Real-congruence canonical forms of real matrices

Fernando De Terán* and Froilán M. Dopico†

Abstract

We present two new canonical forms for real congruence of a real square matrix A. The first one is a direct sum of canonical matrices of four different types and is obtained from the canonical form under *congruence of complex matrices provided by Horn and Sergeichuk in [Linear Algebra Appl. 416 (2006) 1010-1032]. The second one is a direct sum of canonical matrices of three different types, has a block tridiagonal structure and is obtained from the canonical form under *congruence of complex matrices provided by Futorny, Horn and Sergeichuk in [J. Algebra 319 (2008) 2351-2371]. A detailed comparison between both canonical forms is also presented, as well as their relation with the real Kronecker canonical form under strict real equivalence of the matrix pair $(A^\top, A)$. Another canonical form for real congruence was presented by Lee and Weinberg in [Linear Algebra Appl. 249 (1996) 207-215], which consists of a direct sum of eight different types of matrices. In the last part of the paper, we explain the correspondence between the blocks in this canonical form and those in the two new forms introduced in this work.

Keywords: real matrices; congruence; *congruence; real congruence; matrix pencils; palindromic matrix pencils; real bilinear form; canonical form.

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1 Introduction

The classification of general sesquilinear or bilinear forms $\mathcal{A} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ over a field $\mathbb{F}$ is a long-standing issue that has been addressed both from the scope of pure algebra and linear algebra. A classification of bilinear forms over the complex field (namely, when $\mathbb{F} = \mathbb{C}$) is known since, at least, the 1930s [17, p. 139] using a linear algebra approach (see [1] for some historical details). With a pure algebra approach, relevant contributions were obtained in [6,15], for sesquilinear and bilinear forms over arbitrary fields $\mathbb{F}$ by reducing the problem to classifying Hermitian forms over finite extensions of $\mathbb{F}$ (see also the Introduction of [5] for more details). When $\mathbb{F}$ is, respectively, the complex field or the real field, it is natural to look for a classification using, respectively, the *congruence or the real-congruence of matrices, namely the following actions of the groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{R})$ on the sets of complex and real $n \times n$ matrices, respectively:

$$\begin{array}{ccc} \mathrm{GL}_n(\mathbb{C}) \times \mathbb{C}^{n\times n} & \to & \mathbb{C}^{n\times n} \\ (P, A) & \mapsto & PAP^* \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{GL}_n(\mathbb{R}) \times \mathbb{R}^{n\times n} & \to & \mathbb{R}^{n\times n} \\ (P, A) & \mapsto & PAP^\top, \end{array}$$

*Universidad Carlos III de Madrid, ROR: https://ror.org/03ths8210, Departamento de Matemáticas, Avda. de la Universidad 30, 28911 Leganés (Madrid), Spain, fteran@math.uc3m.es

†Universidad Carlos III de Madrid, ROR: https://ror.org/03ths8210, Departamento de Matemáticas, Avda. de la Universidad 30, 28911 Leganés (Madrid), Spain, dopico@math.uc3m.es

where P^* and $P^\top$ denote the conjugate-transpose and the transpose of P, respectively. The reason for using such actions relies on the fact that two *congruent (respectively, real-congruent) complex (resp., real) matrices correspond to the same sesquilinear (resp., bilinear) form over $\mathbb{C}^n \times \mathbb{C}^n$ (resp., $\mathbb{R}^n \times \mathbb{R}^n$) in different bases. Moreover, it is also natural to classify these sesquilinear or bilinear forms by means of a canonical form, which provides a unique representative for every equivalence class (namely, for every single *orbit* under the action of the group $\mathrm{GL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{R})$).

Canonical forms for *congruence of complex matrices have been provided in [10, Theorem 1.1(b)] and later in [5, Theorem 1.2]. In both cases, the canonical forms consist of a direct sum of canonical blocks. In the case of the canonical form in [10] there are three different types of blocks, whereas in the one in [5] there are two different types of blocks, which are tridiagonal. Some of the blocks depend on certain parameters. From these canonical forms, and specializing to real matrices, it is possible to get canonical forms for real congruence, and this is the main goal of this work. In particular, we derive and present in Theorem 4.1 a canonical form for real congruence of real square matrices from the canonical form introduced in [10]. This canonical form for real congruence consists of a direct sum of blocks of four different kinds, which come from considering separately those blocks of the canonical form for *congruence of general complex matrices involving real parameters and those associated to (pairs of conjugate) non-real parameters. In terms of bilinear forms, the canonical form of Theorem 4.1 allows us to classify bilinear forms over the real field. Analogously, we present and derive in Theorem 5.1 a canonical form for real congruence of real square matrices from the canonical form introduced in [5]. The canonical form in Theorem 5.1 is a direct sum of blocks of three different types. The relation between the two canonical forms for real congruence in Theorems 4.1 and 5.1 is presented in Theorem 6.1, which is connected with Theorem 3.2 about the correspondence between the blocks of the canonical forms for *congruence of complex matrices in [10, Theorem 1.1(b)] and [5, Theorem 1.2].

Another canonical form for real congruence of real square matrices was presented in [12, Theorem II]. It consists also of a direct sum of canonical blocks of several types, but in this case the number of different types of blocks is eight, which is twice the number of different types of blocks in the canonical form in Theorem 4.1 and more than twice the number of canonical blocks in the canonical form in Theorem 5.1. In Section 7, we obtain and display the correspondence between the eight types of blocks in the canonical form of [12] and the ones presented in Theorem 4.1 (the correspondence with the blocks in Theorem 5.1 can be obtained via Theorem 6.1). Moreover, in Section 7, we impose restrictions on the values of some of the parameters appearing in the blocks in [12, Theorem II] to make these blocks "truly" canonical (see Remark 7.2).

The real congruence canonical form of a matrix $A \in \mathbb{R}^{n\times n}$ is naturally related with the real Kronecker canonical form under strict real equivalence [11] of the real matrix pair $(A^\top, A)$, or, equivalently, of the matrix pencil $\lambda A^\top + A$ in the variable λ. These particular pencils are called real palindromic pencils and are in the intersection of the families of complex $\top$-palindromic pencils and $*$-palindromic pencils [13]. Although palindromic pencils (without the name) exist in the literature since long time ago, they have received considerable attention in the last two decades, in particular since [13] was published. A small sample of other recent works dealing with palindromic pencils are [1, 2, 14]. Recall in this context that two complex matrix pencils $\lambda N_1 + M_1$ and $\lambda N_2 + M_2$ are said to be *strictly equivalent* if there are two invertible complex matrices P, Q such that $P(\lambda N_1 + M_1)Q = \lambda N_2 + M_2$ and *strictly real equivalent* if P and Q are real. Theorems 4.2 and 5.2 establish, respectively, the relations between the *real congruence* canonical forms of $A \in \mathbb{R}^{n\times n}$ in Theorems 4.1 and 5.1 and the real Kronecker canonical form *under strict real equivalence* of $(A^\top, A)$. More precisely, Theorems 4.2 and 5.2 prove that the real congruence canonical forms fully determine the real Kronecker canonical form but that *the real Kronecker canonical form determines the real congruence canonical forms only up to the sign of* *certain canonical blocks*. Despite this indetermination, we will see throughout the paper that the real Kronecker canonical form under strict real equivalence of $(A^\top, A)$ is a very useful tool in the study of the real congruence canonical forms of A, because the class of real strict equivalence transformations is much larger than the class of real congruence transformations. So, it is often easier to prove that two real pairs $(A^\top, A)$ and $(B^\top, B)$ are strictly real equivalent than to prove that the matrices A and B are real congruent.

2 Notation, basic notions, and basic results

In this section, we present some definitions and results that will be used in the rest of the paper. Most of them are well-known or are direct consequences of well-known results. Others are new.

2.1 Congruence of matrices. Equivalence of matrix pairs

By I_k we denote the $k \times k$ identity matrix, and $\mathfrak{i}$ denotes the imaginary unit (namely $\mathfrak{i}^2 = -1$). The notation $M^\top$ and M^* is used for, respectively, the transpose and the conjugate transpose of the matrix M. The **cosquare* of an invertible matrix M is the matrix $M^{-*}M$, where M^{-*} denotes the conjugate transpose of the inverse of M. By $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ we denote the sets of $m \times n$ complex and real matrices, respectively.

Definition 2.1. *Two matrices $A, B \in \mathbb{C}^{n\times n}$ are*

- *congruent *if there is some invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $SAS^* = B$,*
- congruent *if there is some invertible matrix $S \in \mathbb{C}^{n\times n}$ such that $SAS^\top = B$,*
- real-congruent *if there is some invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $SAS^\top = B$.*

The interesting case for real congruence is when A, B both have real entries.

The following theorem will be fundamental in the proofs of the main results of this paper. It states that if two real matrices are *congruent then they are also real-congruent.

Theorem 2.2. [4, Th. 1.1] *Let $A, B \in \mathbb{R}^{n\times n}$ be such that $PAP^* = B$, for some invertible $P \in \mathbb{C}^{n\times n}$. Then, there exists an invertible $Q \in \mathbb{R}^{n\times n}$ such that $QAQ^\top = B$.*

The counterpart of Theorem 2.2 for the *similarity* of matrices is well-known and can be found in [9, Theorem 1.3.29], for instance. We remark that the proof of Theorem 2.2 is considerably more involved than the one of [9, Theorem 1.3.29].

Matrix pairs, or matrix pencils, will play also an important role in this paper. So, we recall some related definitions.

Definition 2.3. *Let $A, B, C, D \in \mathbb{C}^{m\times n}$. The matrix pairs (A, B) and (C, D) are*

- strictly equivalent *if there are invertible matrices $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$ such that $RAS = C$ and $RBS = D$,*
- strictly real-equivalent *if there are invertible matrices $R \in \mathbb{R}^{m\times m}$ and $S \in \mathbb{R}^{n\times n}$ such that $RAS = C$ and $RBS = D$.*

The interesting case for strict real-equivalence is when A, B, C, D have real entries.

A matrix pair (A, B) can also be seen as a matrix pencil $\lambda A + B$, where λ is a variable [7, Ch. XII]. We will use both views throughout the paper. We emphasize that, in this paper, in the pencil view the first matrix of a pair (A, B) is the leading coefficient of the corresponding pencil and that we use the "+" sign in the definition of the pencil. We will denote strict equivalence of pairs by $\approx$ and strict real-equivalence by $\overset{r}{\approx}$. For simplicity, given matrices R, A, B, S of adequate sizes, we define the product of matrices times matrix pairs as $R(A, B)S := (RAS, RBS)$.

The following result states that if two real pencils are strictly equivalent then they are also strictly real-equivalent. We include the proof since we have not found it in the literature.

