomial matrix with full column rank into a column reduced polynomial matrix via multiplication on the right by unimodular matrices. In order to be self-contained, we include a short proof of this lemma.

Lemma 3.8.

(i) *Let* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ *be a polynomial matrix such that the constant matrix* $L(\lambda_0)$ *has full column rank* r *for all* $\lambda_0 \in \overline{\mathbb{F}}$. *Then,* $L(\lambda)$ *can be factorized as* $L(\lambda) = L_c(\lambda)V(\lambda)$, *where the columns of* $L_c(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ *form a minimal basis of* $\mathcal{C}ol(L)$ *and* $V(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$ *is unimodular. Hence, the degrees of the columns of* $L_c(\lambda)$ *are the minimal indices of* $\mathcal{C}ol(L)$.

(ii) *Let* $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ *be a polynomial matrix such that the constant matrix* $R(\lambda_0)$ *has full row rank* r *for all* $\lambda_0 \in \overline{\mathbb{F}}$. *Then,* $R(\lambda)$ *can be factorized as* $R(\lambda) = U(\lambda)R_r(\lambda)$, *where the rows of* $R_r(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ *form a minimal basis of* $\mathcal{R}ow(R)$ *and* $U(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$ *is unimodular. Hence, the degrees of the rows of* $R_r(\lambda)$ *are the minimal indices of* $\mathcal{R}ow(R)$.

Proof. We only prove item (i), since item (ii) is obtained from item (i) by transposition. The columns of $L(\lambda)$ are a basis of $\mathcal{C}ol(L)$. If the columns of $L_c(\lambda)$ are any minimal basis of $\mathcal{C}ol(L)$, then $L(\lambda) = L_c(\lambda)V(\lambda)$, with $V(\lambda)$ an $r\times r$ polynomial matrix according to [11, p. 495]. In addition, $V(\lambda)$ must be unimodular since, otherwise, $L(\lambda_0) = L_c(\lambda_0)V(\lambda_0)$ would have rank strictly smaller than r for any root λ_0 of $\det V(\lambda_0)$. $\square$

The next example illustrates Lemma 3.8.

Example 3.9. The matrices $L(\lambda)$ and $R(\lambda)$ in (6) can be factorized as follows:

$$\begin{bmatrix} \lambda^8 & \lambda^2 \\ \lambda^6+1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda^6 & 1 \end{bmatrix} =: L_c(\lambda)V(\lambda), \tag{8}$$

$$\begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \\ 0 & \lambda^2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\lambda^5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 1 \end{bmatrix} =: U(\lambda)R_r(\lambda). \tag{9}$$

Theorem 2.4 implies that the columns of $L_c(\lambda)$ are a minimal basis, as well as the rows of $R_r(\lambda)$. Obviously $V(\lambda)$ and $U(\lambda)$ are unimodular matrices.

Theorem 3.10 presents for each polynomial matrix three different types of rank factorizations, with $E(\lambda) = I_r$ in the case of items (ii) and (iii).

Theorem 3.10. *Let* $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ *with* $\min\{m,n\} > \operatorname{rank}(P) = r > 0$. *Then,* $P(\lambda)$ *can be factorized as follows:*

(i) $P(\lambda) = L_c(\lambda)E(\lambda)R_r(\lambda)$, *where* $L_c(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$, $E(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$, $R_r(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$, *the columns of* $L_c(\lambda)$ *form a minimal basis of* $\mathcal{C}ol(P)$, *the rows of* $R_r(\lambda)$ *form a minimal basis of* $\mathcal{R}ow(P)$, *and the invariant polynomials of* $E(\lambda)$ *are the invariant polynomials of* $P(\lambda)$.

(ii) *$P(\lambda) = L_c(\lambda)R(\lambda)$, where $L_c(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$, $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$, the columns of $L_c(\lambda)$ form a minimal basis of $\mathcal{C}ol(P)$ and the invariant polynomials of $R(\lambda)$ are the invariant polynomials of $P(\lambda)$.*

(iii) *$P(\lambda) = L(\lambda)R_r(\lambda)$, where $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$, $R_r(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$, the rows of $R_r(\lambda)$ form a minimal basis of $\mathcal{R}ow(P)$, and the invariant polynomials of $L(\lambda)$ are the invariant polynomials of $P(\lambda)$.*

Proof. Let $P(\lambda) = \widetilde{L}(\lambda)\widetilde{E}(\lambda)\widetilde{R}(\lambda)$ with $\widetilde{L}(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$, $\widetilde{E}(\lambda) \in \mathbb{F}[\lambda]^{r\times r}$, and $\widetilde{R}(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$, be a Smith rank factorization as in Lemma 3.4. Therefore, $\widetilde{L}(\lambda_0)$ and $\widetilde{R}(\lambda_0)$ have, respectively, full column rank and full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, because they are formed by columns and rows, respectively, of unimodular matrices. Then using the factorizations in Lemma 3.8 applied to $\widetilde{L}(\lambda)$ and $\widetilde{R}(\lambda)$, we get the following three expressions,

$$P(\lambda) = L_c(\lambda)\,(V(\lambda)\widetilde{E}(\lambda)U(\lambda))\,R_r(\lambda), \tag{10}$$

$$P(\lambda) = L_c(\lambda)\,(V(\lambda)\widetilde{E}(\lambda)\widetilde{R}(\lambda)), \tag{11}$$

$$P(\lambda) = (\widetilde{L}(\lambda)\widetilde{E}(\lambda)U(\lambda))\,R_r(\lambda). \tag{12}$$

The factorization in (10) proves item (i) with $E(\lambda) = V(\lambda)\widetilde{E}(\lambda)U(\lambda)$, because the r diagonal entries of $\widetilde{E}(\lambda)$ are the invariant polynomials of $P(\lambda)$ and they do not change under multiplications by unimodular matrices. The statements about $\mathcal{C}ol(P)$ and $\mathcal{R}ow(P)$ follow from Lemma 3.2.

The factorization in (11) proves item (ii) with $R(\lambda) = V(\lambda)\widetilde{E}(\lambda)\widetilde{R}(\lambda)$. Note that $\widetilde{R}(\lambda)$ is formed by the first r rows of a unimodular matrix $\widetilde{V}(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, according to Lemma 3.4. Thus,

$$R(\lambda) = V(\lambda)\begin{bmatrix}\widetilde{E}(\lambda) & 0\end{bmatrix}\widetilde{V}(\lambda),$$

and indeed the invariant polynomials of $R(\lambda)$ are the same of those of $\widetilde{E}(\lambda)$, which in turn are those of $P(\lambda)$. The statement about $\mathcal{C}ol(P)$ follows again from Lemma 3.2.

Analogously, the factorization in (12) proves item (iii). □

Definition 3.11. *Any of the three factorizations introduced in Theorem 3.10 is called a minimal rank factorization of $P(\lambda)$.*

The name "minimal rank factorization" in Definition 3.11 reminds us that these factorizations display a minimal basis of $\mathcal{C}ol(P)$ and/or a minimal basis of $\mathcal{R}ow(P)$, in addition to the rank of $P(\lambda)$.

Remark 3.12. The minimal rank factorizations in Theorem 3.10 are not unique. In fact $L_c(\lambda)$ can be any of the infinitely many minimal bases of $\mathcal{C}ol(P)$ and $R_r(\lambda)$ can be any of the infinitely many minimal bases of $\mathcal{R}ow(P)$. However, note that once $L_c(\lambda)$ and/or $R_r(\lambda)$ are chosen, $E(\lambda)$ in item (i) is uniquely determined by this choice, $R(\lambda)$ in item (ii) is uniquely determined by this choice, and $L(\lambda)$ in item (iii) is uniquely determined by this choice.

The next example illustrates Theorem 3.10.

Example 3.13. In this example, the Smith rank factorization in (6) is combined with the factorizations in (8) and (9) to obtain the following minimal rank factorizations of $P(\lambda)$ in (5):

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\lambda^5 \\ \lambda^6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 1 \end{bmatrix} =: L_c(\lambda)F(\lambda)R_r(\lambda), \tag{13}$$

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \\ \lambda^6 & 0 & 0 \end{bmatrix} =: L_c(\lambda)R(\lambda), \tag{14}$$

$$P(\lambda) = \begin{bmatrix} \lambda^8 & 0 \\ \lambda^6 + 1 & -\lambda^5 \\ 1 & -\lambda^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2 & 1 \end{bmatrix} =: L(\lambda)R_r(\lambda). \tag{15}$$

The factorizations in (13), (14) and (15) illustrate, respectively, items (i), (ii) and (iii) of Theorem 3.10. Observe that none of them reveals by inspection the invariant polynomials 1 and λ^{11} of $P(\lambda)$ in (5). However, the degree of $P(\lambda)$, which is 8, is revealed as the largest degree of the terms in each of the expansions of $P(\lambda)$ into a sum of rank one matrices stemming from (13), (14) and (15). These expansions are the following ones:

$$\begin{aligned} P(\lambda) &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\lambda^5 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \end{bmatrix} + \begin{bmatrix} \lambda^2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda^7 & -\lambda^5 \end{bmatrix} + \begin{bmatrix} \lambda^2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \lambda^6 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda^8 \\ \lambda^6 + 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\lambda^5 \\ -\lambda^5 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \end{bmatrix}. \end{aligned}$$

The term with highest degree in each of these expansions has degree 8, which is precisely the degree of the polynomial. This behaviour is in contrast with the degrees of the terms in the expansion (7) coming from the Smith rank factorization (6). This result about degrees holds for any minimal rank factorization and will be proved in Theorem 3.14.

Example 3.13 motivates us to establish in Theorem 3.14 the "predictable degree properties" of certain products of two and three polynomial matrices. This result is inspired by the properties of minimal bases presented in [11] (see also [14, p. 387]). Observe that Theorem 3.14 can be applied, in particular, to minimal rank factorizations but that it holds for much more general products of polynomial matrices. Recall the following notation introduced in Section 2: X_{*i} denotes the ith column of the matrix X and Y_{i*} denotes the ith row of Y.

Theorem 3.14.

(i) *Let $P(\lambda) = L(\lambda)R(\lambda)$, where $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ and $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$. If $L(\lambda)$ is column reduced, then*

$$\deg(P) = \max_{1\le i\le r} \{\deg(L_{*i}) + \deg(R_{i*})\},$$

where we emphasize that $R(\lambda)$ is a completely arbitrary matrix.