Lemma 2.4. *Let $(A, B) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ and $(C, D) \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times n}$ be such that $R(A, B) = (C, D)S$ for some invertible matrices $R \in \mathbb{C}^{m\times m}$ and $S \in \mathbb{C}^{n\times n}$. Then there exist invertible matrices $\widetilde{R} \in \mathbb{R}^{m\times m}$ and $\widetilde{S} \in \mathbb{R}^{n\times n}$ such that $\widetilde{R}(A, B) = (C, D)\widetilde{S}$.*

Proof. Let $R = R_r + \mathfrak{i}R_{\mathfrak{i}}$ with $R_r, R_{\mathfrak{i}} \in \mathbb{R}^{m\times m}$ and $S = S_r + \mathfrak{i}S_{\mathfrak{i}}$ with $S_r, S_{\mathfrak{i}} \in \mathbb{R}^{n\times n}$, i.e., we express R and S in terms of their real and imaginary parts. Then, $R(A, B) = (C, D)S$ implies $R_r(A, B) = (C, D)S_r$ and $R_{\mathfrak{i}}(A, B) = (C, D)S_{\mathfrak{i}}$. So, for any number τ, $(R_r + \tau R_{\mathfrak{i}})(A, B) = (C, D)(S_r + \tau S_{\mathfrak{i}})$. It only remains to prove that we can choose $\tau_0 \in \mathbb{R}$, such that $\det(R_r + \tau_0 R_{\mathfrak{i}}) \neq 0$ and $\det(S_r + \tau_0 S_{\mathfrak{i}}) \neq 0$. For this purpose note that the polynomials in τ, $p(\tau) = \det(R_r + \tau R_{\mathfrak{i}})$ and $q(\tau) = \det(S_r + \tau S_{\mathfrak{i}})$ are not identically zero since $p(\mathfrak{i}) \neq 0$ and $q(\mathfrak{i}) \neq 0$. Moreover, $p(\tau)$ has at most m complex roots and $q(\tau)$ has at most n complex roots. Thus, we can take τ_0 to be equal to any real number which is not a root of $p(\tau)\,q(\tau)$ and $\widetilde{R} = R_r + \tau_0 R_{\mathfrak{i}}$ and $\widetilde{S} = S_r + \tau_0 S_{\mathfrak{i}}$. □

2.2 Canonical blocks

The canonical forms considered in this work are direct sums of certain canonical blocks that are described in this subsection. We also investigate some properties of these canonical blocks. As usual, the direct sum of two matrices is defined as $A \oplus B := \left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$.

The following matrices were introduced in [10, p. 1011], for each integer $k \geq 1$ and $\mu \in \mathbb{C}$:

$$J_k(\mu) := \begin{bmatrix} \mu & 1 & & \\ & \ddots & \ddots & \\ & & \mu & 1 \\ & & & \mu \end{bmatrix}_{k\times k}, \tag{1}$$

$$\Gamma_k := \begin{bmatrix} 0 & & & & & (-1)^{k+1} \\ & & & & \iddots & (-1)^k \\ & & & -1 & \iddots & \\ & & 1 & 1 & & \\ & -1 & -1 & & & \\ 1 & 1 & & & & 0 \end{bmatrix}_{k\times k}, \tag{2}$$

$$H_{2k}(\mu) := \begin{bmatrix} 0 & I_k \\ J_k(\mu) & 0 \end{bmatrix}_{2k\times 2k}. \tag{3}$$

The matrix in (1) is a Jordan block associated with the eigenvalue μ (see, for instance, [9, Def. 3.1.1]). We highlight that $\Gamma_1 = 1$ and $J_1(\mu) = \mu$.

Following the notation in [9, p. 202], for $a, b \in \mathbb{R}$ and $k \geq 1$, we define the matrices

$$C(a, b) := \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad \text{and} \quad C_{2k}(a, b) := \begin{bmatrix} C(a, b) & I_2 & & \\ & C(a, b) & \ddots & \\ & & \ddots & I_2 \\ & & & C(a, b) \end{bmatrix}_{2k\times 2k}. \tag{4}$$

Note that we write $C_{2k}(a,b)$ instead of $C_k(a,b)$ (as in [9]) to highlight that the size of the matrix is $2k \times 2k$, as we have done with $H_{2k}(\mu)$.

We will also need the matrix (again for $a, b \in \mathbb{R}$ and $k \geq 1$):

$$\widehat{H}_{4k}(a,b) := \begin{bmatrix} 0 & I_{2k} \\ C_{2k}(a,b) & 0 \end{bmatrix}. \tag{5}$$

Lemmas 2.5 and 2.6 are slight variants of some identities from [9, pp. 201-202]. They will be used later. Lemma 2.5 follows from a direct computation.

Lemma 2.5. *Let* $\mu = a + \mathrm{i}b$, *with* $a, b \in \mathbb{R}$ *and* $b \neq 0$, *and* $D(\mu) := \left[\begin{smallmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{smallmatrix}\right]$. *Then the unitary matrix* $W = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} -\mathrm{i} & -\mathrm{i} \\ 1 & -1 \end{smallmatrix}\right]$ *satisfies* $WD(\mu)W^* = C(a,b)$.

In the proof of the next lemma, as well as in many other parts of the paper, we use the Kronecker product $A \otimes B$ of two matrices and some of its properties. The reader can find a lot of information about it in [8, Ch. 4].

Lemma 2.6. *Let* $\mu = a + \mathrm{i}b$, *with* $a, b \in \mathbb{R}$ *and* $b \neq 0$. *Then there is a unitary matrix* $U \in \mathbb{C}^{2k \times 2k}$ *such that* $U \left[\begin{smallmatrix} J_k(\mu) & 0 \\ 0 & J_k(\overline{\mu}) \end{smallmatrix}\right] U^* = C_{2k}(a,b)$.

Proof. According to [9, p. 201], there is a permutation matrix P such that

$$(I_k \otimes W) P \left[\begin{smallmatrix} J_k(\mu) & 0 \\ 0 & J_k(\overline{\mu}) \end{smallmatrix}\right] P^\top (I_k \otimes W^*) = (I_k \otimes W) \begin{bmatrix} D(\mu) & I_2 & & \\ & D(\mu) & \ddots & \\ & & \ddots & I_2 \\ & & & D(\mu) \end{bmatrix} (I_k \otimes W^*) = C_{2k}(a,b),$$

where $D(\mu)$ and W are as in Lemma 2.5. Setting $U = (I_k \otimes W)P$ we get the result. $\square$

The following tridiagonal matrices were introduced in [5, eqs. (6) and (7)], for each integer $k \geq 1$ and $\mu \in \mathbb{C}$:

$$T_k(\mu) := \begin{bmatrix} 0 & 1 & & & 0 \\ \mu & 0 & 1 & & \\ & \mu & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & \mu & 0 \end{bmatrix}_{k \times k} \qquad (T_1(\mu) = 0), \tag{6}$$

$$\widetilde{\Gamma}_k := \begin{bmatrix} 1 & 1 & & & & 0 \\ -1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & -1 & 0 & 1 & \\ & & & 1 & 0 & \ddots \\ 0 & & & & \ddots & \ddots \end{bmatrix}_{k \times k} \qquad (\widetilde{\Gamma}_1 = 1). \tag{7}$$

From $T_k(\mu)$ in (6) and the first matrix in (4), we define, for $a, b \in \mathbb{R}$ and $k \geq 1$, the matrix:

$$\widehat{T}_{2k}(a,b) := \begin{bmatrix} 0 & I_2 & & & 0 \\ C(a,b) & 0 & I_2 & & \\ & C(a,b) & 0 & \ddots & \\ & & \ddots & \ddots & I_2 \\ 0 & & & C(a,b) & 0 \end{bmatrix}_{2k \times 2k} \qquad (\widehat{T}_2(a,b) = 0_{2\times 2}). \tag{8}$$

We warn the reader that in the main results of this paper where the matrix in (8) plays a key role, i.e., Theorems 5.1, 5.2, and 6.1, the parameter k defining the number of 2×2 blocks is an even number and so the matrix appears written as $\widehat{T}_{4k}(a,b)$.

The next lemma relates the matrices (6) and (8). It resembles Lemma 2.6.

Lemma 2.7. *Let $\mu = a + \mathfrak{i}b$, with $a, b \in \mathbb{R}$ and $b \neq 0$. Then there is a unitary matrix $V \in \mathbb{C}^{2k \times 2k}$ such that $V \left[\begin{smallmatrix} T_k(\mu) & 0 \\ 0 & T_k(\overline{\mu}) \end{smallmatrix}\right] V^* = \widehat{T}_{2k}(a,b)$.*

Proof. Let $P \in \mathbb{C}^{2k \times 2k}$ be the permutation matrix which corresponds to permuting the rows $\begin{bmatrix} 1, & 2, & \ldots, & k, & k+1, & k+2, & \ldots, & 2k \end{bmatrix}$ of I_{2k} as follows

$$\begin{bmatrix} 1, & k+1, & 2 & k+2, & 3, & k+3, & \ldots, & k, & 2k \end{bmatrix},$$

i.e., the first k rows of I_{2k} are in the odd positions in P and the last k rows of I_{2k} are in the even positions, preserving in both cases the relative order. If W and $D(\mu)$ are the matrices in Lemma 2.5, then

$$\begin{aligned} (I_k \otimes W) P \begin{bmatrix} T_k(\mu) & 0 \\ 0 & T_k(\overline{\mu}) \end{bmatrix} P^\top (I_k \otimes W^*) &= (I_k \otimes W) \begin{bmatrix} 0 & I_2 & & & 0 \\ D(\mu) & 0 & I_2 & & \\ & D(\mu) & 0 & \ddots & \\ & & \ddots & \ddots & I_2 \\ 0 & & & D(\mu) & 0 \end{bmatrix} (I_k \otimes W^*) \\ &= \widehat{T}_{2k}(a,b). \end{aligned}$$

Setting $V = (I_k \otimes W)P$ concludes the proof. ☐

We will also use the following result.

Lemma 2.8. *If A and B are square $n \times n$ similar matrices, then $\left[\begin{smallmatrix} 0 & I_n \\ A & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & I_n \\ B & 0 \end{smallmatrix}\right]$ are congruent and *congruent.*

Proof. Let S be nonsingular such that $SAS^{-1} = B$. Then $M \left[\begin{smallmatrix} 0 & I_n \\ A & 0 \end{smallmatrix}\right] M^\star = \left[\begin{smallmatrix} 0 & I_n \\ B & 0 \end{smallmatrix}\right]$, where $M = \left[\begin{smallmatrix} S^{-\star} & 0 \\ 0 & S \end{smallmatrix}\right]$, and with $\star$ being either $\top$ or $*$. ☐

Lemma 2.9 states two properties of the matrices in (2) and (3) which will be often used. The proof is omitted because it is essentially provided in [10, p. 1016], with $H_{2k}(\mu)^*$ replaced by $H_{2k}(\mu)^\top$.

Lemma 2.9. *Let Γ_k and $H_{2k}(\mu)$ be the matrices defined in* (2) *and* (3)*. Then*

1. *$\Gamma_k^{-\top}\Gamma_k$ is similar to $J_k((-1)^{k+1})$, and*

2. $H_{2k}(\mu)^{-*}H_{2k}(\mu)$ *is similar to* $\begin{bmatrix} J_k(\mu) & 0 \\ 0 & J_k(1/\overline{\mu}) \end{bmatrix}$.

The key property of the Kronecker product in Lemma 2.10 will be applied in several proofs. It is a particular case of [8, Cor. 4.3.10].

Lemma 2.10. *Let* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{p\times p}$*. Then there is a permutation matrix* $P(n,p) \in \mathbb{C}^{np\times np}$*, depending only on the dimensions* n *and* p*, such that* $A\otimes B = P(n,p)\,(B\otimes A)\,P(n,p)^{\top}$.