(ii) *Let* $P(\lambda) = L(\lambda)R(\lambda)$*, where* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ *and* $R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$*. If* $R(\lambda)$ *is row reduced, then*

$$\deg(P) = \max_{1\leq i\leq r}\{\deg(L_{*i}) + \deg(R_{i*})\},$$

where we emphasize that $L(\lambda)$ *is a completely arbitrary matrix.*

(iii) *Let* $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$*, where* $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$*,* $E(\lambda) \in \mathbb{F}[\lambda]^{r\times s}$ *and* $R(\lambda) \in \mathbb{F}[\lambda]^{s\times n}$*. If* $L(\lambda)$ *is column reduced and* $R(\lambda)$ *is row reduced, then*

$$\deg(P) = \max_{\substack{1\leq i\leq r\\ 1\leq j\leq s}} \{\deg(L_{*i}) + \deg(e_{ij}) + \deg(R_{j*}),$$

where we emphasize that $E(\lambda)$ *is a completely arbitrary matrix.*

Proof. Proof of item (i). The result is obviously true if $R(\lambda) = 0$. Then, we will assume that $R(\lambda) \neq 0$. Let $c_1,\ldots,c_r$ be the degrees of the columns of $L(\lambda)$ and define $D(\lambda) = \mathrm{diag}(\lambda^{c_1},\ldots,\lambda^{c_r}) \in \mathbb{F}[\lambda]^{r\times r}$. Then $P(\lambda) = (L(\lambda)D(\lambda)^{-1})(D(\lambda)R(\lambda))$. If L_{hc} is the highest-column-degree coefficient matrix of $L(\lambda)$, which has full column rank since $L(\lambda)$ is column reduced, then $(L(\lambda)D(\lambda)^{-1}) = L_{hc} + \frac{1}{\lambda}L_{-1} + \frac{1}{\lambda^2}L_{-2} + \cdots + \frac{1}{\lambda^{c_{max}}}L_{-c_{max}}$, where $c_{max} := \max\{c_1,\ldots,c_r\}$ and $L_{-j} \in \mathbb{F}^{m\times r}$ are constant matrices for $j = 1,\ldots,c_{max}$. Moreover, $D(\lambda)R(\lambda)$ is a polynomial matrix whose degree is the maximum degree d of its rows, that is, $d = \max_{1\leq i\leq r}\{c_i + \deg(R_{i*})\}$. Thus, $D(\lambda)R(\lambda) = \lambda^d\widetilde{R}_d + \cdots + \lambda\widetilde{R}_1 + \widetilde{R}_0$, with $\widetilde{R}_d \neq 0$ and $\widetilde{R}_j \in \mathbb{F}^{r\times n}$ are constant matrices for $j = 0,1,\ldots,d$. With these results at hand, we get

$$P(\lambda) = (L(\lambda)D(\lambda)^{-1})(D(\lambda)R(\lambda)) = L_{hc}\widetilde{R}_d\,\lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0,$$

with $P_j \in \mathbb{F}^{m\times n}$ for $j = 0,1,\ldots,d-1$. Note that $L_{hc}\widetilde{R}_d \neq 0$, since L_{hc} has full column rank and $\widetilde{R}_d \neq 0$, which implies that $\deg(P) = d = \max_{1\leq i\leq r}\{c_i + \deg(R_{i*})\}$ and the result is proved.

Proof of item (ii). Simply apply item (i) to the transposed polynomial matrix $P(\lambda)^T$.

Proof of item (iii). The result is obviously true if $E(\lambda) = 0$. Then, we will assume that $E(\lambda) \neq 0$. Let $c_1,\ldots,c_r$ be the degrees of the columns of $L(\lambda)$ and define $D_L(\lambda) = \mathrm{diag}(\lambda^{c_1},\ldots,\lambda^{c_r}) \in \mathbb{F}[\lambda]^{r\times r}$ and $c_{max} := \max\{c_1,\ldots,c_r\}$. Let $r_1,\ldots,r_s$ be the degrees of the rows of $R(\lambda)$ and define $D_R(\lambda) = \mathrm{diag}(\lambda^{r_1},\ldots,\lambda^{r_s}) \in \mathbb{F}[\lambda]^{s\times s}$ and $r_{max} := \max\{r_1,\ldots,r_s\}$. Then

$$P(\lambda) = (L(\lambda)D_L(\lambda)^{-1})\,(D_L(\lambda)E(\lambda)D_R(\lambda))\,(D_R(\lambda)^{-1}R(\lambda)).$$

If L_{hc} and R_{hr} are, respectively, the highest-column-degree coefficient matrix of $L(\lambda)$ and the highest-row-degree coefficient matrix of $R(\lambda)$, which have full column and full row ranks, respectively, since $L(\lambda)$ is column reduced and $R(\lambda)$ is row reduced, then

$$\begin{aligned} L(\lambda)D_L(\lambda)^{-1} &= L_{hc} + \frac{1}{\lambda}L_{-1} + \frac{1}{\lambda^2}L_{-2} + \cdots + \frac{1}{\lambda^{c_{max}}}L_{-c_{max}}\,, \\ D_R(\lambda)^{-1}R(\lambda) &= R_{hr} + \frac{1}{\lambda}R_{-1} + \frac{1}{\lambda^2}R_{-2} + \cdots + \frac{1}{\lambda^{r_{max}}}R_{-r_{max}}, \end{aligned}$$

where $L_{-j} \in \mathbb{F}^{m\times r}$ is a constant matrix for $j = 1, \dots, c_{max}$ and $R_{-j} \in \mathbb{F}^{s\times n}$ is a constant matrix for $j = 1, \dots, r_{max}$. Moreover, $D_L(\lambda)E(\lambda)D_R(\lambda)$ is a polynomial matrix whose degree is the maximum degree d of its entries, that is,

$$d = \max_{\substack{1 \le i \le r \\ 1 \le j \le s}} (c_i + \deg(e_{ij}) + r_j).$$

Thus, $D_L(\lambda)E(\lambda)D_R(\lambda) = \lambda^d \widetilde{E}_d + \cdots + \lambda \widetilde{E}_1 + \widetilde{E}_0$, with $\widetilde{E}_d \neq 0$ and $\widetilde{E}_j \in \mathbb{F}^{r\times s}$ for $j = 0, 1, \dots, d$, and

$$\begin{aligned} P(\lambda) &= (L(\lambda)D_L(\lambda)^{-1})(D_L(\lambda)E(\lambda)D_R(\lambda))(D_R(\lambda)^{-1}R(\lambda)) \\ &= \lambda^d L_{hc}\widetilde{E}_d R_{hr} + P_{d-1}\lambda^{d-1} + \cdots + P_0, \end{aligned}$$

with $P_j \in \mathbb{F}^{m\times n}$ for $j = 0, 1, \dots, d-1$. Note that $L_{hc}\widetilde{E}_d R_{hr} \neq 0$, since L_{hc} has full column rank, R_{hr} has full row rank, and $\widetilde{E}_d \neq 0$, which implies that $\deg(P) = d$. □

Remark 3.15. Observe that $P(\lambda)$ in items (i) and (ii) of Theorem 3.14 can be expanded as a sum of rank one polynomial matrices as $P(\lambda) = \sum_{i=1}^r L_{*i}(\lambda)\, R_{i*}(\lambda)$, while $P(\lambda)$ in item (iii) can be expanded as a sum of rank one polynomial matrices as $P(\lambda) = \sum_{i=1}^r \sum_{j=1}^s L_{*i}(\lambda)\, e_{ij}(\lambda)\, R_{j*}(\lambda)$. Thus, taking into account Lemma 3.7, Theorem 3.14 states that the degree of $P(\lambda)$ is precisely the degree of the term(s) with highest degree in such expansions.

In the last part of this section, we study minimal rank factorizations of polynomial matrices that have no finite or infinite eigenvalues. The motivation for this study comes from Theorem 2.11, which shows that rank deficient polynomial matrices have no finite or infinite eigenvalues, generically, when $\mathbb{F} = \mathbb{C}$. We will see that the minimal rank factorizations have very simple properties in this case.

Theorem 3.16. *Let $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ and r be an integer such that $0 < r < \min\{m, n\}$. $P(\lambda)$ has normal rank r, degree exactly d, and has no finite or infinite eigenvalues if and only if $P(\lambda)$ can be factorized as*

$$P(\lambda) = L(\lambda)R(\lambda), \quad L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}, \quad R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}, \tag{16}$$

where the columns of $L(\lambda)$ are a minimal basis, the rows of $R(\lambda)$ are a minimal basis, and

$$\deg(L_{*i}) + \deg(R_{i*}) = d, \quad \text{for } i = 1, \dots, r. \tag{17}$$

Proof. Sufficiency. If $P(\lambda)$ satisfies (16) and (17) with $L(\lambda)$ and $R(\lambda)$ minimal bases, then $\operatorname{rank}(P) = r$ follows from Lemma 3.3 with $E(\lambda) = I_r$, and $\deg(P) = d$ follows from Theorem 3.14. Lemma 3.2 implies that the degrees of the columns of $L(\lambda)$ are the minimal indices of $\mathcal{C}ol(P)$ and that the degrees of the rows of $R(\lambda)$ are the minimal indices of $\mathcal{R}ow(P)$, and (17) implies that the sum of all these minimal indices is equal to rd. Combining this fact with Corollary 2.10, we see that all the terms of the partial multiplicity sequence at ∞ of $P(\lambda)$ must be zero, as well as all the degrees of the invariant polynomials of $P(\lambda)$. This is equivalent to state that $P(\lambda)$ has no finite or infinite eigenvalues.

Necessity. If $P(\lambda)$ has normal rank r, degree d and has no finite or infinite eigenvalues, then we start from a minimal rank factorization of $P(\lambda)$ given by Theorem 3.10-(ii). That is, $P(\lambda) = L(\lambda)\widehat{R}(\lambda)$, where the columns of $L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}$ form a minimal basis of $\mathcal{C}ol(P)$ and the degrees of these columns, denoted by $c_1, \dots, c_r$, are the minimal indices of $\mathcal{C}ol(P)$. Note also that Lemma 3.2 guarantees that the rows of $\widehat{R}(\lambda) \in \mathbb{F}[\lambda]^{r\times n}$ form a basis of $\mathcal{R}ow(P)$. Let $\hat{r}_1, \dots, \hat{r}_r$ be the degrees of the rows of $\widehat{R}(\lambda)$, which are not necessarily the minimal indices $\mathcal{R}ow(P)$. Therefore, their sum is larger than or equal to the sum of the minimal indices $r_1, \dots, r_r$ of $\mathcal{R}ow(P)$ by the definition of minimal basis. That is

$$\sum_{i=1}^{r} \hat{r}_i \geq \sum_{i=1}^{r} r_i \tag{18}$$

and, simultaneously, from Theorem 3.14-(i),

$$d \geq c_i + \hat{r}_i \qquad \text{for } i = 1, \dots, r. \tag{19}$$

On the other hand Corollary 2.10 implies

$$\sum_{i=1}^{r} c_i + \sum_{j=1}^{r} r_j = rd, \tag{20}$$

since $P(\lambda)$ has no finite or infinite eigenvalues, which is equivalent to $\sum_{k=1}^{r} \gamma_k + \sum_{\ell=1}^{r} \delta_\ell = 0$. The combination of (18), (19) and (20) leads to

$$\sum_{i=1}^{r} c_i + \sum_{j=1}^{r} \hat{r}_j \geq \sum_{i=1}^{r} c_i + \sum_{j=1}^{r} r_j = rd \geq \sum_{i=1}^{r} c_i + \sum_{j=1}^{r} \hat{r}_j \,. \tag{21}$$

Therefore, $\sum_{i=1}^{r} \hat{r}_i = \sum_{i=1}^{r} r_i$, which implies that $\widehat{R}(\lambda)$ is a minimal basis of $\mathcal{R}ow(P)$. Moreover, (21) implies $\sum_{i=1}^{r} c_i + \sum_{j=1}^{r} \hat{r}_j = rd$, which combined with (19) yields

$$c_i + \hat{r}_i = d, \quad i = 1, \dots, r \,.$$

This completes the proof. □

Remark 3.17. Observe that the hypothesis that $P(\lambda)$ has degree exactly d in Theorem 3.16 is redundant, because Lemma 2.2 combined with the hypothesis that $P(\lambda)$ has not eigenvalues at ∞ implies that $\deg(P) = d$. We have included this redundant hypothesis for emphasizing this key property of the polynomial matrices satisfying Theorem 3.16.

We remark that the proof of the *necessity* in Theorem 3.16 proves, in fact, that *for any* minimal rank factorization as in Theorem 3.10-(ii) of a polynomial matrix $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ with normal rank r, with degree exactly d, and without finite or infinite eigenvalues, the factor $R(\lambda)$ must be a minimal basis and that the degree constraints (17) must be satisfied. A complementary result can be proved *for any* minimal rank factorization as in Theorem 3.10-(iii) just by transposing the argument above. These discussions can be formalized into the following theorem.

Theorem 3.18. *Let $P(\lambda) \in \mathbb{F}[\lambda]_d^{m\times n}$ be a polynomial matrix with normal rank r, $0 < r < \min\{m, n\}$, with degree exactly d, and without eigenvalues, finite or infinite. Then, the following statements hold:*

(i) *If the minimal rank factorization $P(\lambda) = L(\lambda)E(\lambda)R(\lambda)$ satisfies the properties in Theorem 3.10-(i), then the rows of $\widehat{R}(\lambda) = E(\lambda)R(\lambda)$ form a minimal basis of $\mathcal{R}ow(P)$ and the columns of $\widehat{L}(\lambda) = L(\lambda)E(\lambda)$ form a minimal basis of $\mathcal{C}ol(P)$. Moreover,*

$$\deg(L_{*i}) + \deg(\widehat{R}_{i*}) = \deg(\widehat{L}_{*i}) + \deg(R_{i*}) = d, \quad \text{for } i = 1, \dots r.$$

(ii) *If the minimal rank factorization $P(\lambda) = L(\lambda)R(\lambda)$ satisfies the properties in Theorem 3.10-(ii), then the rows of $R(\lambda)$ form a minimal basis of $\mathcal{R}ow(P)$. Moreover,*

$$\deg(L_{*i}) + \deg(R_{i*}) = d, \quad \text{for } i = 1, \dots r.$$

(iii) *If the minimal rank factorization $P(\lambda) = L(\lambda)R(\lambda)$ satisfies the properties in Theorem 3.10-(iii), then the columns of $L(\lambda)$ form a minimal basis of $\mathcal{C}ol(P)$. Moreover,*

$$\deg(L_{*i}) + \deg(R_{i*}) = d, \quad \text{for } i = 1, \dots r.$$

The next example illustrates that polynomial matrices without finite nor infinite eigenvalues have rank factorizations that are not minimal, that do not satisfy the degree conditions (17) and whose factors can have arbitrarily large degrees. Thus, for polynomial matrices without eigenvalues, minimal rank factorizations are clearly preferable.

Example 3.19. Consider the following polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]_6^{3\times 3}$ with $\operatorname{rank}(P) = 2$ and its following factorizations:

$$P(\lambda) = \begin{bmatrix} \lambda^6 & \lambda^5 & 0 \\ \lambda & \lambda^6+1 & \lambda^2 \\ 0 & \lambda^4 & 1 \end{bmatrix} = \begin{bmatrix} \lambda^5 & 0 \\ 1 & \lambda^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda^4 & 1 \end{bmatrix}, \tag{22}$$

$$= \begin{bmatrix} \lambda^5 & 0 \\ \lambda^{p+2}+1 & \lambda^2 \\ \lambda^p & 1 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ -\lambda^{p+1} & -\lambda^p+\lambda^4 & 1 \end{bmatrix}, \tag{23}$$

where $p > 3$ is an integer. Both factors in the factorization (22) are minimal bases, according to Theorem 2.4, and they satisfy (17). This proves that $P(\lambda)$ has no finite or infinite eigenvalues. In contrast, the left and right factors in (23) are not, respectively, column and row reduced polynomial matrices. So, they are not minimal bases. Nevertheless, the factorization in (23) is a rank factorization of $P(\lambda)$. Its factors have arbitrarily high degrees for arbitrarily large values of p.

By Theorem 2.11, rank deficient complex polynomial matrices have, generically, no finite or infinite eigenvalues. Combining this with Theorem 3.16, we obtain that, generically, rank deficient complex polynomial matrices have minimal rank factorizations as simple as those appearing in Theorem 3.16. However, the degree condition (17) still allows for a lot of freedom for the possible degrees of the columns of $L(\lambda)$ and the rows of $R(\lambda)$ when d is large. In the next section, we restrict this freedom considerably and prove that, generically, the degrees of the columns of $L(\lambda)$ differ at most by one and that the degrees of the rows of $R(\lambda)$ also differ at most by one.

4. Generic minimal rank factorizations in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$ and related results

In this section, we prove that arbitrarily close (in the distance defined in (3)) to any polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}_{d,r}$ there is another polynomial matrix $Q(\lambda) \in \mathbb{C}[\lambda]^{m\times n}_{d,r}$ that can be factorized as $Q(\lambda) = L(\lambda)R(\lambda)$, with $L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}$, $R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}$, with the degrees of the columns of $L(\lambda)$ differing at most by one and with the degrees of the rows of $R(\lambda)$ also differing at most by one. Moreover, we will see that $L(\lambda)$ and $R(\lambda)$ can be chosen to be minimal bases satisfying (17). Finally, we will relate the sets of polynomial matrices in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$ that can be factorized in these specific ways with the orbits $\mathcal{O}(K_a)$ of Theorem 2.11.

Before stating the first result in this section, we recall that the degree of the zero polynomial has been defined to be $-\infty$. Therefore, an expression as $\deg(L_{*i}) + \deg(R_{i*}) = d$ for the ith column and row of the factors in $Q(\lambda) = L(\lambda)R(\lambda)$ implies that $L_{*i}(\lambda) \neq 0$, $R_{i*}(\lambda) \neq 0$, $0 \leq \deg(L_{*i}) \leq d$, and $0 \leq \deg(R_{i*}) \leq d$. In contrast, an expression as $\deg(L_{*i}) + \deg(R_{i*}) \leq d$ without further conditions does not imply that $\deg(L_{*i}) \leq d$ and $\deg(R_{i*}) \leq d$, because it might be possible that $\deg(L_{*i}) = -\infty$ and $\deg(R_{i*})$ is arbitrarily large, or vice versa.

The first result in this section is a simple consequence of the results in Section 3 and states that every polynomial matrix in $\mathbb{F}[\lambda]^{m\times n}_{d,r}$ can be factorized into two factors that reveal the maximum possible rank r and such that the sums of the degrees of their corresponding columns and rows is bounded by d.

Theorem 4.1. *Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $0 < r < \min\{m, n\}$. Then*

$$\mathbb{F}[\lambda]^{m\times n}_{d,r} = \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}, \; R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}, \\ \deg(L_{*i}) \leq d, \quad \deg(R_{i*}) \leq d, \\ \deg(L_{*i}) + \deg(R_{i*}) \leq d, \quad \textit{for } i = 1, \ldots, r \end{array} \right\}.$$

Proof. For brevity, in this proof we use the notation:

$$\mathcal{S} := \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{F}[\lambda]^{m\times r}, \; R(\lambda) \in \mathbb{F}[\lambda]^{r\times n}, \\ \deg(L_{*i}) \leq d, \quad \deg(R_{i*}) \leq d, \\ \deg(L_{*i}) + \deg(R_{i*}) \leq d, \quad \text{for } i = 1, \ldots, r \end{array} \right\}.$$

Proof of $\mathbb{F}[\lambda]^{m\times n}_{d,r} \subseteq \mathcal{S}$. If $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}_{d,r}$ and $P(\lambda) = 0$, then trivially $P(\lambda) = 0_{m\times r}0_{r\times n} \in \mathcal{S}$. If $P(\lambda) \in \mathbb{F}[\lambda]^{m\times n}_{d,r}$ and $P(\lambda) \neq 0$, then $0 \leq \deg(P) = \widetilde{d} \leq d$ and $0 < \operatorname{rank}(P) = \widetilde{r} \leq r$. Then, Theorem 3.10-(ii) and Theorem 3.14-(i) imply that $P(\lambda)$ can be factorized as $P(\lambda) = \widetilde{L}(\lambda)\widetilde{R}(\lambda)$, with $\widetilde{L}(\lambda) \in \mathbb{F}[\lambda]^{m\times \widetilde{r}}$, $\widetilde{R}(\lambda) \in \mathbb{F}[\lambda]^{\widetilde{r}\times n}$, and $0 \leq \deg(\widetilde{L}_{*i}) + \deg(\widetilde{R}_{i*}) \leq \widetilde{d} \leq d$ for $i = 1, \ldots, \widetilde{r}$. If $\widetilde{r} = r$, this proves that $P(\lambda) \in \mathcal{S}$. If $\widetilde{r} < r$, then we pad $\widetilde{L}(\lambda)$ and $\widetilde{R}(\lambda)$ with zeros and define

$$L(\lambda) := \begin{bmatrix} \widetilde{L}(\lambda) & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{m\times r} \quad \text{and} \quad R(\lambda) := \begin{bmatrix} \widetilde{R}(\lambda) \\ 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{r\times n},$$

which satisfy $P(\lambda) = L(\lambda)R(\lambda)$, with $\deg(L_{*i}) \leq d$, $\deg(R_{i*}) \leq d$, and $\deg(L_{*i}) + \deg(R_{i*}) \leq d$ for $i = 1, \ldots, r$. Therefore, $P(\lambda) \in \mathcal{S}$. This proves $\mathbb{F}[\lambda]^{m\times n}_{d,r} \subseteq \mathcal{S}$.