2.3 Kronecker and real-Kronecker canonical forms of matrix pairs

We revise in this section the Kronecker Canonical Form (KCF) of a complex matrix pair under strict equivalence and the real Kronecker Canonical Form (real-KCF) of a real matrix pair under strict real-equivalence. The reason is that we will relate the canonical forms under *congruence of a matrix $A \in \mathbb{C}^{n\times n}$ (resp. under real-congruence of a matrix $A \in \mathbb{R}^{n\times n}$) with the KCF of the complex pair (A^*, A) (resp. with the real-KCF of the real pair $(A^{\top}, A)$). We will need the following two additional matrices for defining the KCF:

$$F_k := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{k\times(k+1)} \quad \text{and} \quad G_k := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}_{k\times(k+1)}, \tag{9}$$

where $F_0 = G_0$ is the 0×1 matrix. The direct sum of two matrix pairs is defined in a natural way as $(A,B)\oplus(C,D) = (A\oplus C, B\oplus D)$.

Theorem 2.11. (KCF, [7, Ch. XII, Theorem 5]) *Each matrix pair* $(A,B) \in \mathbb{C}^{m\times n}\times\mathbb{C}^{m\times n}$ *is strictly equivalent to a direct sum, uniquely determined up to permutation of summands, of canonical pairs of the following four types*

Regular pairs for finite eigenvalues	$(I_k, J_k(\mu))$ *with* $\mu \in \mathbb{C}$
Regular pairs for infinite eigenvalues	$(J_k(0), I_k)$
Right singular pairs	(F_k, G_k)
Left singular pairs	$(F_k^{\top}, G_k^{\top})$

The direct sum asserted in Theorem 2.11 is (up to permutation of its direct summands) the KCF of (A,B) and it will be denoted by KCF(A,B).

Theorem 2.12. (real-KCF, [11, Theorem 3.2]) *Each matrix pair* $(A,B) \in \mathbb{R}^{m\times n}\times\mathbb{R}^{m\times n}$ *is strictly real-equivalent to a direct sum, uniquely determined up to permutation of summands, of real canonical pairs of the following five types*

Regular pairs for finite real eigenvalues	$(I_k, J_k(\mu))$ *with* $\mu \in \mathbb{R}$
Regular pairs for infinite eigenvalues	$(J_k(0), I_k)$
Regular pairs for finite complex-conjugate eigenvalues	$(I_{2k}, C_{2k}(a,b))$ *with* $a,b \in \mathbb{R}, b>0$
Right singular pairs	(F_k, G_k)
Left singular pairs	$(F_k^{\top}, G_k^{\top})$

The direct sum asserted in Theorem 2.12 is (up to permutation of its direct summands) the real-KCF of (A,B) and it will be denoted by real-KCF(A,B).

2.4 Horn-Sergeichuk canonical form for *congruence

Due to its relevance in this paper, we reproduce here the canonical form for *congruence of a matrix $A \in \mathbb{C}^{n\times n}$ provided in [10, Theorem 1.1 (b)]. Moreover, we relate such canonical form with the KCF of (A^*, A) under strict equivalence. We call the direct sum asserted in Theorem 2.13 (a) the *Horn-Sergeichuk canonical form of* A and it will be denoted as HSCF(A), where it must be understood that HSCF(A) is defined up to permutation of the direct summands. We emphasize that Theorem 2.13 (b) establishes that KCF(A^*, A) determines HSCF(A) up to the signs of a particular type of canonical blocks. This may be useful in determining HSCF(A) because the class of strict equivalence transformations is larger than the class of *congruence transformations.

Theorem 2.13.

(a) [10, Th. 1.1 (b)] *Each square complex matrix A is *congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the following three types*

Name	*Block*	*Conditions*
Type 0	$J_k(0)$	–
Type I	$\mu\Gamma_k$	$\mu \in \mathbb{C},\ \lvert\mu\rvert = 1$
Type II	$H_{2k}(\mu)$	$\mu \in \mathbb{C},\ \lvert\mu\rvert > 1$

(b) *The direct sum asserted in (a) determines the* KCF(A^*, A) *under strict equivalence uniquely up to permutation of its direct summands. Conversely, the* KCF(A^*, A) *under strict equivalence determines the direct sum asserted in (a) uniquely up to permutation of summands and multiplication of any direct summand of Type I by* -1*. For any direct summand B of Types I, II, or III in part (a), the* KCF(B^*, B) *under strict equivalence is given in the following table:*

Block B in HSCF	KCF(B^*, B)
$J_k(0)$	$(F_\ell, G_\ell) \oplus (F_\ell^\top, G_\ell^\top)$ *if* $k = 2\ell + 1$ $(J_\ell(0), I_\ell) \oplus (I_\ell\, J_\ell(0))$ *if* $k = 2\ell$
$\mu\Gamma_k$, $\mu \in \mathbb{C},\ \lvert\mu\rvert = 1$	$\left(I_k\,,\ J_k\left((-1)^{k+1}\mu^2\right)\right)$
$H_{2k}(\mu)$, $\mu \in \mathbb{C},\ \lvert\mu\rvert > 1$	$(I_k\,,\ J_k(\mu)) \oplus (I_k\,,\ J_k(1/\overline{\mu}))$

Proof of (b). Observe that if $A = P(B_1 \oplus \cdots \oplus B_q)P^*$ is the *congruence asserted in part (a), where each B_i is a canonical matrix of Type 0, I, or II, then $(A^*, A) = P\left((B_1^*, B_1) \oplus \cdots \oplus (B_q^*, B_q)\right) P^*$. So, $(A^*, A) \approx (B_1^*, B_1) \oplus \cdots \oplus (B_q^*, B_q)$ and KCF$(A^*, A) =$ KCF$(B_1^*, B_1) \oplus \cdots \oplus$ KCF(B_q^*, B_q). Thus, HSCF(A) and the table in part (b) determine completely KCF(A^*, A). On the other hand, from the table in part (b), it is obvious that KCF(A^*, A) determines HSCF(A) only up to multiplication of any direct summand of Type I by -1. Therefore, it only remains to prove the table in part (b).

The KCF of $(J_k(0)^*, J_k(0))$ follows from [5, Theorem 1.2 (b)] by taking $\lambda = 0$ in that reference. It can be also easily deduced via standard arguments of matrix pencils. For $\mathrm{KCF}((\mu\Gamma_k)^*, \mu\Gamma_k)$ with $|\mu| = 1$, we proceed as follows

$$((\mu\Gamma_k)^*, \mu\Gamma_k)) \approx (I_k, (\mu\Gamma_k)^{-*}\mu\Gamma_k)) = \left(I_k, \frac{\mu}{\overline{\mu}}\Gamma_k^{-\top}\Gamma_k\right) \approx \left(I_k, J_k\left((-1)^{k+1}\frac{\mu}{\overline{\mu}}\right)\right)$$
$$= \left(I_k, J_k\left((-1)^{k+1}\mu^2\right)\right),$$

where the last strict equivalence follows from Lemma 2.9 and the last equality from the fact that $\mu^2 = \mu/\overline{\mu}$ if $|\mu| = 1$. Analogously, for $\mathrm{KCF}(H_{2k}(\mu)^*, H_{2k}(\mu))$, observe that

$$(H_{2k}(\mu)^*, H_{2k}(\mu)) \approx (I_{2k}, H_{2k}(\mu)^{-*}H_{2k}(\mu)) \approx (I_k\,,\, J_k(\mu)\,) \oplus (I_k\,,\, J_k\,(1/\overline{\mu})\,),$$

where the last strict equivalence follows from Lemma 2.9. □

Remark 2.14. *We emphasize that the condition $|\mu| > 1$ in the Type II blocks in Theorem* 2.13 *(a) can be replaced by $0 < |\mu| < 1$. To see this note that $\left[\begin{smallmatrix} 0 & J_k(\mu)^{-1} \\ I & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & I \\ J_k(\mu) & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & J_k(\mu)^{-1} \\ I & 0 \end{smallmatrix}\right]^* = \left[\begin{smallmatrix} 0 & I \\ J_k(\mu)^{-*} & 0 \end{smallmatrix}\right]$ and, since $J_k(\mu)^{-*}$ is similar to $J_k(1/\overline{\mu})$, we can use Lemma* 2.8 *to replace μ by $1/\overline{\mu}$.*

2.5 Futorny-Horn-Sergeichuk tridiagonal canonical form for *congruence

Another interesting canonical form for *congruence of a matrix $A \in \mathbb{C}^{n\times n}$ is the tridiagonal one introduced in [5, Theorem 1.2], which requires fewer canonical blocks than the HSCF described in Theorem 2.13. We reproduce here [5, Theorem 1.2], since we will also develop a real counterpart of this canonical form. We remark that [5, Theorem 1.2] is valid for any algebraically closed field with a nonidentity involution, but, taking into account the purposes of this work, we will state it over $\mathbb{C}$. The direct sum asserted in Theorem 2.15 (a) will be called the *Futorny-Horn-Sergeichuk canonical form of A* and will be denoted by FHSCF(A), which is defined up to permutation of the direct summands. As in the case of Theorem 2.13, the FHSCF(A) will be related to KCF(A^*, A), although in this case such a relationship was already established in [5, Theorem 1.2]. Also in this case, KCF(A^*, A) determines FHSCF(A) up to the signs of a particular type of canonical blocks. It is natural to wonder about the precise relationship between the HSCF and the FHSCF. We postpone the answer to this question to Section 3, as it requires careful analysis. We warn the reader that we have used $\mu^2 = \overline{\mu}^{-1}\mu$ if $|\mu| = 1$, and $(J_k(\overline{\alpha}), I_k) \approx (I_k, J_k(1/\overline{\alpha}))$ if $\alpha \neq 0$ in Theorem 2.15 (b).

Theorem 2.15. [5, Theorem 1.2]

(a) *Each square complex matrix A is *congruent to a direct sum, uniquely determined up to permutation of summands, of tridiagonal canonical matrices of the following two types*

Name	*Block*	*Conditions*
Type Tri-I	$T_k(\mu)$	$\mu \in \mathbb{C}$, $|\mu| \neq 1$, *each nonzero μ is determined up to replacement by $\overline{\mu}^{-1}$,* $\mu = 0$ *if k is odd*
Type Tri-II	$\mu\widetilde{\Gamma}_k$	$\mu \in \mathbb{C}$, $|\mu| = 1$

(b) *The direct sum asserted in (a) determines the* KCF(A^*, A) *under strict equivalence uniquely up to permutation of its direct summands. Conversely, the* KCF(A^*, A) *under strict equivalence determines the direct sum asserted in (a) uniquely up to permutation of summands and multiplication of any direct summand of Type Tri-II by* -1*. For any direct summand* B *of Types Tri-I or Tri-II in part (a), the* KCF(B^*, B) *under strict equivalence is given in the following table:*

Block B *in* FHSCF	KCF(B^*, B)
$T_k(\mu)$, $\mu \in \mathbb{C}$, $\lvert\mu\rvert \neq 1$, $\mu = 0$ *if* k *is odd*	$(F_\ell, G_\ell) \oplus (F_\ell^\top, G_\ell^\top)$ *if* $k = 2\ell + 1$ $(J_\ell(0), I_\ell) \oplus (I_\ell, J_\ell(0))$ *if* $k = 2\ell$ *and* $\mu = 0$ $(I_\ell\,,\, J_\ell(\mu)) \oplus (I_\ell\,,\, J_\ell(1/\overline{\mu}))$ *if* $k = 2\ell$ *and* $\mu \neq 0$
$\mu\widetilde{\Gamma}_k$, $\mu \in \mathbb{C}$, $\lvert\mu\rvert = 1$	$\left(I_k\,,\, J_k\left((-1)^{k+1}\mu^2\right)\right)$

3 Relation between Horn-Sergeichuk and Futorny-Horn- Sergeichuk canonical forms

We present in Theorem 3.2 the precise *congruence relations between the canonical blocks in the HSCF of Theorem 2.13 (a) and those in the FHSCF of Theorem 2.15 (a). For that purpose, we need the auxiliary Lemma 3.1, which shows that Γ_k is real-congruent to either $\widetilde{\Gamma}_k$, when $k \equiv 1, 2 \pmod 4$, or $-\widetilde{\Gamma}_k$, when $k \equiv 0, 3 \pmod 4$. Moreover, it also provides the explicit real-congruence between these two matrices, which is the product of a permutation matrix times a diagonal signature matrix and, so, is orthogonal. The proof of Lemma 3.1 is postponed to Appendix A.

Lemma 3.1. *For any* $k \geq 1$*, let* $P_k \in \mathbb{R}^{k\times k}$ *be the permutation matrix which corresponds to permuting the columns* $\begin{bmatrix}1, & \ldots, & k\end{bmatrix}$ *of* I_k *as follows*

$$\left[\tfrac{k}{2}+1,\ \ \tfrac{k}{2},\ \ \tfrac{k}{2}+2,\ \ \tfrac{k}{2}-1,\ \ \ldots,\ \ k-1,\ \ 2,\ \ k,\ \ 1\right] \quad \textit{if } k \textit{ is even, and} \tag{10}$$

$$\left[\tfrac{k+1}{2},\ \ \tfrac{k+3}{2},\ \ \tfrac{k-1}{2},\ \ \tfrac{k+5}{2},\ \ \tfrac{k-3}{2},\ \ \ldots,\ \ k-1,\ \ 2,\ \ k,\ \ 1\right] \quad \textit{if } k \textit{ is odd.} \tag{11}$$

Let also $S_k := \operatorname{diag}(s_1, \ldots, s_k)$*, where*

$$s_i = \begin{cases} -1 & \textit{if } i \equiv 3 \pmod 4, \\ 1 & \textit{otherwise.} \end{cases}$$

Then

$$(P_k S_k)^\top \Gamma_k (P_k S_k) = \begin{cases} \widetilde{\Gamma}_k & \textit{if } k \equiv 1, 2 \pmod 4, \\ -\widetilde{\Gamma}_k & \textit{if } k \equiv 0, 3 \pmod 4. \end{cases}$$

Theorem 3.2 is a direct corollary of Lemma 3.1 and part (b) in Theorems 2.13 and 2.15, so the proof is omitted.

Theorem 3.2. *Let* $J_k(0)$*,* Γ_k*,* $H_{2k}(\mu)$*,* $T_k(\mu)$*, and* $\widetilde{\Gamma}_k$ *be the matrices in (1), (2), (3), (6), and (7), respectively. Then the* **congruences described in the following table hold.*

Block B *in* HSCF	*is* **congruent to block* C *in* FHSCF
$J_k(0)$	$T_k(0)$
$\mu\Gamma_k, \quad \mu \in \mathbb{C},\ \lvert\mu\rvert = 1$	$\mu\widetilde{\Gamma}_k \quad$ *if* $k \equiv 1, 2 \pmod 4$ $-\mu\widetilde{\Gamma}_k \quad$ *if* $k \equiv 0, 3 \pmod 4$
$H_{2k}(\mu), \quad \mu \in \mathbb{C},\ \lvert\mu\rvert > 1$	$T_{2k}(\mu)$

4 First canonical form of real matrices for real-congruence

Theorem 4.1 establishes a real counterpart of the HSCF presented in Theorem 2.13 (a). It is one of the two main results of this paper. Theorem 4.1 will be complemented in Theorem 4.2 with the relation between the first canonical form of $A \in \mathbb{R}^{n\times n}$ for real-congruence and the real-KCF of $(A^\top, A)$.

Theorem 4.1. *Each square matrix $A \in \mathbb{R}^{n\times n}$ is real-congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the following four types*

Name	*Block*	*Conditions*
Type (i)	$J_k(0)$	–
Type (ii)	$\Gamma_k \otimes N$	$N = \pm 1$ *or* $N = C(a, b)$, *with* $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, *and* $b > 0$
Type (iii)	$H_{2k}(a)$	$a \in \mathbb{R}$, $0 < \lvert a\rvert < 1$
Type (iv)	$\widehat{H}_{4k}(a, b)$	$a, b \in \mathbb{R}$, $a^2 + b^2 < 1$, *and* $b > 0$

In the matrices of Type (ii), *$\Gamma_k \otimes N$ can be replaced by $N \otimes \Gamma_k$. In the matrices of Type* (iii), *the condition $0 < |a| < 1$ can be replaced by $|a| > 1$ or, in other words, each a is determined up to replacement by $1/a$. In the matrices of Type* (iv), *the condition $a^2 + b^2 < 1$ can be replaced by $a^2 + b^2 > 1$ or, in other words, each pair (a, b) is determined up to replacement by $\Big(a/(a^2 + b^2)\,,\, b/(a^2 + b^2)\Big)$.*

Proof. According to Theorem 2.13 (a), there is a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $A = P\,C_A P^*$, where C_A is a direct sum of canonical matrices of the types $J_k(0), \mu\Gamma_k$ with $|\mu| = 1$, and $H_{2k}(\mu)$ with $|\mu| > 1$ (which can be replaced by $0 < |\mu| < 1$, see Remark 2.14), where $J_k(0), \Gamma_k$, and $H_{2k}(\mu)$ are as in (1), (2), and (3). By taking conjugates in $A = P\,C_A P^*$ and using that A is real, we get $A = \overline{P}\,\overline{C}_A P^\top$. This means that $\overline{C}_A$ is *congruent to A. Since $\overline{C}_A$ is also a direct sum of canonical matrices $J_k(0), \widetilde{\mu}\,\Gamma_k$ with $|\widetilde{\mu}| = 1$, and $H_{2k}(\widetilde{\mu})$ with $|\widetilde{\mu}| > 1$, the uniqueness (up to permutation of summands) of the direct sum in Theorem 2.13 (a) implies that, for $\mu \in \mathbb{C} \setminus \mathbb{R}$, if a block $\mu\Gamma_k$ appears in C_A, the block $\overline{\mu}\,\Gamma_k$ must appear as well. In other words, the blocks $\mu\Gamma_k, \overline{\mu}\,\Gamma_k$, with $\mu \in \mathbb{C} \setminus \mathbb{R}$ are paired up in C_A. The same happens with the blocks $H_{2k}(\mu)$ and $H_{2k}(\overline{\mu})$, with $\mu \in \mathbb{C} \setminus \mathbb{R}$.

Now, we divide the proof in several steps. In Step 1 we see that each pair $\mu\Gamma_k \oplus \overline{\mu}\,\Gamma_k$, with $\mu \in \mathbb{C} \setminus \mathbb{R}$, is *congruent to a real block of Type (ii) with $N = C(a,b)$ in the statement, whereas in Step 2 we do the same with the pairs of blocks $H_{2k}(\mu) \oplus H_{2k}(\overline{\mu})$, with $\mu \in \mathbb{C} \setminus \mathbb{R}$, and blocks of Type (iv) in the statement. In Step 3 we show that then there is a real congruence leading A to the direct sum of the blocks produced in Steps 1–2, together with the real blocks of Types (i), (ii) with $N = \pm 1$, and (iii) that were already in C_A. Finally, in Step 4 we prove that this final direct sum is unique up to permutation of its direct summands.

Step 1: Let $\mu = a + \mathrm{i}b$, with $|\mu| = 1$ (so $a^2 + b^2 = 1$) and $b \neq 0$. Then

$$\begin{bmatrix} \mu\Gamma_k & 0 \\ 0 & \overline{\mu}\,\Gamma_k \end{bmatrix} = \begin{bmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{bmatrix} \otimes \Gamma_k = \Pi \left(\Gamma_k \otimes \begin{bmatrix} \mu & 0 \\ 0 & \overline{\mu} \end{bmatrix} \right) \Pi^\top = \Pi (I_k \otimes W)^* \left(\Gamma_k \otimes \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \right) (I_k \otimes W) \Pi^\top,$$

where Π is the permutation matrix in Lemma 2.10, and W is as in Lemma 2.5. As a consequence, $\left[\begin{smallmatrix} \mu\Gamma_k & 0 \\ 0 & \overline{\mu}\,\Gamma_k \end{smallmatrix}\right]$ is (unitarily) *congruent to $\Gamma_k \otimes C(a,b)$.

If $b < 0$ then $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] C(a,b) \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] = C(a,-b)$, so $C(a,-b)$ is *congruent to $C(a,b)$. Therefore, we can restrict ourselves to $b > 0$.

Hence every block $\left[\begin{smallmatrix} \mu\Gamma_k & 0 \\ 0 & \overline{\mu}\,\Gamma_k \end{smallmatrix}\right]$ is *congruent to a block of Type (ii) in the statement, with $N = C(a,b)$ and $b > 0$. Lemma 2.10 proves also that $\Gamma_k \otimes N$ can be replaced by $N \otimes \Gamma_k$ in any direct summand of Type (ii).

Step 2: Let $\mu = a + \mathrm{i}b$ with $b > 0$. Then, the block $\left[\begin{smallmatrix} H_{2k}(\mu) & 0 \\ 0 & H_{2k}(\overline{\mu}) \end{smallmatrix}\right]$ is *congruent to $\widehat{H}_{4k}(a,b)$. To see this, let U be the unitary matrix in Lemma 2.6 and let P be the 4×4 block permutation matrix of size $4k \times 4k$ which exchanges the second and third block rows (or columns). Then

$$\left(\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} P \right) \begin{bmatrix} H_{2k}(\mu) & 0 \\ 0 & H_{2k}(\overline{\mu}) \end{bmatrix} \left(\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} P \right)^*$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_k \end{bmatrix} \begin{bmatrix} 0 & I_k & 0 & 0 \\ J_k(\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & J_k(\overline{\mu}) & 0 \end{bmatrix} \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_k & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_k \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}^*$$

$$= \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \left[\begin{array}{cc|cc} 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_k \\ \hline J_k(\mu) & 0 & 0 & 0 \\ 0 & J_k(\overline{\mu}) & 0 & 0 \end{array} \right] \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}^* = \begin{bmatrix} 0 & I_{2k} \\ C_{2k}(a,b) & 0 \end{bmatrix} = \widehat{H}_{4k}(a,b),$$

where, to get the last-but-one identity, we use that U is unitary (namely, $UU^* = I$).

Moreover, note that, if $b < 0$, we can consider $\widehat{H}_{4k}(a,-b)$, since $\widehat{H}_{4k}(a,b)$ is real-congruent to $\widehat{H}_{4k}(a,-b)$. To see this, set $\Delta_2 := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, and then $(I_{2k} \otimes \Delta_2)\widehat{H}_{4k}(a,b)(I_{2k} \otimes \Delta_2)^\top = \widehat{H}_{2k}(a,-b)$.

Finally, according to Remark 2.14, $H_{2k}(\mu)$ with $|\mu| > 1$ can be replaced by $H_{2k}(\mu)$ with $0 < |\mu| < 1$. In the first case, the argument above leads to $\widehat{H}_{4k}(a,b)$ with $a^2 + b^2 > 1$ and in the second to $\widehat{H}_{4k}(a,b)$ with $a^2 + b^2 < 1$.