Proof of $\mathcal{S} \subseteq \mathbb{F}[\lambda]^{m\times n}_{d,r}$. If $P(\lambda) = L(\lambda)R(\lambda) \in \mathcal{S}$, then $\operatorname{rank}(P) \leq \min\{\operatorname{rank}(L), \operatorname{rank}(R)\} \leq r$. In addition, the expansion $P(\lambda) = \sum_{i=1}^{r} L_{*i}(\lambda)R_{i*}(\lambda)$ and Lemma 3.7 imply

$\deg(P) \leq \max_{1\leq i\leq r}\{\deg(L_{*i}) + \deg(R_{i*})\} \leq d$. Thus $P(\lambda) \in \mathbb{F}[\lambda]_{d,r}^{m\times n}$, and the proof is completed. $\square$

The rest of the results of this section are valid only over the field $\mathbb{C}$ since they use limits and topological concepts with respect to the distance in (3). This will allow us to prove that every polynomial matrix in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ is the limit of a sequence of polynomial matrices in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ that can be factorized into two factors such that the degrees of their columns and rows have very specific properties when compared with those in Theorem 4.1. The first result in this direction is Theorem 4.2.

Theorem 4.2. *Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $0 < r < \min\{m, n\}$ and define the sets*

$$\mathcal{A}_{d,r}^{m\times n} := \left\{ L(\lambda)R(\lambda) \, : \, \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}, \; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg(L_{*i}) + \deg(R_{i*}) = d, \quad \textit{for } i = 1, \ldots, r \end{array} \right\}.$$

Then

$$\mathbb{C}[\lambda]_{d,r}^{m\times n} = \overline{\mathcal{A}_{d,r}^{m\times n}}\,.$$

Proof. From Theorem 4.1 it is obvious that $\mathcal{A}_{d,r}^{m\times n} \subseteq \mathbb{C}[\lambda]_{d,r}^{m\times n}$. Moreover, $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ is a closed subset of $\mathbb{C}[\lambda]_{d}^{m\times n}$ and the closure of $\mathcal{A}_{d,r}^{m\times n}$ is the smallest closed set that contains $\mathcal{A}_{d,r}^{m\times n}$. Therefore, $\mathcal{A}_{d,r}^{m\times n} \subseteq \overline{\mathcal{A}_{d,r}^{m\times n}} \subseteq \mathbb{C}[\lambda]_{d,r}^{m\times n}$.

In the rest of the proof, we prove that $\mathbb{C}[\lambda]_{d,r}^{m\times n} \subseteq \overline{\mathcal{A}_{d,r}^{m\times n}}$. If $P(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$, then Theorem 4.1 implies that $P(\lambda) = L(\lambda)R(\lambda)$ with $L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}$, $R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}$, $\deg(L_{*i}) \leq d$, $\deg(R_{i*}) \leq d$, and $\deg(L_{*i}) + \deg(R_{i*}) \leq d$, for $i = 1, \ldots, r$. Moreover, without loss of generality, we take $L_{*j}(\lambda) = 0$ whenever $R_{j*}(\lambda) = 0$. If $\deg(L_{*i}) + \deg(R_{i*}) = d$, for $i = 1, \ldots, r$, then $P(\lambda) \in \mathcal{A}_{d,r}^{m\times n} \subseteq \overline{\mathcal{A}_{d,r}^{m\times n}}$. Otherwise, let us consider the set of indices corresponding to strict inequalities, that is,

$$\mathcal{I} := \{j \, : \, 1 \leq j \leq r \text{ and } \deg(L_{*j}) + \deg(R_{j*}) < d\}.$$

Then, consider any two sequences of constant nonzero vectors $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^{m\times 1}$ such that $\lim_{k\to\infty} v_k = 0$ and $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^{1\times n}$ such that $\lim_{k\to\infty} w_k = 0$, and construct the following two sequences of polynomial matrices: (1) $L_k(\lambda) = L(\lambda) + F_k(\lambda)$, where the columns of $F_k(\lambda)$ are constructed as follows

$$(F_k)_{*j}(\lambda) = \begin{cases} 0, & \text{if } j \notin \mathcal{I}, \\ \lambda^{d-\deg(R_{j*})} v_k, & \text{if } j \in \mathcal{I} \text{ and } R_{j*}(\lambda) \neq 0, \\ \lambda^d v_k, & \text{if } j \in \mathcal{I} \text{ and } R_{j*}(\lambda) = 0, \end{cases}$$

and (2) $R_k(\lambda) = R(\lambda) + G_k(\lambda)$, where the rows of $G_k(\lambda)$ are constructed as follows

$$(G_k)_{j*}(\lambda) = \begin{cases} 0, & \text{if } j \notin \mathcal{I}, \\ 0, & \text{if } j \in \mathcal{I} \text{ and } R_{j*}(\lambda) \neq 0, \\ w_k, & \text{if } j \in \mathcal{I} \text{ and } R_{j*}(\lambda) = 0. \end{cases}$$

Then, $P_k(\lambda) := L_k(\lambda)R_k(\lambda) \in \mathcal{A}_{d,r}^{m\times n}$ and $\lim_{k\to\infty} P_k(\lambda) = P(\lambda)$, which implies that $P(\lambda) \in \overline{\mathcal{A}_{d,r}^{m\times n}}$. $\square$

Next, we consider some subsets of the set $\mathcal{A}_{d,r}^{m\times n}$ introduced in Theorem 4.2 that will be fundamental auxiliary tools for getting the main results of this section. More precisely, we express in the next theorem the set $\mathcal{A}_{d,r}^{m\times n}$ as the union of such subsets, and $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ as the union of their closures.

Theorem 4.3. *Let $\mathcal{A}_{d,r}^{m\times n}$ be the set defined in Theorem 4.2 and for each natural number $a = 0, 1, ..., rd$ define the following subsets of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$*

$$\mathcal{A}_{d,r,a}^{m\times n} := \left\{ L(\lambda)R(\lambda) \, : \, \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}, \; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg(L_{*i}) + \deg(R_{i*}) = d, \quad \textit{for } i = 1, \dots, r, \\ \sum_{i=1}^{r} \deg(R_{i*}) = a \end{array} \right\}.$$

Then

(i) $\mathcal{A}_{d,r}^{m\times n} = \bigcup_{0\le a\le rd} \mathcal{A}_{d,r,a}^{m\times n}$,

(ii) $\mathbb{C}[\lambda]_{d,r}^{m\times n} = \bigcup_{0\le a\le rd} \overline{\mathcal{A}_{d,r,a}^{m\times n}}$,

(iii) *for every $P(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$, there exists an integer a such that $P(\lambda) \in \overline{\mathcal{A}_{d,r,a}^{m\times n}}$.*

Proof. Item (i). Let us prove first that $\mathcal{A}_{d,r}^{m\times n} \subseteq \bigcup_{0\le a\le rd} \mathcal{A}_{d,r,a}^{m\times n}$. If $P(\lambda) \in \mathcal{A}_{d,r}^{m\times n}$, then $P(\lambda) = L(\lambda)R(\lambda)$ with $L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}$, $R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}$, and $\deg(L_{*i}) + \deg(R_{i*}) = d$, for $i = 1, \dots, r$. So, $\sum_{i=1}^{r} \deg(L_{*i}) + \sum_{i=1}^{r} \deg(R_{i*}) = rd$, which implies that $0 \le \sum_{i=1}^{r} \deg(R_{i*}) \le rd$. Therefore, $P(\lambda) \in \mathcal{A}_{d,r,a}^{m\times n}$ for some $a = 0, 1, \dots, rd$ and $P(\lambda) \in \bigcup_{0\le a\le rd} \mathcal{A}_{d,r,a}^{m\times n}$.

The reverse inclusion $\bigcup_{0\le a\le rd} \mathcal{A}_{d,r,a}^{m\times n} \subseteq \mathcal{A}_{d,r}^{m\times n}$ holds by definition.

Item (ii). It is an immediate consequence of Theorem 4.2, item (i), and the basic fact that "the closure of the union of a finite number of sets is the union of the closures of such sets".

Item (iii) is just another expression of item (ii). □

We present next the key technical result of this section, Theorem 4.6, which deals with some subsets of $\mathcal{A}_{d,r,a}^{m\times n}$ that are introduced in Definition 4.4.

Definition 4.4. *Let m, n, r and d be integers such that $m, n \ge 2$, $d \ge 1$ and $0 < r < \min\{m, n\}$, a be an integer such that $0 \le a \le rd$ and $(r_1, \dots, r_r)$ be any list of integers such that $0 \le r_i \le d$, for $i = 1, 2, \dots, r$, and $\sum_{i=1}^{r} r_i = a$. The following subsets of $m \times n$ polynomial matrices are defined:*

$$\mathcal{A}_{d,r,a}^{m\times n}(r_1, \dots, r_r) := \left\{ L(\lambda)R(\lambda) : \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r}, \; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg(R_{i*}) = r_i, \quad \deg(L_{*i}) = d - r_i, \; \textit{for } i = 1, \dots, r, \\ 0 \le r_i \le d, \quad \sum_{i=1}^{r} r_i = a \end{array} \right\}.$$

It is obvious that the following proposition holds.

Proposition 4.5. *Let $\mathcal{A}_{d,r,a}^{m\times n}$ be the set defined in Theorem 4.3 and $\mathcal{A}_{d,r,a}^{m\times n}(r_1, \dots, r_r)$ be any of the sets defined in Definition 4.4. Then*

$$\mathcal{A}_{d,r,a}^{m\times n}(r_1, \dots, r_r) \subset \mathcal{A}_{d,r,a}^{m\times n}.$$

Theorem 4.6. *Let $\mathcal{A}_{d,r,a}^{m\times n}$ be the set defined in Theorem 4.3 and $\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_r)$ be any of the sets defined in Definition 4.4. Then the following statements hold:*

(i) *If $(\sigma_1,\dots,\sigma_r)$ is any permutation of $(1,\dots,r)$, then*

$$\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_r)=\mathcal{A}_{d,r,a}^{m\times n}(r_{\sigma_1},\dots,r_{\sigma_r}).$$

(ii) *If $r_j-r_k\ge 2$, then*

$$\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_j,\dots,r_k,\dots,r_r)\subseteq\overline{\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_j-1,\dots,r_k+1,\dots,r_r)}\,.$$

(iii) *If $d_R=\lfloor a/r\rfloor$ and $t_R=a \bmod r$, then*

$$\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_r)\subseteq\overline{\mathcal{A}_{d,r,a}^{m\times n}(\underbrace{d_R+1,\dots,d_R+1}_{t_R},\underbrace{d_R,\dots,d_R}_{r-t_R})}\,.$$

Proof. Proof of item (i). If $L(\lambda)R(\lambda)\in\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_r)$ and Π is an $r\times r$ permutation matrix such that the ith row of $\Pi R(\lambda)$ is the σ_ith row of $R(\lambda)$, for $i=1,\dots,r$, then $L(\lambda)R(\lambda)=(L(\lambda)\Pi^T)(\Pi R(\lambda))\in\mathcal{A}_{d,r,a}^{m\times n}(r_{\sigma_1},\dots,r_{\sigma_r})$. Therefore, $\mathcal{A}_{d,r,a}^{m\times n}(r_1,\dots,r_r)\subseteq\mathcal{A}_{d,r,a}^{m\times n}(r_{\sigma_1},\dots,r_{\sigma_r})$. The "reverse" inclusion is proved in a similar manner using the "reverse" permutation.

Proof of item (ii). As a consequence of item (i), we can assume without loss of generality that $j=1$ and $k=2$. Let $P(\lambda)=L(\lambda)R(\lambda)\in\mathcal{A}_{d,r,a}^{m\times n}(r_1,r_2,\dots,r_r)$ with $r_1-r_2\ge 2$. Then, the first row of $R(\lambda)$ and the second column of $L(\lambda)$ can be written as follows:

$$R_{1*}(\lambda)=\lambda^{r_1}v_{r_1}+\widetilde{R}_{1*}(\lambda),\qquad\text{with } 0\ne v_{r_1}\in\mathbb{C}^{1\times n}\text{ and }\deg(\widetilde{R}_{1*})<r_1,\tag{24}$$

$$L_{*2}(\lambda)=\lambda^{d-r_2}w_{d-r_2}+\widetilde{L}_{*2}(\lambda),\quad\text{with } 0\ne w_{d-r_2}\in\mathbb{C}^{m\times 1}\text{ and }\deg(\widetilde{L}_{*2})<d-r_2.\tag{25}$$

Next, for any sequence $\{\epsilon_k\}_{k\in\mathbb{N}}\subset\mathbb{C}$ of *nonzero* numbers such that $\lim_{k\to\infty}\epsilon_k=0$, we define two sequences of polynomial matrices $\{L_k(\lambda)\}_{k\in\mathbb{N}}\subseteq\mathbb{C}[\lambda]^{m\times r}$ and $\{R_k(\lambda)\}_{k\in\mathbb{N}}\subseteq\mathbb{C}[\lambda]^{r\times n}$ (via their columns and rows, respectively) as follows

$$\begin{array}{ll}(L_k)_{*1}(\lambda):=-\epsilon_k\lambda^{d-r_1+1}w_{d-r_2}+L_{*1}(\lambda), & (R_k)_{2*}(\lambda):=\epsilon_k\lambda^{r_2+1}v_{r_1}+R_{2*}(\lambda),\\ (L_k)_{*i}(\lambda):=L_{*i}(\lambda),\quad 1<i\le r, & (R_k)_{i*}(\lambda):=R_{i*}(\lambda),\quad i\ne 2,\ 1\le i\le r.\end{array}\tag{26}$$

From these sequences, we define the sequence $\{P_k(\lambda)\}_{k\in\mathbb{N}}:=\{L_k(\lambda)R_k(\lambda)\}_{k\in\mathbb{N}}\subseteq\mathbb{C}[\lambda]^{m\times n}$, which obviously satisfies $\lim_{k\to\infty}P_k(\lambda)=P(\lambda)$. In the rest of the proof, we will prove that there exists an index k_0 such that for every $k\ge k_0$,

$$P_k(\lambda)=L_k(\lambda)R_k(\lambda)\in\mathcal{A}_{d,r,a}^{m\times n}(r_1-1,r_2+1,r_3,\dots,r_r),$$

which implies that $P(\lambda)\in\overline{\mathcal{A}_{d,r,a}^{m\times n}(r_1-1,r_2+1,r_3,\dots,r_r)}$. For this purpose, we define

$$D_k(\lambda):=\begin{bmatrix}1 & -\frac{1}{\epsilon_k}\lambda^{r_1-r_2-1}\\ 0 & 1\end{bmatrix}\oplus I_{r-2},$$

whose inverse is

$$D_k(\lambda)^{-1}:=\begin{bmatrix}1 & \frac{1}{\epsilon_k}\lambda^{r_1-r_2-1}\\ 0 & 1\end{bmatrix}\oplus I_{r-2}\,.$$

Therefore,

$$P_k(\lambda) = (L_k(\lambda)D_k(\lambda)^{-1})(D_k(\lambda)R_k(\lambda)). \tag{27}$$

The ith row of $(D_k(\lambda)R_k(\lambda))$ is equal to the ith row of $R_k(\lambda)$ for $i = 2, \ldots, r$, and, taking into account (24) and (26), the first row is

$$(D_k(\lambda)R_k(\lambda))_{1*} = \widetilde{R}_{1*}(\lambda) - \frac{1}{\epsilon_k}\lambda^{r_1-r_2-1}R_{2*}(\lambda),$$

which has degree $r_1 - 1$ for ϵ_k sufficiently close to zero or equivalently for all k sufficiently large. In summary, there exists an index k' such that for all $k \geq k'$

$$\text{the degrees of the rows of } D_k(\lambda)R_k(\lambda) \text{ are} \quad r_1 - 1, r_2 + 1, r_3, r_4, \ldots, r_r\,. \tag{28}$$

On the other hand, the ith column of $L_k(\lambda)D_k(\lambda)^{-1}$ is equal to the ith column of $L_k(\lambda)$ for $i = 1, 3, 4 \ldots, r$, and, taking into account (25) and (26), the second column is

$$(L_k(\lambda)D_k(\lambda)^{-1})_{*2} = \frac{1}{\epsilon_k}\lambda^{r_1-r_2-1}\, L_{*1}(\lambda) + \widetilde{L}_{*2}(\lambda),$$

which has degree $d-r_2-1$ for ϵ_k sufficiently close to zero or equivalently for all k sufficiently large. In summary, there exists an index k'' such that for all $k \geq k''$

$$\text{the degrees of the columns of } L_k(\lambda)D_k(\lambda)^{-1} \text{ are} \quad d-r_1+1, d-r_2-1, d-r_3, d-r_4, \ldots, d-r_r\,. \tag{29}$$

Combining (27), (28), and (29) we get that

$$P_k(\lambda) = L_k(\lambda)R_k(\lambda) \in \mathcal{A}^{m\times n}_{d,r,a}(r_1 - 1, r_2 + 1, r_3, \ldots, r_r)$$

for all $k \geq \max\{k', k''\} = k_0$ and the proof is completed.

Proof of item (iii). Observe that item (ii) and the fact that "the closure of a set is the smallest closed set that includes it" imply

$$\overline{\mathcal{A}^{m\times n}_{d,r,a}(r_1, \ldots, r_j, \ldots, r_k, \ldots, r_r)} \subseteq \overline{\mathcal{A}^{m\times n}_{d,r,a}(r_1, \ldots, r_j - 1, \ldots, r_k + 1, \ldots, r_r)}\,.$$

Therefore, we can apply again this result to the set on the right hand side of the equation above (permuting if necessary the indices by using the result in item (i)) as long as for at least two of the indices in $(r_1, \ldots, r_j - 1, \ldots, r_k + 1, \ldots, r_r)$ the absolute value of their difference is larger than or equal to two. We can construct in this way a chain of subset inclusions until the indices r_i differ at most by one unit, that is,

$$\begin{aligned}
\mathcal{A}^{m\times n}_{d,r,a}(r_1, \ldots, r_j, \ldots, r_k, \ldots, r_r) &\subseteq \overline{\mathcal{A}^{m\times n}_{d,r,a}(r_1, \ldots, r_j, \ldots, r_k, \ldots, r_r)} \\
&\subseteq \overline{\mathcal{A}^{m\times n}_{d,r,a}(r_1, \ldots, r_j - 1, \ldots, r_k + 1, \ldots, r_r)} \\
&\subseteq \cdots \subseteq \overline{\mathcal{A}^{m\times n}_{d,r,a}(\underbrace{d_R + 1, \ldots, d_R + 1}_{t_R}, \underbrace{d_R, \ldots, d_R}_{r-t_R})}\,.
\end{aligned}$$

We emphasize that the values of d_R and t_R are completely determined by the fact that the sum of the r indices of all the subsets in the chain above is always a and that the indices in the last subset differ at most by one (in absolute value). □

Example 4.7. In order to illustrate the proof and the statement of Theorem 4.6, we consider the following polynomial matrix

$$P(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 1 & \lambda^2 & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix} \in \mathcal{A}^{3\times 3}_{2,2,2}(2,0) \subset \mathbb{C}[\lambda]^{3\times 3}_{2,2}. \tag{30}$$

Since $a = 2$ and $r = 2$, the quantities in Theorem 4.6-(iii) are $d_R = 1$ and $t_R = 0$. One might wonder whether $P(\lambda)$ might be factorized in a form different from the one in (30) in such a way that $P(\lambda) \in \mathcal{A}^{3\times 3}_{2,2,2}(1,1)$. However, it is easy to see that $P(\lambda) \notin \mathcal{A}^{3\times 3}_{2,2,2}(1,1)$ as follows. Observe first that in the factorization $P(\lambda) = L(\lambda)R(\lambda)$ given in (30) both factors are minimal bases by Theorem 2.4. Thus, the minimal indices of $\mathcal{R}ow(P)$ are 2 and 0. If $P(\lambda) \in \mathcal{A}^{3\times 3}_{2,2,2}(1,1)$, then there would exist a factorization $P(\lambda) = \widetilde{L}(\lambda)\widetilde{R}(\lambda)$ with $\widetilde{L}(\lambda) \in \mathbb{C}[\lambda]^{3\times 2}$ and $\widetilde{R}(\lambda) \in \mathbb{C}[\lambda]^{2\times 3}$ with the degrees of both rows of $\widetilde{R}(\lambda)$ equal to 1 and, since $\operatorname{rank}(P) = 2$, Lemma 3.2-(iv) would imply that the rows of $\widetilde{R}(\lambda)$ form a polynomial basis of $\mathcal{R}ow(P)$ with the sum of the degrees of its vectors equal to 2. Therefore, the rows of $\widetilde{R}(\lambda)$ would be a minimal basis of $\mathcal{R}ow(P)$ and the minimal indices of this rational subspace would be 1 and 1, which contradicts that the minimal indices of $\mathcal{R}ow(P)$ are 2 and 0.

Consider any sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of nonzero numbers with $\lim_{k\to\infty} \epsilon_k = 0$ and construct from $P(\lambda)$ the following sequence of polynomial matrices via the strategy in (26):

$$P_k(\lambda) = \begin{bmatrix} -\epsilon_k\lambda & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & \epsilon_k\lambda & 0 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 & -\epsilon_k\lambda \\ 1 & \lambda^2 + \epsilon_k\lambda & 1 \\ 0 & \lambda^2 & 1 \end{bmatrix},$$

which satisfies $\lim_{k\to\infty} P_k(\lambda) = P(\lambda)$. Proceeding as in (27), $P_k(\lambda)$ can be written as:

$$\begin{aligned} P_k(\lambda) &= \begin{bmatrix} -\epsilon_k\lambda & \lambda^2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{\epsilon_k}\lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{\epsilon_k}\lambda \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \lambda^2 & 1 \\ 1 & \epsilon_k\lambda & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\epsilon_k\lambda & 0 \\ 1 & \frac{1}{\epsilon_k}\lambda + 1 \\ 1 & \frac{1}{\epsilon_k}\lambda \end{bmatrix} \begin{bmatrix} -\frac{1}{\epsilon_k}\lambda & 0 & 1 \\ 1 & \epsilon_k\lambda & 0 \end{bmatrix} \in \mathcal{A}^{3\times 3}_{2,2,2}(1,1) \subset \mathbb{C}[\lambda]^{3\times 3}_{2,2}. \end{aligned}$$

The set $\mathcal{A}^{m\times n}_{d,r,a}(d_R+1,\ldots,d_R+1,d_R,\ldots,d_R)$ appearing in Theorem 4.6-(iii), with t_R entries equal to d_R+1 and $r-t_R$ entries equal to d_R in the list, plays a crucial role in the main results of this section. Therefore, we redefine it in Definition 4.8 and introduce a simpler notation for it.