Step 3: As a consequence of Steps 1–2, the matrix A is *congruent to a direct sum of blocks of Types (i)–(iv) in the statement, that we denote by C_A^r. Since both A and C_A^r have real entries, Theorem 2.2 guarantees that they are also real-congruent.

Step 4 (uniqueness): Assume that there are two different direct sums of blocks of Types (i)–(iv), denoted by C_1 and C_2, which are real-congruent. Then, they are also *congruent. But, as we have seen before, blocks of Type (i)–(iv) are *congruent to a direct sum of blocks of Types 0, I, and II in Theorem 2.13. More precisely, blocks of Type-(i) correspond to Type-0 blocks; blocks of Type-(ii) correspond to either blocks of Type-I with $\mu = \pm 1$ or to a direct sum of Type-I blocks when $\mu = a + \mathfrak{i}b \neq \pm 1$; blocks of Type-(iii) are particular cases of Type-II blocks; and blocks of Type-(iv) are *congruent to a direct sum of Type-II blocks. Note also that the Type-I blocks corresponding to different blocks of Type-(ii) are not *congruent to each other, since they correspond to different complex numbers μ with $|\mu| = 1$, and the same happens with the Type-II blocks associated with different blocks of Types (iii) and (iv). Let us denote by $\widehat{C}_1$ and $\widehat{C}_2$ the direct sum corresponding to the Type 0, I, and II blocks associated with the blocks of C_1 and C_2, respectively (in the same order). By Theorem 2.13, the blocks in $\widehat{C}_2$ are a permutation of the blocks in $\widehat{C}_1$ and, then, the blocks in C_2 are also a permutation of the blocks in C_1. □

4.1 Relation of the first canonical form for real-congruence of A and the real-KCF of $(A^\top, A)$

Theorem 4.2 establishes that real-KCF$(A^\top, A)$ determines the canonical form in Theorem 4.1 up to the signs of some parameters in the blocks of Type (ii).

Theorem 4.2. *Let $A \in \mathbb{R}^{n\times n}$. The direct sum asserted in Theorem* 4.1 *determines* real-KCF$(A^\top, A)$ *under strict real-equivalence uniquely up to permutation of its direct summands. Conversely, the* real-KCF$(A^\top, A)$ *under strict real-equivalence determines the direct sum asserted in Theorem* 4.1 *uniquely up to permutation of summands, multiplication of any direct summand of type $\Gamma_k \otimes [\pm 1]$ by -1, and multiplication of the parameter a in any direct summand of type $\Gamma_k \otimes C(a, b)$ by -1. For any direct summand B of* Types (i), (ii), (iii), and (iv) *in Theorem* 4.1, *the* real-KCF$(B^\top, B)$ *under strict real-equivalence is given in the following table:*

Block B in Th. 4.1	real-KCF$(B^\top, B)$
$J_k(0)$	$(F_\ell, G_\ell) \oplus (F_\ell^\top, G_\ell^\top)$ *if* $k = 2\ell + 1$ $(J_\ell(0), I_\ell) \oplus (I_\ell, J_\ell(0))$ *if* $k = 2\ell$
$\Gamma_k \otimes (\pm 1)$	$\left(I_k\, ,\, J_k\left((-1)^{k+1}\right)\right)$
$\Gamma_k \otimes C(a, b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, $b > 0$	$\left(I_k\, ,\, J_k\left((-1)^k\right)\right) \oplus \left(I_k\, ,\, J_k\left((-1)^k\right)\right)$ *if* $a = 0, b = 1$ $\left(I_{2k}, C_{2k}(\,(-1)^{k+1}(a^2 - b^2)\,,\, 2\lvert ab\rvert\,)\right)$ *if* $a \neq 0$
$H_{2k}(a)$, $a \in \mathbb{R}$, $0 < \lvert a\rvert < 1$	$(I_k\, ,\, J_k(a)\,) \oplus (I_k\, ,\, J_k\,(1/a)\,)$
$\widehat{H}_{4k}(a, b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 < 1$, $b > 0$	$(I_{2k}, C_{2k}(a, b)) \oplus \left(I_{2k}, C_{2k}\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)\right)$

Remark 4.3. *Observe that, given a matrix pair $(I_{2k}, C_{2k}(c, d))$, $c, d \in \mathbb{R}$, $c^2 + d^2 = 1$, and $d > 0$, the equality*

$$(I_{2k}, C_{2k}(c, d)) = \left(I_{2k}\, ,\, C_{2k}(\,(-1)^{k+1}(a^2 - b^2)\,,\, 2|ab|\,)\right),$$

with $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, *and* $b > 0$, *holds if and only if*

$$a = \pm\sqrt{\frac{1 + (-1)^{k+1}c}{2}} \quad \textit{and} \quad b = \frac{d}{\sqrt{2\left(1 + (-1)^{k+1}c\right)}}.$$

This allows us to determine explicitly, up to the sign of a*, the block* $B = \Gamma_k \oplus C(a, b)$ *from the* real-KCF$(B^\top, B)$*.*

Proof of Theorem 4.2. An argument analogous to that at the beginning of the proof of Theorem 2.13 (b) shows that it suffices to prove the results in the table in the statement. For this purpose, we will perform general strict equivalences, i.e., multiplications by invertible matrices that may be complex, on each real pair $(B^\top, B)$ in the table until we get the desired real target pair. Then, we apply Lemma 2.4 to conclude that $(B^\top, B)$ and the target pair are strictly real-equivalent.

(1) The real-KCF of $(J_k(0)^\top, J_k(0))$ follows from the first row in the table of Theorem 2.13 (b) and Lemma 2.4.

(2) The real-KCF of $(\,(\Gamma_k \otimes N)^\top, \Gamma_k \otimes N)$ with $N = \pm 1$ follows from the second row in the table of Theorem 2.13 (b) with $\mu = \pm 1$ and Lemma 2.4.

(3) Next, we obtain the real-KCF of $(\,(\Gamma_k \otimes N)^\top, \Gamma_k \otimes N)$ with $N = C(a, b)$, with $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, and $b > 0$. Observe that in this case $C(a, b)^\top = C(a, b)^{-1}$. Thus,

$$\begin{aligned}(\,(\Gamma_k \otimes C(a, b))^\top, \Gamma_k \otimes C(a, b)) = (\Gamma_k^\top \otimes C(a, b)^\top, \Gamma_k \otimes C(a, b)) &\approx (I_{2k}, \Gamma_k^{-\top}\Gamma_k \otimes C(a, b)^2) \\ &\approx (I_{2k}, C(a, b)^2 \otimes \Gamma_k^{-\top}\Gamma_k), \qquad (12)\end{aligned}$$

where the last strict equivalence follows from Lemma 2.10. If W and $D(\mu)$ are the matrices in Lemma 2.5, $C(a, b)^2 = WD(\mu)^2W^*$, which combined with (12) and Lemma 2.9, yields

$$(\,(\Gamma_k \otimes C(a, b))^\top, \Gamma_k \otimes C(a, b)) \approx (I_{2k}, D(\mu)^2 \otimes J_k((-1)^{k+1})). \qquad (13)$$

If $a = 0$, then $b = 1$, $\mu^2 = (\mathrm{i}b)^2 = -1$, $\overline{\mu}^2 = (-\mathrm{i}b)^2 = -1$, $D(\mu)^2 = -I_2$, and (13) reads

$$(\,(\Gamma_k \otimes C(a, b))^\top, \Gamma_k \otimes C(a, b)) \approx (I_{2k}, J_k((-1)^k) \oplus J_k((-1)^k)),$$

which is the desired real-KCF. Then, the use of Lemma 2.4 completes the proof. On the other hand, if $a \neq 0$, then (13) implies

$$\begin{aligned}(\,(\Gamma_k \otimes C(a, b))^\top, \Gamma_k \otimes C(a, b)) &\approx (I_{2k}\,,\; J_k((-1)^{k+1}\,\mu^2) \oplus J_k((-1)^{k+1}\,\overline{\mu}^2)) \\ &\approx (\,I_{2k}\,,\; C_{2k}(\,(-1)^{k+1}(a^2 - b^2)\,,\, (-1)^{k+1}\,(2ab)\,)\,),\end{aligned}$$

where the last strict equivalence follows from Lemma 2.6 with μ replaced by $(-1)^{k+1}\,\mu^2$. If $(-1)^{k+1}\,(2ab) = 2|ab|$, we have obtained the target real-KCF. If not, multiply the pair above on the left and on the right by $I_k \otimes \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. In one case or in another, the use of Lemma 2.4 completes again the proof.

(4) The real-KCF of $(H_{2k}(a)^\top, H_{2k}(a))$, with $a \in \mathbb{R}$, $0 < |a| < 1$, follows from the last row in the table of Theorem 2.13 (b) and Lemma 2.4 (recall Remark 2.14).

(5) Finally, we obtain the real-KCF of $(\widehat{H}_{4k}(a,b)^\top, \widehat{H}_{4k}(a,b))$, with $a, b \in \mathbb{R}$, $a^2+b^2<1$, and $b>0$. For this purpose, note that from **Step 2** in the proof of Theorem 4.1 (observe that $\widehat{H}_{4k}(a,b)^\top = \widehat{H}_{4k}(a,b)^*$), Lemma 2.9 and Lemma 2.6, we have the following

$$\begin{aligned}
\left(\widehat{H}_{4k}(a,b)^\top, \widehat{H}_{4k}(a,b)\right) &\approx \left(\begin{bmatrix} H_{2k}(\mu)^* & 0 \\ 0 & H_{2k}(\overline{\mu})^* \end{bmatrix}, \begin{bmatrix} H_{2k}(\mu) & 0 \\ 0 & H_{2k}(\overline{\mu}) \end{bmatrix} \right) \\
&\approx \left(I_{4k} \, , \begin{bmatrix} H_{2k}(\mu)^{-*} H_{2k}(\mu) & 0 \\ 0 & H_{2k}(\overline{\mu})^{-*} H_{2k}(\overline{\mu}) \end{bmatrix} \right) \\
&\approx (\, I_{4k} \, , \, J_k(\mu) \oplus J_k(1/\overline{\mu}) \oplus J_k(\overline{\mu}) \oplus J_k(1/\mu) \,) \\
&\approx (\, I_{4k} \, , \, J_k(\mu) \oplus J_k(\overline{\mu}) \oplus J_k(1/\overline{\mu}) \oplus J_k(1/\mu) \,) \\
&\approx \left(I_{4k} \, , \, C_{2k}(a,b) \oplus C_{2k}\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2} \right) \right),
\end{aligned}$$

which is the target real-KCF. Again, the use of Lemma 2.4 completes the proof. □

5 Second canonical form of real matrices for real-congruence: block tridiagonal form

Theorem 5.1 establishes a real counterpart of the FHSCF presented in Theorem 2.15 (a). It is one of the two main results of this paper. Theorem 5.1 will be complemented in Theorem 5.2 with the relation between the second canonical form of $A \in \mathbb{R}^{n\times n}$ for real-congruence and the real-KCF of $(A^\top, A)$. The relation between the real canonical forms in Theorems 4.1 and 5.1 is established in Section 6. Recall that the matrix $\widehat{T}_{4k}(a,b)$ is the matrix defined in (8) with k replaced by $2k$.