Definition 4.8. *Let m,n,r and d be integers such that $m,n \geq 2$, $d \geq 1$ and $0 < r < \min\{m,n\}$, and a be an integer such that $0 \leq a \leq rd$. Let us define $d_R := \lfloor a/r \rfloor$, $t_R := a \bmod r$ and the following subset of polynomial matrices*

$$\mathcal{B}^{m\times n}_{d,r,a} := \left\{ L(\lambda)R(\lambda) : \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\ R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg(R_{i*}) = d_R + 1, \quad \textit{for } i = 1,\ldots,t_R, \\ \deg(R_{i*}) = d_R, \quad \textit{for } i = t_R+1,\ldots,r, \\ \deg(L_{*i}) = d - \deg(R_{i*}), \quad \textit{for } i = 1,\ldots,r \end{array} \right\} \in \mathbb{C}[\lambda]^{m\times n}_{d,r}.$$

Remark 4.9. Observe that the sets introduced in Definition 4.8 and in Theorem 4.6-(iii) are equal, that is,

$$\mathcal{B}_{d,r,a}^{m\times n} = \mathcal{A}_{d,r,a}^{m\times n}(\underbrace{d_R+1,\ldots,d_R+1}_{t_R},\underbrace{d_R,\ldots,d_R}_{r-t_R}).$$

As a simple consequence of the developments above, we prove the first main result of this section.

Theorem 4.10. *Let $\mathcal{A}_{d,r,a}^{m\times n}$ and $\mathcal{B}_{d,r,a}^{m\times n}$ be the sets of polynomial matrices defined in Theorem 4.3 and in Definition 4.8, respectively. Then,*

(i) $\mathcal{B}_{d,r,a}^{m\times n} \subseteq \mathcal{A}_{d,r,a}^{m\times n}$ *for* $a = 0, 1, \ldots, rd$,

(ii) $\overline{\mathcal{B}_{d,r,a}^{m\times n}} = \overline{\mathcal{A}_{d,r,a}^{m\times n}}$ *for* $a = 0, 1, \ldots, rd$,

(iii) $\mathbb{C}[\lambda]_{d,r}^{m\times n} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_{d,r,a}^{m\times n}}$, *and*

(iv) *for every* $P(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$*, there exists an integer a such that* $P(\lambda) \in \overline{\mathcal{B}_{d,r,a}^{m\times n}}$.

Proof. Item (i) is obvious by definition. Item (i) implies $\overline{\mathcal{B}_{d,r,a}^{m\times n}} \subseteq \overline{\mathcal{A}_{d,r,a}^{m\times n}}$. Next, suppose $L(\lambda)R(\lambda) \in \mathcal{A}_{d,r,a}^{m\times n}$. Then $L(\lambda)R(\lambda) \in \mathcal{A}_{d,r,a}^{m\times n}(r_1,\ldots,r_r)$ for some integers $(r_1,\ldots,r_r)$ such that $0 \le r_i \le d$, for $i = 1,\ldots,r$, and $\sum_{i=1}^{r} r_i = a$, and, by Theorem 4.6-(iii), $L(\lambda)R(\lambda) \in \overline{\mathcal{B}_{d,r,a}^{m\times n}}$. Therefore, $\mathcal{A}_{d,r,a}^{m\times n} \subseteq \overline{\mathcal{B}_{d,r,a}^{m\times n}}$, which implies $\overline{\mathcal{A}_{d,r,a}^{m\times n}} \subseteq \overline{\mathcal{B}_{d,r,a}^{m\times n}}$. This proves item (ii). Finally, items (iii) and (iv) follow from item (ii) and the items (ii) and (iii), respectively, of Theorem 4.3. □

Theorem 4.10 proves the promised result that arbitrarily close to any polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$, there is another polynomial matrix $Q(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$ that can be factorized as $Q(\lambda) = L(\lambda)R(\lambda)$, with the degrees of the columns of $L(\lambda)$ differing at most by one and with the degrees of the rows of $R(\lambda)$ also differing at most by one. However, the factorization of $Q(\lambda)$ is not necessarily a minimal rank factorization, according to the definition of $\mathcal{B}_{d,r,a}^{m\times n}$. Next, we prove in Theorem 4.12 that arbitrarily close to any polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]_{d,r}^{m\times n}$ there is a polynomial matrix $Q(\lambda)$ that can be factorized as $Q(\lambda) = L(\lambda)R(\lambda)$ with factors satisfying the conditions of Theorem 3.16, and, moreover, with the degrees of the columns of $L(\lambda)$ differing at most by one and with the degrees of the rows of $R(\lambda)$ also differing at most by one. In addition, the minimal indices of $\mathcal{N}_\ell(Q)$ and $\mathcal{N}_r(Q)$ are as those in Theorem 2.11. For that purpose we introduce first the following definitions.

Definition 4.11. *Let m,n,r and d be integers such that $m,n \ge 2$, $d \ge 1$ and $0 < r < \min\{m,n\}$, and a be an integer such that $0 \le a \le rd$. Let us define $d_R := \lfloor a/r \rfloor$, $t_R := a \bmod r$, $\alpha := \lfloor a/(n-r) \rfloor$, $s := a \bmod (n-r)$, $\beta := \lfloor (rd-a)/(m-r) \rfloor$, and $t := (rd-a) \bmod (m-r)$ and the following subsets of $\mathbb{C}[\lambda]_{d,r}^{m\times n}$*

$$\mathcal{M}_{d,r,a}^{m\times n} := \left\{ L(\lambda)R(\lambda) : \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\ R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ L(\lambda) \textit{ and } R(\lambda) \textit{ are minimal bases}, \\ \deg(R_{i*}) = d_R + 1, \quad \textit{for } i = 1,\ldots,t_R, \\ \deg(R_{i*}) = d_R, \quad \textit{for } i = t_R+1,\ldots,r, \\ \deg(L_{*i}) = d - \deg(R_{i*}), \quad \textit{for } i = 1,\ldots,r \end{array} \right\},$$

$$\mathcal{MH}^{m\times n}_{d,r,a} := \left\{ L(\lambda)R(\lambda) \;:\; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\ R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ L(\lambda) \textit{ and } R(\lambda) \textit{ are minimal bases,} \\ \mathcal{N}_\ell(L) \textit{ has minimal indices } \{\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-r-t}\}, \\ \mathcal{N}_r(R) \textit{ has minimal indices } \{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-r-s}\}, \\ \deg(R_{i*}) = d_R+1, \quad \textit{for } i=1,\ldots,t_R, \\ \deg(R_{i*}) = d_R, \quad \textit{for } i=t_R+1,\ldots,r, \\ \deg(L_{*i}) = d-\deg(R_{i*}), \quad \textit{for } i=1,\ldots,r \end{array} \right\}.$$

With respect to the definition of $\mathcal{MH}^{m\times n}_{d,r,a}$, it is important to recall that Lemma 3.2 implies that $\mathcal{N}_\ell(L) = \mathcal{N}_\ell(L(\lambda)R(\lambda))$ and that $\mathcal{N}_r(R) = \mathcal{N}_r(L(\lambda)R(\lambda))$.

Theorem 4.12. *Let $\mathcal{B}^{m\times n}_{d,r,a}$, $\mathcal{M}^{m\times n}_{d,r,a}$ and $\mathcal{MH}^{m\times n}_{d,r,a}$ be the sets of polynomial matrices introduced in Definitions 4.8 and 4.11. Then,*

(i) $\mathcal{MH}^{m\times n}_{d,r,a} \subseteq \mathcal{M}^{m\times n}_{d,r,a} \subseteq \mathcal{B}^{m\times n}_{d,r,a}$ *for* $a = 0,1,\ldots,rd$,

(ii) $\overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \overline{\mathcal{M}^{m\times n}_{d,r,a}} = \overline{\mathcal{B}^{m\times n}_{d,r,a}}$ *for* $a = 0,1,\ldots,rd$,

(iii) $\mathbb{C}[\lambda]^{m\times n}_{d,r} = \bigcup_{0\le a\le rd} \overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \bigcup_{0\le a\le rd} \overline{\mathcal{M}^{m\times n}_{d,r,a}}$, *and*

(iv) *for every $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}_{d,r}$, there exists an integer a such that $P(\lambda) \in \overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \overline{\mathcal{M}^{m\times n}_{d,r,a}}$.*

Proof. Item (i) is obvious from the definitions of the involved sets.

Proof of item (ii). First note that item (i) implies immediately that

$$\overline{\mathcal{MH}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{M}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{B}^{m\times n}_{d,r,a}}.$$

With this result at hand, observe that if we prove $\mathcal{B}^{m\times n}_{d,r,a} \subseteq \overline{\mathcal{MH}^{m\times n}_{d,r,a}}$, then $\overline{\mathcal{B}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{MH}^{m\times n}_{d,r,a}}$ immediately follows, which implies $\overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \overline{\mathcal{B}^{m\times n}_{d,r,a}}$, which in turn implies the result in item (ii). Therefore, we focus on proving $\mathcal{B}^{m\times n}_{d,r,a} \subseteq \overline{\mathcal{MH}^{m\times n}_{d,r,a}}$. If $L(\lambda)R(\lambda) \in \mathcal{B}^{m\times n}_{d,r,a}$, then

$$L(\lambda)^T \in \mathbb{C}[\lambda]^{r\times m}_{\underline{\mathbf{f}}},\ \text{with } \underline{\mathbf{f}} = (\underbrace{d-d_R-1,\ldots,d-d_R-1}_{t_R},\underbrace{d-d_R,\ldots,d-d_R}_{r-t_R}), \tag{31}$$

$$R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}_{\underline{\mathbf{g}}},\ \text{with } \underline{\mathbf{g}} = (\underbrace{d_R+1,\ldots,d_R+1}_{t_R},\underbrace{d_R,\ldots,d_R}_{r-t_R}). \tag{32}$$

Therefore, Theorem 2.13 applied to $L(\lambda)$ and $R(\lambda)$ implies that there exist sequences of polynomial matrices $\{L_k(\lambda)\}_{k\in\mathbb{N}} \subset \mathbb{C}[\lambda]^{m\times r}$ and $\{R_k(\lambda)\}_{k\in\mathbb{N}} \subset \mathbb{C}[\lambda]^{r\times n}$, such that

(1) $\lim_{k\to\infty} L_k(\lambda) = L(\lambda)$ and $\lim_{k\to\infty} R_k(\lambda) = R(\lambda)$,

(2) each polynomial matrix $L_k(\lambda)$ is a minimal basis, $\mathcal{N}_\ell(L_k)$ has minimal indices equal to $\{\underbrace{\beta+1,\ldots,\beta+1}_{t},\underbrace{\beta,\ldots,\beta}_{m-r-t}\}$, and $\deg((L_k)_{*i}) = d-d_R-1$ for $i=1,\ldots,t_R$, and $\deg((L_k)_{*i}) = d-d_R$ for $i = t_R+1,\ldots,r$,

(3) each polynomial matrix $R_k(\lambda)$ is a minimal basis, $\mathcal{N}_r(R_k)$ has minimal indices equal to $\{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-r-s}\}$, and $\deg((R_k)_{i*}) = d_R + 1$ for $i = 1,\ldots,t_R$, and $\deg((R_k)_{i*}) = d_R$ for $i = t_R+1,\ldots,r$.

This means that $\{L_k(\lambda)R_k(\lambda)\}_{k\in\mathbb{N}} \subset \mathcal{MH}^{m\times n}_{d,r,a}$ and that $\lim_{k\to\infty} L_k(\lambda)R_k(\lambda) = L(\lambda)R(\lambda)$. So, $L(\lambda)R(\lambda) \in \overline{\mathcal{MH}^{m\times n}_{d,r,a}}$ and $\mathcal{B}^{m\times n}_{d,r,a} \subseteq \overline{\mathcal{MH}^{m\times n}_{d,r,a}}$ is proved.

Items (iii) and (iv) follow from item (ii) and items (iii) and (iv) in Theorem 4.10. □

To compare the results we are obtaining for polynomial matrices with degree at most d, where $d \geq 1$, with those in [4] for matrix pencils, that is, for $d = 1$, we introduce some additional sets of polynomial matrices and prove for them a result similar to Theorem 4.10.

Definition 4.13. *Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $0 < r < \min\{m, n\}$, and a be an integer such that $0 \leq a \leq rd$. Let us define $d_R := \lfloor a/r \rfloor$, $t_R := a \bmod r$, and the following subset of $\mathbb{C}[\lambda]^{m\times n}_{d,r}$*

$$\mathcal{C}^{m\times n}_{d,r,a} := \left\{ L(\lambda)R(\lambda) \;:\; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg(R_{i*}) \leq d_R + 1, \quad \text{for } i = 1,\ldots,t_R, \\ \deg(R_{i*}) \leq d_R, \quad \text{for } i = t_R+1,\ldots,r, \\ \deg(L_{*i}) \leq d - d_R - 1, \quad \text{for } i = 1,\ldots,t_R, \\ \deg(L_{*i}) \leq d - d_R, \quad \text{for } i = t_R+1,\ldots,r \end{array} \right\}.$$

Theorem 4.14. *Let $\mathcal{B}^{m\times n}_{d,r,a}$ and $\mathcal{C}^{m\times n}_{d,r,a}$ be the sets of polynomial matrices introduced in Definitions 4.8 and 4.13, respectively. Then,*

(i) $\mathcal{B}^{m\times n}_{d,r,a} \subseteq \mathcal{C}^{m\times n}_{d,r,a}$ *for $a = 0, 1, \ldots, rd$,*

(ii) $\overline{\mathcal{B}^{m\times n}_{d,r,a}} = \overline{\mathcal{C}^{m\times n}_{d,r,a}}$ *for $a = 0, 1, \ldots, rd$,*

(iii) $\mathbb{C}[\lambda]^{m\times n}_{d,r} = \bigcup_{0\leq a\leq rd} \overline{\mathcal{C}^{m\times n}_{d,r,a}}$, *and*

(iv) *for every $P(\lambda) \in \mathbb{C}[\lambda]^{m\times n}_{d,r}$, there exists an integer a such that $P(\lambda) \in \overline{\mathcal{C}^{m\times n}_{d,r,a}}$.*

Proof. Item (i) is obvious from the definitions of the involved sets.

Proof of item (ii). From item (i), we get that $\overline{\mathcal{B}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{C}^{m\times n}_{d,r,a}}$. Next, we prove that $\mathcal{C}^{m\times n}_{d,r,a} \subseteq \overline{\mathcal{B}^{m\times n}_{d,r,a}}$. Let $L(\lambda)R(\lambda) \in \mathcal{C}^{m\times n}_{d,r,a}$, but $L(\lambda)R(\lambda) \notin \mathcal{B}^{m\times n}_{d,r,a}$. This means that the degrees of some rows of $R(\lambda)$ and/or of some columns of $L(\lambda)$ are strictly less than the corresponding quantities d_R+1, d_R, $d-d_R-1$, $d-d_R$ appearing in Definition 4.13. Using any sequences of constant nonzero vectors $\{v_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^{m\times 1}$ and/or $\{w_k\}_{k\in\mathbb{N}} \subset \mathbb{C}^{1\times n}$, such that $\lim_{k\to\infty} v_k = 0$ and $\lim_{k\to\infty} w_k = 0$, we sum to the rows of $R(\lambda)$ with degrees strictly less than d_R+1 and/or d_R polynomial vectors $\lambda^{d_R+1}w_k$ and/or $\lambda^{d_R}w_k$, and sum to the columns of $L(\lambda)$ with degrees strictly less than $d-d_R-1$ and/or $d-d_R$ polynomial vectors $\lambda^{d-d_R-1}v_k$ and/or $\lambda^{d-d_R}v_k$. This allows us to construct a sequence $\{L_k(\lambda)R_k(\lambda)\}_{k\in\mathbb{N}} \subset \mathcal{B}^{m\times n}_{d,r,a}$ such that $\lim_{k\to\infty} L_k(\lambda)R_k(\lambda) = L(\lambda)R(\lambda)$. This proves $L(\lambda)R(\lambda) \in \overline{\mathcal{B}^{m\times n}_{d,r,a}}$ and $\mathcal{C}^{m\times n}_{d,r,a} \subseteq \overline{\mathcal{B}^{m\times n}_{d,r,a}}$, which implies $\overline{\mathcal{C}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{B}^{m\times n}_{d,r,a}}$. This proves item (ii).

Items (iii) and (iv) follow from item (ii) and items (iii) and (iv) in Theorem 4.10. □

Remark 4.15. (Comparisons with results for matrix pencils) For $d = 1$, i.e., for matrix pencils, the sets $\mathcal{C}_{1,r,a}^{m\times n}$, for $a = 0, 1, \dots, r$, in Definition 4.13 are exactly the sets $\mathcal{C}_a^r$ in [4, Lemma 4]. However, by using the Kronecker canonical form of pencils, Lemma 4 in [4] proves that $\mathbb{C}[\lambda]_{1,r}^{m\times n} = \bigcup_{0\le a\le r} \mathcal{C}_{1,r,a}^{m\times n}$, which is a result stronger than Theorem 4.14-(iii) because it does not involve closures. This raises the question whether for $d \ge 2$ the closures can be removed in Theorem 4.14-(iii). Unfortunately, this is not possible as the next example shows.

Example 4.16. Consider the polynomial matrix $P(\lambda) \in \mathbb{C}[\lambda]_{2,2}^{3\times 3}$ in (30). We are going to show that $P(\lambda) \notin \bigcup_{0\le a\le 4} \mathcal{C}_{2,2,a}^{3\times 3}$. For this purpose, we follow an argument similar to that in Example 4.7. Note first that the two factors $L(\lambda)$ and $R(\lambda)$ of $P(\lambda)$ in (30) are minimal bases. Thus, the minimal indices of $\mathcal{C}ol(P)$ are 2 and 0 and the minimal indices of $\mathcal{R}ow(P)$ are also 2 and 0. Moreover, since $\operatorname{rank}(P) = 2$, any factorization $P(\lambda) = \widetilde{L}(\lambda)\widetilde{R}(\lambda)$ with $\widetilde{L}(\lambda) \in \mathbb{C}[\lambda]^{3\times 2}$ and $\widetilde{R}(\lambda) \in \mathbb{C}[\lambda]^{2\times 3}$ must satisfy $\operatorname{rank}(\widetilde{L}) = \operatorname{rank}(\widetilde{R}) = 2$ and, so, the columns of $\widetilde{L}(\lambda)$ are a polynomial basis of $\mathcal{C}ol(P)$ and the rows of $\widetilde{R}(\lambda)$ are a polynomial basis of $\mathcal{R}ow(P)$. This means that the sum of the degrees of the columns of $\widetilde{L}(\lambda)$ must be larger than or equal to 2 and that the sum of the degrees of the rows of $\widetilde{R}(\lambda)$ must be larger than or equal to 2. Therefore, $P(\lambda) \notin \mathcal{C}_{2,2,0}^{3\times 3}$ and $P(\lambda) \notin \mathcal{C}_{2,2,1}^{3\times 3}$, because in both cases the sum of the degrees of the rows of $\widetilde{R}(\lambda)$ would be smaller than 2, and also that $P(\lambda) \notin \mathcal{C}_{2,2,3}^{3\times 3}$ and $P(\lambda) \notin \mathcal{C}_{2,2,4}^{3\times 3}$, because in both cases the sum of the degrees of the columns of $\widetilde{L}(\lambda)$ would be smaller than 2. Then, the only remaining option is $P(\lambda) \in \mathcal{C}_{2,2,2}^{3\times 3}$ but in this case $d_R = 1$ and $t_R = 0$, which implies that both rows of $\widetilde{R}(\lambda)$ must have degree exactly 1, and that they will be a minimal basis of $\mathcal{R}ow(P)$, which is impossible because the minimal indices of $\mathcal{R}ow(P)$ are 2 and 0.

4.1. Relation between factorizations and generic complete eigenstructures in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$

A glance to the results in Theorems 2.11, 4.3, 4.10, 4.12 and 4.14 hints a relationship between the closures of the orbits $\mathcal{O}(K_a)$ of polynomial matrices with generic eigenstructures and those of the sets defined before in Section 4. To establish this relationship, we characterize $\mathcal{O}(K_a)$ as a set of factorized polynomial matrices in the next theorem.