Theorem 5.1. *Each square matrix $A \in \mathbb{R}^{n\times n}$ is real-congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the following three types*

Name	*Block*	*Conditions*
Type Tri-(i)	$T_k(a)$	*$a \in \mathbb{R}$, $a \neq \pm 1$,* *each nonzero a is determined* *up to replacement by a^{-1},* *$a = 0$ if k is odd*
Type Tri-(ii)	$\widehat{T}_{4k}(a,b)$	*$a, b \in \mathbb{R}$, $a^2+b^2 \neq 1$, and $b>0$,* *(a,b) is determined up to* *replacement by $\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)$*
Type Tri-(iii)	$\widetilde{\Gamma}_k \otimes N$	*where $N = \pm 1$ or* *$N = C(a,b)$, with $a, b \in \mathbb{R}$, $a^2+b^2=1$, and $b>0$*

In the matrices of Type Tri-(iii), *$\widetilde{\Gamma}_k \otimes N$ can be replaced by $N \otimes \widetilde{\Gamma}_k$.*

Proof. The proof is very similar to that of Theorem 4.1, but based on the FHSCF in Theorem 2.15 (a) instead of the HSCF of Theorem 2.13 (a). Therefore, we focus on the relevant differences and sketch very briefly the similar parts. According to Theorem 2.15 (a) there exists a nonsingular matrix $P \in \mathbb{C}^{n\times n}$ such that $A = PF_AP^*$, where F_A is a direct sum of canonical matrices of types Tri-I and Tri-II. The fact that A is real implies via the same argument in the first paragraph of the proof of Theorem 4.1 that for $\mu \in \mathbb{C}\setminus\mathbb{R}$ the blocks $T_k(\mu)$, $T_k(\overline{\mu})$ are paired up in F_A (note that k is even in such $T_k(\mu)$ because $0 \notin \mathbb{C}\setminus\mathbb{R}$), as well as the blocks $\mu\widetilde{\Gamma}_k, \overline{\mu}\widetilde{\Gamma}_k$.

Then, we divide the proof in four steps as in Theorem 4.1. Steps 3 and 4 are identical to those in Theorem 4.1, and Step 1 is also equal except for the fact that Γ_k is now replaced by $\widetilde{\Gamma}_k$. This step proves that every block $\left[\begin{smallmatrix}\mu\widetilde{\Gamma}_k & 0\\ 0 & \overline{\mu}\widetilde{\Gamma}_k\end{smallmatrix}\right]$, with $\mu \in \mathbb{C}\setminus\mathbb{R}$, is *congruent to a block $\widetilde{\Gamma}_k \otimes C(a,b)$ with $b>0$ as in the statement. Then, the only difference with respect to the proof of Theorem 4.1 is in Step 2, which now deals with $\left[\begin{smallmatrix}T_k(\mu) & 0\\ 0 & T_k(\overline{\mu})\end{smallmatrix}\right]$, where $\mu = a + \mathfrak{i}b$, $a,b\in\mathbb{R}$, $b\neq 0$, $|\mu|\neq 1$, and k even. According to Lemma 2.7, $\left[\begin{smallmatrix}T_k(\mu) & 0\\ 0 & T_k(\overline{\mu})\end{smallmatrix}\right]$ is (unitarily) *congruent to $\widehat{T}_{2k}(a,b)$. If $b>0$, the proof is complete. Otherwise, perform the (unitary) *congruence $(I_k \otimes \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])\,\widehat{T}_{2k}(a,b)\,(I_k\otimes\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])^\top = \widehat{T}_{2k}(a,-b)$. Since k is even, we replace $\widehat{T}_{2k}(a,b)$ by $\widehat{T}_{4k}(a,b)$ in the statement. □

5.1 Relation of the second canonical form for real-congruence of A and the real-KCF of $(A^\top, A)$

Theorem 5.2 establishes that real-KCF$(A^\top, A)$ determines the canonical form in Theorem 5.1 up to the signs of some parameters in the blocks of Type Tri-(iii).

Theorem 5.2. *Let $A\in\mathbb{R}^{n\times n}$. The direct sum asserted in Theorem* 5.1 *determines the* real-KCF$(A^\top, A)$ *under strict real-equivalence uniquely up to permutation of its direct summands. Conversely, the* real-KCF$(A^\top, A)$ *under strict real-equivalence determines the direct sum asserted in Theorem* 5.1 *uniquely up to permutation of summands, multiplication of any direct summand of type $\widetilde{\Gamma}_k\otimes[\pm1]$ by -1, and multiplication of the parameter a in any direct summand of type $\widetilde{\Gamma}_k\otimes C(a,b)$ by -1. For any direct summand B of* Types Tri-(i), Tri-(ii), and Tri-(iii) *in Theorem* 5.1, *the* real-KCF$(B^\top, B)$ *under strict real-equivalence is given in the following table:*

Block B in Th. 5.1	real-KCF$(B^\top, B)$
$T_k(a)$, $a \in \mathbb{R}$, $a \neq \pm 1$, $a = 0$ *if* k *is odd*	$(F_\ell, G_\ell) \oplus (F_\ell^\top, G_\ell^\top)$ *if* $k = 2\ell + 1$ $(J_\ell(0), I_\ell) \oplus (I_\ell, J_\ell(0))$ *if* $k = 2\ell$, $a = 0$ $(I_\ell, J_\ell(a)) \oplus (I_\ell\,,\, J_\ell\,(1/a))$ *if* $k = 2\ell$, $a \neq 0$
$\widehat{T}_{4k}(a,b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 1$, $b > 0$	$(I_{2k}, C_{2k}(a,b)) \oplus \left(I_{2k}, C_{2k}\left(\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2}\right)\right)$
$\widetilde{\Gamma}_k \otimes [\pm 1]$	$\left(I_k\,,\, J_k\left((-1)^{k+1}\right)\right)$
$\widetilde{\Gamma}_k \otimes C(a,b)$, $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, $b > 0$	$\left((I_k\,,\, J_k\left((-1)^k\right)) \oplus (I_k\,,\, J_k\left((-1)^k\right)\right)$ *if* $a = 0, b = 1$ $\left(I_{2k}, C_{2k}(\,(-1)^{k+1}(a^2 - b^2)\,,\, 2\lvert ab\rvert\,)\right)$ *if* $a \neq 0$

Proof. As in the proof of Theorem 4.2, it suffices to prove the results in the table in the statement. We will proceed case by case via analogous manipulations to those in Theorem 4.2.

Before we start note that the second row in the table of Theorem 2.15 (b) with $\mu = 1$ implies $(\widetilde{\Gamma}_k^\top, \widetilde{\Gamma}_k) \approx (I_k, J_k((-1)^{k+1}))$. On the other hand, $(\widetilde{\Gamma}_k^\top, \widetilde{\Gamma}_k) \approx (I_k, \widetilde{\Gamma}_k^{-\top}\widetilde{\Gamma}_k)$. Thus, $(I_k, \widetilde{\Gamma}_k^{-\top}\widetilde{\Gamma}_k) \approx (I_k, J_k((-1)^{k+1}))$, which implies that $\widetilde{\Gamma}_k^{-\top}\widetilde{\Gamma}_k$ is similar to $J_k((-1)^{k+1})$. We will use this fact in the proof without explicitly referring to.

(1) The real-KCF of $(T_k(a)^\top, T_k(a))$ with $a \in \mathbb{R}$, $a \neq \pm 1$ follows from the first row in the table in Theorem 2.15 (b) with $\mu = a$ and Lemma 2.4.

(2) We obtain the real-KCF of $(\widehat{T}_{4k}(a,b)^\top, \widehat{T}_{4k}(a,b))$, $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 1$, $b > 0$, from Lemma 2.7, the table in Theorem 2.15 (b) with $\mu = a + ib$, and Lemma 2.6 as follows

$$
\begin{aligned}
(\widehat{T}_{4k}(a,b)^\top, \widehat{T}_{4k}(a,b)) &\approx (T_{2k}(\mu)^* \oplus T_{2k}(\overline{\mu})^*, T_{2k}(\mu) \oplus T_{2k}(\overline{\mu})) \\
&= (T_{2k}(\mu)^*, T_{2k}(\mu)) \oplus (T_{2k}(\overline{\mu})^*, T_{2k}(\overline{\mu})) \\
&\approx (I_k\,,\, J_k(\mu)\,) \oplus (I_k\,,\, J_k\,(1/\overline{\mu})\,) \oplus (I_k\,,\, J_k(\overline{\mu})\,) \oplus (I_k\,,\, J_k\,(1/\mu)\,) \\
&\approx (I_k\,,\, J_k(\mu)\,) \oplus (I_k\,,\, J_k(\overline{\mu})\,) \oplus (I_k\,,\, J_k\,(1/\overline{\mu})\,) \oplus (I_k\,,\, J_k\,(1/\mu)\,) \\
&\approx (I_{2k}, C_{2k}(a,b)) \oplus \left(I_{2k}, C_{2k}\left(a/(a^2+b^2), b/(a^2+b^2)\right)\right).
\end{aligned}
$$

The proof is completed by applying Lemma 2.4.

(3) The real-KCF of $(\,(\widetilde{\Gamma}_k \otimes N)^\top, \widetilde{\Gamma}_k \otimes N)$ with $N = \pm 1$ follows from the second row in the table of Theorem 2.15 (b) with $\mu = \pm 1$ and Lemma 2.4.

(4) Next, we obtain the real-KCF of $(\,(\widetilde{\Gamma}_k \otimes C(a,b))^\top, \widetilde{\Gamma}_k \otimes C(a,b))$, with $a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, and $b > 0$. We proceed as in the proof of Theorem 4.2 for $\Gamma_k \otimes C(a,b)$, i.e., case (3) in that proof. So, we skip most of the details. Recall that for these parameters a, b, $C(a,b)^\top = C(a,b)^{-1}$. Thus,

$$
\begin{aligned}
(\,(\widetilde{\Gamma}_k \otimes C(a,b))^\top, \widetilde{\Gamma}_k \otimes C(a,b)) &\approx (I_{2k}, \widetilde{\Gamma}_k^{-\top}\widetilde{\Gamma}_k \otimes C(a,b)^2) \approx (I_{2k}, C(a,b)^2 \otimes \widetilde{\Gamma}_k^{-\top}\widetilde{\Gamma}_k) \\
&\approx (I_{2k}, D(\mu)^2 \otimes J_k((-1)^{k+1})),
\end{aligned}
$$

where $D(\mu)$ is the matrix in Lemma 2.5. Note that this strict equivalence is as that in (13). Therefore, the rest of the proof is exactly the same as the corresponding one in Theorem 4.2. □

6 Relation between the first and the second canonical forms for real congruence

Theorem 6.1 presents the precise real-congruence relations between the canonical blocks in Theorems 4.1 and 5.1. It is a direct consequence of Lemma 3.1 and Theorems 4.2 and 5.2, so the proof is omitted.

Theorem 6.1. *Let $J_k(0), \Gamma_k, H_{2k}(a), C(a,b), \widehat{H}_{4k}(a,b), T_k(a), \widetilde{\Gamma}_k$, and $\widehat{T}_{2k}(a,b)$ be the matrices in (1)-(8), and*

$$\varepsilon := \begin{cases} 1 & \text{if } k \equiv 1,2 \pmod 4, \\ -1 & \text{if } k \equiv 0,3 \pmod 4. \end{cases}$$

Then the real-congruences described in the following table hold.