Theorem 4.17. *Let m, n, r and d be integers such that $m, n \ge 2$, $d \ge 1$ and $0 < r < \min\{m, n\}$, and a be an integer such that $0 \le a \le rd$. Let us define $\alpha := \lfloor a/(n-r) \rfloor$, $s := a \mod (n-r)$, $\beta := \lfloor (rd-a)/(m-r) \rfloor$, and $t := (rd-a) \mod (m-r)$. Let $\mathcal{O}(K_a)$ be the orbit of polynomial matrices in $\mathbb{C}[\lambda]_{d,r}^{m\times n}$ appearing in Theorem 2.11. Then*

$$\mathcal{O}(K_a) = \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ L(\lambda) \textit{ and } R(\lambda) \textit{ are minimal bases}, \\ \mathcal{N}_\ell(L) \textit{ has minimal indices } \{\underbrace{\beta+1, \dots, \beta+1}_{t}, \underbrace{\beta, \dots, \beta}_{m-r-t}\}, \\ \mathcal{N}_r(R) \textit{ has minimal indices } \{\underbrace{\alpha+1, \dots, \alpha+1}_{s}, \underbrace{\alpha, \dots, \alpha}_{n-r-s}\}, \\ \deg(L_{*i}) + \deg(R_{i*}) = d, \quad \textit{for } i = 1, \dots, r \end{array} \right\}.$$

Proof. The result is an immediate corollary of Theorem 3.16 and the facts that $\mathcal{N}_\ell(L) = \mathcal{N}_\ell(L(\lambda)R(\lambda))$ and $\mathcal{N}_r(R) = \mathcal{N}_r(L(\lambda)R(\lambda))$ according to Lemma 3.2. $\square$

With this result at hand, we get the next theorem.

Theorem 4.18. *Let $\mathcal{A}^{m\times n}_{d,r,a}$, $\mathcal{B}^{m\times n}_{d,r,a}$, $\mathcal{M}^{m\times n}_{d,r,a}$, $\mathcal{MH}^{m\times n}_{d,r,a}$ and $\mathcal{C}^{m\times n}_{d,r,a}$ be the sets of polynomial matrices introduced in Theorem 4.3 and in Definitions 4.8, 4.11 and 4.13. Let $\mathcal{O}(K_a)$ be the orbit of polynomial matrices in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$ appearing in Theorem 2.11. Then,*

(i) $\mathcal{O}(K_a) \subseteq \mathcal{A}^{m\times n}_{d,r,a}$ *for* $a = 0, 1, \dots, rd$,

(ii) $\mathcal{MH}^{m\times n}_{d,r,a} \subseteq \mathcal{O}(K_a)$ *for* $a = 0, 1, \dots, rd$,

(iii) $\overline{\mathcal{O}(K_a)} = \overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \overline{\mathcal{M}^{m\times n}_{d,r,a}} = \overline{\mathcal{B}^{m\times n}_{d,r,a}} = \overline{\mathcal{C}^{m\times n}_{d,r,a}} = \overline{\mathcal{A}^{m\times n}_{d,r,a}}$ *for* $a = 0, 1, \dots, rd$.

Proof. Proof of item (i). If $P(\lambda) \in \mathcal{O}(K_a)$, then $P(\lambda) = L(\lambda)R(\lambda)$ with the factors $L(\lambda)$ and $R(\lambda)$ satisfying the properties described in Theorem 4.17. These properties imply that the degrees of the rows of $R(\lambda)$ are the minimal indices of $\mathcal{R}ow(P)$ by Lemma 3.2. Combining this result with the fact that $\mathcal{N}_r(R) = \mathcal{N}_r(P)$, again by Lemma 3.2, and with Corollary 2.9, we get

$$\sum_{i=1}^{r} \deg(R_{i*}) = s(\alpha+1) + (n-r-s)\alpha = (n-r)\alpha + s = a.$$

This implies that $P(\lambda) \in \mathcal{A}^{m\times n}_{d,r,a}$ and, so, item (i).

Item (ii) follows from the definitions of the involved sets.

Proof of item (iii). Item (i) implies $\overline{\mathcal{O}(K_a)} \subseteq \overline{\mathcal{A}^{m\times n}_{d,r,a}}$. Combining this inclusion with Theorems 4.10-(ii), 4.12-(ii) and 4.14-(ii), we get

$$\overline{\mathcal{O}(K_a)} \subseteq \overline{\mathcal{MH}^{m\times n}_{d,r,a}} = \overline{\mathcal{M}^{m\times n}_{d,r,a}} = \overline{\mathcal{B}^{m\times n}_{d,r,a}} = \overline{\mathcal{C}^{m\times n}_{d,r,a}} = \overline{\mathcal{A}^{m\times n}_{d,r,a}}.$$

On the other hand, item (ii) implies $\overline{\mathcal{MH}^{m\times n}_{d,r,a}} \subseteq \overline{\mathcal{O}(K_a)}$, which combined with the equation above yields the result in item (iii). □

The inclusion relationships presented in Theorem 4.18-(i) and (ii) between $\mathcal{O}(K_a)$ and the other sets involved in this theorem are the only ones that hold in general. We illustrate this statement in the next example.

Example 4.19. Consider the following polynomial matrix

$$P(\lambda) = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & \lambda^2 & \lambda^2 \\ 1 & \lambda^2 & 2\lambda^4 & \lambda^4 \\ 1 & \lambda^2 & \lambda^4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda^2 & 0 \\ \lambda^4 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & \lambda^2 & \lambda^4 & 0 \end{bmatrix} =: L(\lambda)R(\lambda). \tag{33}$$

$P(\lambda)$ belongs to $\mathbb{C}[\lambda]^{4\times 4}_{4,2}$. Moreover, the factors $L(\lambda)$ and $R(\lambda)$ are minimal bases by Theorem 2.4. Consider also the following polynomial matrices

$$\widehat{L}(\lambda) = \begin{bmatrix} \lambda^2 & 0 \\ -1 & \lambda^2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \widehat{R}(\lambda) = \begin{bmatrix} \lambda^2 & -1 & 0 & 0 \\ 0 & \lambda^2 & -1 & 1 \end{bmatrix}. \tag{34}$$

It is easy to check that the columns of $\widehat{L}(\lambda)$ are a minimal basis of $\mathcal{N}_r(R) = \mathcal{N}_r(P)$ and that the rows of $\widehat{R}(\lambda)$ are a minimal basis of $\mathcal{N}_\ell(L) = \mathcal{N}_\ell(P)$. Therefore, $P(\lambda) \in \mathcal{O}(K_4)$, by Theorem 4.17. However, $P(\lambda) \notin \mathcal{B}^{4\times 4}_{4,2,4}$, $P(\lambda) \notin \mathcal{M}^{4\times 4}_{4,2,4}$, $P(\lambda) \notin \mathcal{MH}^{4\times 4}_{4,2,4}$ and $P(\lambda) \notin \mathcal{C}^{4\times 4}_{4,2,4}$. To see this, we need to check that no factorization of $P(\lambda)$ as $P(\lambda) = \widetilde{L}(\lambda)\widetilde{R}(\lambda)$, with $\widetilde{L}(\lambda) \in \mathbb{C}[\lambda]^{4\times 2}$ and $\widetilde{R}(\lambda) \in \mathbb{C}[\lambda]^{2\times 4}$, satisfies the conditions of the definitions of these sets. Note that in any of these factorizations $P(\lambda) = \widetilde{L}(\lambda)\widetilde{R}(\lambda)$ the rows of $\widetilde{R}(\lambda)$ are a polynomial basis of $\mathcal{R}ow(P)$. Therefore, combining the "Strong Minimality Property of Minimal Indices" in [15, Theorem 4.2] with the fact that the minimal indices of $\mathcal{R}ow(P)$ are $0, 4$, we obtain that $\deg(\widetilde{R}) \geq 4$. But, $d_R = \lfloor a/r \rfloor = \lfloor 4/2 \rfloor = 2$ and $t_R = 0$, which implies that any polynomial matrix in any of the sets $\mathcal{B}^{4\times 4}_{4,2,4}$, $\mathcal{M}^{4\times 4}_{4,2,4}$, $\mathcal{MH}^{4\times 4}_{4,2,4}$ and $\mathcal{C}^{4\times 4}_{4,2,4}$ can be factorized as $L_S(\lambda)R_S(\lambda)$ with $L_S(\lambda) \in \mathbb{C}[\lambda]^{4\times 2}$, $R_S(\lambda) \in \mathbb{C}[\lambda]^{2\times 4}$ and $\deg(R_S) \leq 2$. Thus, $P(\lambda)$ does not belong to any of these sets.

Next, consider the polynomial matrix

$$Q(\lambda) = \begin{bmatrix} \lambda^2 & 0 \\ -1 & \lambda^2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^2 & -1 & 0 & 0 \\ 0 & \lambda^2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \lambda^4 & -\lambda^2 & 0 & 0 \\ -\lambda^2 & \lambda^4+1 & -\lambda^2 & \lambda^2 \\ 0 & -\lambda^2 & 1 & -1 \\ 0 & \lambda^2 & -1 & 1 \end{bmatrix}, \tag{35}$$

which has been constructed as $Q(\lambda) = \widehat{L}(\lambda)\widehat{R}(\lambda)$ with the matrices in (34). Observe that $Q(\lambda) \in \mathcal{M}^{4\times 4}_{4,2,4} \subseteq \mathcal{B}^{4\times 4}_{4,2,4} \subseteq \mathcal{C}^{4\times 4}_{4,2,4}$ and $Q(\lambda) \in \mathcal{A}^{4\times 4}_{4,2,4}$. However, $Q(\lambda) \notin \mathcal{O}(K_4)$ because the minimal indices of $\mathcal{N}_r(\widehat{R}) = \mathcal{N}_r(Q)$ are 0 and 4, since the columns of $L(\lambda)$ in (33) are a minimal basis of $\mathcal{N}_r(\widehat{R})$.

Remark 4.20. (Comparisons with results for matrix pencils) For $d = 1$, it was proved in [4, Theorem 6] that $\overline{\mathcal{O}(K_a)} = \mathcal{C}^{m\times n}_{1,r,a}$, while Theorem 4.18 only proves the weaker result $\overline{\mathcal{O}(K_a)} = \overline{\mathcal{C}^{m\times n}_{1,r,a}}$. For $d \geq 2$, the result $\overline{\mathcal{O}(K_a)} = \overline{\mathcal{C}^{m\times n}_{d,r,a}}$ cannot be improved, since, in general, $\overline{\mathcal{O}(K_a)} \neq \mathcal{C}^{m\times n}_{d,r,a}$. The polynomial matrix in (33) illustrates this inequality.

5. Conclusions

We have established many results on rank factorizations and minimal rank factorizations of polynomial matrices, which, as far as we know, are completely new in the literature. In addition, the generic degree properties in the set $\mathbb{C}[\lambda]^{m\times n}_{d,r}$ of complex $m \times n$ polynomial matrices of degree at most d and rank at most r of such factorizations have been carefully studied and several dense subsets of factorized polynomial matrices have been identified. Some of these subsets allow us to approximate any polynomial matrix in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$ as the limit of a sequence of factorized polynomial matrices that can be easily and efficiently generated due to the particular degree properties of their factorizations, which have left factors with columns whose degrees differ at most by one and right factors with rows whose degrees differ at most by one. Apart from their fundamental nature in the theory of polynomial matrices, we hope that these results will have applications in the solution of different nearness problems involving polynomial matrices in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$. Possible lines of future research include exploring the development of structured rank factorizations and minimal rank factorizations of classes of structured polynomial matrices appearing in applications [16], and verifying if some of the dense subsets of polynomial matrices in Section 4 are also open in $\mathbb{C}[\lambda]^{m\times n}_{d,r}$.

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