Block B in Theorem 4.1	*is real-congruent to block C in* Theorem 5.1
$J_k(0)$	$T_k(0)$
$\Gamma_k \otimes [\pm 1]$	$\widetilde{\Gamma}_k \otimes [\pm\varepsilon]$
$\Gamma_k \otimes C(a,b)$, $a, b \in \mathbb{R}$, $a^2+b^2=1$, $b>0$	$\widetilde{\Gamma}_k \otimes C(\varepsilon a, b)$
$H_{2k}(a)$, $a \in \mathbb{R}$, $0<\lvert a\rvert<1$	$T_{2k}(a)$
$\widehat{H}_{4k}(a,b)$, $a, b \in \mathbb{R}$, $a^2+b^2<1$, $b>0$	$\widehat{T}_{4k}(a,b)$

7 Correspondence with the blocks in the Lee-Weinberg canonical form

In this section, we relate the blocks in the first canonical form for real-congruence in Theorem 4.1 with those in the real-congruence canonical form in [12, Theorem II, p. 213]. Once this result is established, the correspondence of the blocks in [12, Theorem II, p. 213] with those in the second block-tridiagonal canonical form for real-congruence in Theorem 5.1 follows immediately from Theorem 6.1. For brevity, such correspondence is not explicitly stated.

To describe the canonical form in [12], several structured matrices must be defined. We use the same notation as in [12], although additional restrictions are imposed on some parameters to make the form in [12] truly canonical (see Remark 7.2). We start with the following auxiliary matrices:

$$L_k := \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}_{(k+1)\times k}, \quad L_k^+ := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{k\times(k+1)},$$

$$\Delta_k := \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}_{k\times k}, \quad \Lambda_k := \begin{bmatrix} & & & 0 \\ & & 0 & 1 \\ & \iddots & 1 & \\ 0 & 1 & & \end{bmatrix}_{k\times k} \qquad (\Lambda_1 := 0),$$

$$S\Delta_k := \begin{cases} \begin{bmatrix} 0 & \Delta_{k/2} \\ -\Delta_{k/2} & 0 \end{bmatrix}_{k\times k} & (k \text{ even}), \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta_{(k-1)/2} \\ 0 & -\Delta_{(k-1)/2} & 0 \end{bmatrix}_{k\times k} & (k \text{ odd}). \end{cases}$$

The canonical form for real-congruence of real square matrices in [12] is a direct sum of the following eight types of blocks:

$$m_3' := \begin{bmatrix} 0 & L_k \\ L_k^+ & 0 \end{bmatrix}_{(2k+1)\times(2k+1)} \qquad (m_3' := 0 \text{ if } k = 0), \tag{14}$$

$$\infty_4' := \varepsilon\,(S\Delta_k + \Lambda_k) \qquad (\varepsilon = \pm 1,\ k \text{ even}), \tag{15}$$

$$\infty_5' := \begin{bmatrix} 0 & \Delta_k + \Lambda_k \\ -\Delta_k + \Lambda_k & 0 \end{bmatrix}_{2k\times 2k} \qquad (k \text{ odd}), \tag{16}$$

$$o_3' := \varepsilon\,(\Delta_k + S\Delta_k) \qquad (\varepsilon = \pm 1,\ k \text{ odd}), \tag{17}$$

$$o_4' := \begin{bmatrix} 0 & \Delta_k + \Lambda_k \\ \Delta_k - \Lambda_k & 0 \end{bmatrix}_{2k\times 2k} \qquad (k \text{ even}), \tag{18}$$

$$\alpha_3' := \begin{bmatrix} 0 & (\alpha+1)\Delta_k + \Lambda_k \\ (-\alpha+1)\Delta_k - \Lambda_k & 0 \end{bmatrix}_{2k\times 2k} \qquad (\alpha \in \mathbb{R},\ \alpha > 0), \tag{19}$$

$$\beta_4' := \varepsilon \begin{bmatrix} & & & R \\ & & R & S \\ & \iddots & \iddots & \\ R & S & & \end{bmatrix}_{2k\times 2k} \qquad (\varepsilon = \pm 1), \tag{20}$$

$$\beta_5' := \left[\begin{array}{cccc|cccc} & & & & & & & R' \\ & & & & & & R' & S' \\ & & & & & \iddots & \iddots & \\ & & & & R' & S' & & \\ \hline & & & -T & & & & \\ & & -T & -S' & & & & \\ & \iddots & \iddots & & & & & \\ -T & -S' & & & & & & \end{array}\right]_{4k\times 4k}, \tag{21}$$

where

$$R = \begin{bmatrix} 1 & |b| \\ -|b| & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} b & a-1 \\ a-1 & -b \end{bmatrix}, \quad R' = \begin{bmatrix} b & a+1 \\ a+1 & -b \end{bmatrix}, \quad S' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with

$$a, b \in \mathbb{R} \quad \text{and} \quad a \neq 0,\, b \neq 0.$$

Theorem 7.1 is the main result in [12].

Theorem 7.1. [12, Theorem II, p. 213] *Each square real matrix is real-congruent to a direct sum, uniquely determined up to permutation of summands, of matrices of types* $m_3', \infty_4', \infty_5', o_3', o_4', \alpha_3', \beta_4'$*, and* β_5'.

Remark 7.2. (On the values of the parameters α in (19) and a, b in (20)–(21)) *In* [12] *the parameter* α *in* (19) *is just required to be "finite and nonzero". We have imposed* $\alpha > 0$*, because otherwise the block would not be "truly" canonical. The reason is that* α_3' *is real-congruent to the matrix obtained by replacing* α *by* $-\alpha$ *in* α_3'*. To see this, set* $S_k := \operatorname{diag}(1, -1, 1, -1, \dots, (-1)^{k-1}) \in \mathbb{R}^{k\times k}$ *and* $\widetilde{S}_k := \operatorname{diag}((-1)^{k-1}, \dots, -1, 1, -1, 1) \in \mathbb{R}^{k\times k}$*, i.e.,* $\widetilde{S}_k$ *has the same diagonal entries as* S_k *but in reversed order. Then*

$$\begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix} \begin{bmatrix} S_k & 0 \\ 0 & \widetilde{S}_k \end{bmatrix} \alpha_3' \begin{bmatrix} S_k & 0 \\ 0 & \widetilde{S}_k \end{bmatrix} \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix} = \begin{bmatrix} 0 & (-\alpha+1)\Delta_k + \Lambda_k \\ (\alpha+1)\Delta_k - \Lambda_k & 0 \end{bmatrix}.$$

In [12]*, the parameters* a *and* b *in* β_4'*, and* β_5' *are just required to be real. However,* [12, Theorem II, p. 213] *is obtained from* [16, Theorem 2 (c), p. 344] *and this result requires* $a \neq 0 \neq b$*. Therefore, we also have required* $a \neq 0 \neq b$*. In fact, if* $a = 0$ *or* $b = 0$*, then* β_4' *and* β_5' *would not be canonical blocks since they would be real-congruent to direct sums of other canonical blocks in Theorem* 7.1*. To see this, compare the table in Appendix* B *with Table* 1.

Theorem 7.3 is the main theorem of this section.

Theorem 7.3. *Let* $m_3', \infty_4', \infty_5', o_3', o_4', \alpha_3', \beta_4'$*, and* β_5' *be the matrices in* (14)–(21)*, and* $J_k(0)$*,* Γ_k*,* $H_{2k}(a), C(a,b)$*, and* $\widehat{H}_{4k}(a,b)$ *be the matrices defined in* (1)–(5) *and appearing in Theorem* 4.1*. Then the real congruences described in Table* 1 *hold.*

The proof of Theorem 7.3 is postponed to Subsection 7.1, where it is obtained as a corollary of a sequence of lemmas, each proving the real-congruence described in one of the rows of Table 1. We hasten to admit that the precise sign of the parameters in two of the blocks in the right column of Table 1 has not been determined. These are the blocks corresponding to β_4'.

We see in Table 1 that the difference in the number of distinct canonical blocks in the canonical forms of Theorems 4.1 and 7.1 comes from the blocks $\infty_4', \infty_5', o_3', o_4'$, and β_4', which are gathered in the two variants $\Gamma_k \otimes [\pm 1]$ and $\Gamma_k \otimes C(a,b)$ of the blocks of Type (ii) in Theorem 4.1.

7.1 Proof of Theorem 7.3

In addition to performing direct real-congruences, we will often use the following approach for proving the results in this section: given a block B in the left column of Table 1, we will compute the real-KCF$(B^\top, B)$ and we will make use of the table in Theorem 4.2 to determine the block in Theorem 4.1 to which B is real-congruent. Recall that Lemma 2.4 allows us to compute real-KCFs by means of intermediate complex strict equivalences.

Lemma 7.4. *The matrix* m_3' *in Table* 1 *is real-congruent to* $J_{2k+1}(0)$.

Block B in Theorem 7.1	is real-congruent to block C in Theorem 4.1
m_3'	$J_{2k+1}(0)$
∞_4'	$\Gamma_k \otimes (\varepsilon\,(-1)^{\frac{k}{2}+1})$ (k even)
∞_5'	$\Gamma_k \otimes C(0,1)$ (k odd)
o_3'	$\Gamma_k \otimes (\varepsilon\,(-1)^{\frac{k-1}{2}})$ (k odd)
o_4'	$\Gamma_k \otimes C(0,1)$ (k even)
α_3'	$J_{2k}(0)$ if $\alpha = 1$ $H_{2k}\left(\frac{1-\alpha}{1+\alpha}\right)$ if $\alpha \neq 1$
β_4'	$\Gamma_k \otimes C\left(\pm\frac{\lvert b\rvert}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+b^2}}\right)$ if k is even $\Gamma_k \otimes C\left(\pm\frac{1}{\sqrt{1+b^2}}, \frac{\lvert b\rvert}{\sqrt{1+b^2}}\right)$ if k is odd
β_5'	$\widehat{H}_{4k}\left(\frac{1-(a^2+b^2)}{(1+\lvert a\rvert)^2+b^2}, \frac{2\lvert b\rvert}{(1+\lvert a\rvert)^2+b^2}\right)$

Table 1: Real-congruences between the canonical blocks in Theorem 7.1 and those in Theorem 4.1. The sizes of the blocks in Theorem 7.1 are those indicated in (14)–(21). The parameter α in α_3' satisfies $\alpha > 0$ and a, b in β_4' and β_5' satisfy $a \neq 0 \neq b$.

Proof. To see this, we compute the real-KCF of the pair $((m_3')^\top, m_3')$:

$$((m_3')^\top, m_3') = \left(\begin{bmatrix} 0 & (L_k^+)^\top \\ L_k^\top & 0 \end{bmatrix}, \begin{bmatrix} 0 & L_k \\ L_k^+ & 0 \end{bmatrix} \right) \overset{r}{\approx} ((L_k^+)^\top, L_k) \oplus (L_k^\top, L_k^+).$$

Thus, real-KCF$((m_3')^\top, m_3')$ = real-KCF$((L_k^+)^\top, L_k) \oplus$ real-KCF$(L_k^\top, L_k^+)$. We first compute real-KCF$(L_k^\top, L_k^+)$. For this purpose, we consider the pencil in the variable λ

$$\lambda L_k^\top + L_k^+ = \begin{bmatrix} 1+\lambda & \lambda - 1 & & \\ & \ddots & \ddots & \\ & & 1+\lambda & \lambda - 1 \end{bmatrix}_{(k+1)\times k}.$$

This pencil has normal rank k and has no finite or infinite eigenvalues, since the rank is k when λ is replaced by any number and the rank of $L_k^\top$ is also k. Moreover, it has no left minimal indices and has only one right minimal index. By the Index Sum Theorem [3, Theorem 6.5], this right minimal index must be k. Therefore, real-KCF$(L_k^\top, L_k^+) = (F_k, G_k)$. A similar argument proves that real-KCF$((L_k^+)^\top, L_k) = (F_k^\top, G_k^\top)$. Therefore, real-KCF$((m_3')^\top, m_3') = (F_k, G_k) \oplus (F_k^\top, G_k^\top)$ and the result follows from the first row in the table of Theorem 4.2. □

Lemma 7.5. *The matrix* ∞_4' *in Table* 1 *is real-congruent to* $\Gamma_k \otimes \left(\varepsilon\,(-1)^{\frac{k}{2}+1}\right)$.

Proof. Taking into account that k is even note that

$$\infty_4' = \varepsilon \left[\begin{array}{cccc|cccc} & & & & & & & 1 \\ & & & & & & 1 & 1 \\ & & & & & \iddots & \iddots & \\ & & & & 1 & 1 & & \\ \hline & & & -1 & 1 & & & \\ & & -1 & 1 & & & & \\ & \iddots & \iddots & & & & & \\ -1 & 1 & & & & & & \end{array} \right]_{k\times k}.$$

Now, if we set $S := \operatorname{diag}((-1)^{k/2}, (-1)^{k/2+1}, (-1)^{k/2}, (-1)^{k/2+1}, \ldots)_{k/2\times k/2} \oplus I_{k/2}$, then $S\infty_4' S^\top = \varepsilon\,(-1)^{\frac{k}{2}+1}\Gamma_k$ (namely, we just change the sign of the first $k/2$ rows and columns with odd or even indices, depending on the parity of $k/2$). □

Lemma 7.6. *The matrix* ∞_5' *in Table* 1 *is real-congruent to* $\Gamma_k \otimes C(0,1)$.

Proof. Note that

$$\infty_5' = \left[\begin{array}{cccc|cccc} & & & & & & & 1 \\ & & & & & & 1 & 1 \\ & & & & & \iddots & \iddots & \\ & & & & 1 & 1 & & \\ \hline & & & -1 & & & & \\ & & -1 & 1 & & & & \\ & \iddots & \iddots & & & & & \\ -1 & 1 & & & & & & \end{array} \right]_{2k\times 2k}.$$

Taking into account that k is odd define $S := \operatorname{diag}(1,-1,1,-1,\ldots,1,-1,1)_{k\times k}\oplus I_k$. Then

$$S\infty'_5 S^\top = \left[\begin{array}{cccccc|cccccc} &&&&&&&&&&&1\\ &&&&&&&&&&-1&-1\\ &&&&&&&&&\iddots&\iddots&\\ &&&&&&&&-1&-1&&\\ &&&&&&1&1&&&&\\ \hline &&&&&-1&&&&&&\\ &&&&1&1&&&&&&\\ &&&-1&-1&&&&&&&\\ &&\iddots&\iddots&&&&&&&&\\ &1&1&&&&&&&&&\\ -1&-1&&&&&&&&&&\end{array}\right]_{2k\times 2k} = C(0,1)\otimes\Gamma_k,$$

and the result follows from Lemma 2.10. ☐

Lemma 7.7. *The matrix o'_3 in Table 1 is real-congruent to $\Gamma_k\otimes\left(\varepsilon\,(-1)^{\frac{k-1}{2}}\right)$.*

Proof. Note that

$$o'_3 = \varepsilon\left[\begin{array}{ccccc|ccccc} &&&&&&&&&1\\ &&&&&&&&1&1\\ &&&&&&&\iddots&\iddots&\\ &&&&&1&1&&&\\ &&&&1&1&&&&\\ \hline &&&1&-1&&&&&\\ &&1&-1&&&&&&\\ &\iddots&\iddots&&&&&&&\\ 1&-1&&&&&&&&\end{array}\right]_{k\times k}.$$

Taking into account that k is odd, if

$$S := \operatorname{diag}((-1)^{(k-1)/2},(-1)^{(k+1)/2},(-1)^{(k-1)/2},(-1)^{(k+1)/2},\ldots)_{\frac{k-1}{2}\times\frac{k-1}{2}}\oplus I_{\frac{k+1}{2}},$$

then $So'_3S^\top = \varepsilon\,(-1)^{\frac{k-1}{2}}\Gamma_k$. ☐

Lemma 7.8. *The matrix o'_4 in Table 1 is real-congruent to $\Gamma_k\otimes C(0,1)$.*

Proof. Note that

$$o'_4 = \left[\begin{array}{cccc|cccc} &&&&&&&1\\ &&&&&&1&1\\ &&&&&\iddots&\iddots&\\ &&&&1&1&&\\ \hline &&&1&&&&\\ &&1&-1&&&&\\ &\iddots&\iddots&&&&&\\ 1&-1&&&&&&\end{array}\right]_{2k\times 2k}.$$

Proceeding as for ∞_5' but with k even, if $S := \operatorname{diag}(-1,1,\dots,-1,1)_{k\times k} \oplus I_k$, then

$$
So_4'S^\top = \left[\begin{array}{c|c}
 & \begin{array}{ccccc} & & & & -1 \\ & & & 1 & 1 \\ & & \iddots & \iddots & \\ & -1 & -1 & & \\ 1 & 1 & & & \end{array} \\ \hline
\begin{array}{cccccc} & & & & & 1 \\ & & & & -1 & -1 \\ & & & 1 & 1 & \\ & & \iddots & \iddots & & \\ & 1 & 1 & & & \\ -1 & -1 & & & & \end{array} &
\end{array}\right]_{2k\times 2k} = C(0,1)\otimes \Gamma_k,
$$

and the result follows from Lemma 2.10. $\square$

Lemma 7.9. *The matrix α_3' in Table 1 is real-congruent to $J_{2k}(0)$ if $\alpha = 1$ and to $H_{2k}\left(\frac{1-\alpha}{1+\alpha}\right)$ if $\alpha \neq 1$.*

Proof. We prove this result by computing the real-KCF$((\alpha_3')^\top, \alpha_3')$. To this purpose, note first that $\Delta_k((\alpha+1)\Delta_k + \Lambda_k) = J_k(\alpha+1)$ and $\Delta_k((-\alpha+1)\Delta_k - \Lambda_k) = -J_k(\alpha-1)$. Taking into account that $J_k(\alpha+1)$ is invertible (because $\alpha > 0$), we proceed as follows:

$$
\begin{aligned}
((\alpha_3')^\top, \alpha_3') &= \left(\begin{bmatrix} 0 & (-\alpha+1)\Delta_k - \Lambda_k \\ (\alpha+1)\Delta_k + \Lambda_k & 0 \end{bmatrix}, \begin{bmatrix} 0 & (\alpha+1)\Delta_k + \Lambda_k \\ (-\alpha+1)\Delta_k - \Lambda_k & 0 \end{bmatrix} \right) \\
&\approx \left(\begin{bmatrix} 0 & -J_k(\alpha-1) \\ J_k(\alpha+1) & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_k(\alpha+1) \\ -J_k(\alpha-1) & 0 \end{bmatrix} \right) \\
&\approx \left(\begin{bmatrix} -J_k(\alpha+1)^{-1}J_k(\alpha-1) & 0 \\ 0 & I_k \end{bmatrix}, \begin{bmatrix} I_k & 0 \\ 0 & -J_k(\alpha+1)^{-1}J_k(\alpha-1) \end{bmatrix} \right).
\end{aligned}
$$

The Jordan canonical form of the matrix $C = -J_k(\alpha+1)^{-1}J_k(\alpha-1)$ is $J_k\left(\frac{1-\alpha}{1+\alpha}\right)$, because C is upper triangular with all diagonal entries equal to $\frac{1-\alpha}{1+\alpha}$. Hence, C has only one eigenvalue equal to $\frac{1-\alpha}{1+\alpha}$ with algebraic multiplicity k. Moreover, its geometric multiplicity is 1 because

$$
\begin{aligned}
\operatorname{rank}\left(C - \frac{1-\alpha}{1+\alpha}I_k\right) &= \operatorname{rank}\left(J_k(\alpha-1) + \frac{1-\alpha}{1+\alpha}J_k(\alpha+1)\right) \\
&= \operatorname{rank}\left((1+\alpha)J_k(\alpha-1) + (1-\alpha)J_k(\alpha+1)\right) \\
&= \operatorname{rank}\begin{bmatrix} 0 & 2 & & \\ & \ddots & \ddots & \\ & & \ddots & 2 \\ & & & 0 \end{bmatrix} = k-1.
\end{aligned}
$$

Thus,

$$
((\alpha_3')^\top, \alpha_3') = \left(J_k\left(\frac{1-\alpha}{1+\alpha}\right), I_k \right) \oplus \left(I_k, J_k\left(\frac{1-\alpha}{1+\alpha}\right) \right).
$$

The result follows from the table in Theorem 4.2. Namely, from its first row for $\alpha = 1$ and from its third row for $\alpha \neq 1$. $\square$

We will need Lemma 7.10 in the proofs of Lemmas 7.11 and 7.12. In the statement of Lemma 7.10, $\mathcal{N}(A)$ and $\mathcal{C}(A)$ denote the null and column space of a matrix A, respectively.

Lemma 7.10. *Let $A, B \in \mathbb{C}^{n\times n}$, with B invertible, $k \geq 2$ be an integer, and*

$$Z_k(A,B) := \begin{bmatrix} & & & & A \\ & & & \iddots & B \\ & & \iddots & \iddots & \\ & A & B & & \\ A & B & & & \end{bmatrix} \in \mathbb{C}^{nk\times nk}.$$

Then:

1. $\dim \mathcal{N}(Z_k(A,B)) = \dim \mathcal{N}(A)$ *if and only if*
$$\mathcal{N}(Z_k(A,B)) = \{[x_1^\top, 0, \ldots, 0]^\top \in \mathbb{C}^{nk} \, : \, x_1 \in \mathcal{N}(A)\}.$$
2. $\dim \mathcal{N}(Z_k(A,B)) = \dim \mathcal{N}(A)$ *if and only if* $\mathcal{N}(A) \cap \mathcal{C}(B^{-1}A) = \{0\}$.

Proof. It is clear that the following inclusion and equalities hold

$$\{[x_1^\top, 0, \ldots, 0]^\top \in \mathbb{C}^{nk} \, : \, x_1 \in \mathcal{N}(A)\} \subseteq \mathcal{N}(Z_k(A,B)), \tag{22}$$

$$\dim\{[x_1^\top, 0, \ldots, 0]^\top \in \mathbb{C}^{nk} \, : \, x_1 \in \mathcal{N}(A)\} = \dim \mathcal{N}(A), \tag{23}$$

$$\mathcal{N}(Z_k(A,B)) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \\ x_k \end{bmatrix} : x_i \in \mathbb{C}^n \text{ and } \begin{array}{ll} Ax_k & = 0, \\ x_k & = -B^{-1}A\,x_{k-1}, \\ \vdots & \vdots \\ x_3 & = -B^{-1}A\,x_2, \\ x_2 & = -B^{-1}A\,x_1. \end{array} \right\}. \tag{24}$$

Proof of part 1. If $\dim \mathcal{N}(Z_k(A,B)) = \dim \mathcal{N}(A)$, then (22) and (23) together imply that $\{[x_1^\top, 0, \ldots, 0]^\top \in \mathbb{C}^{nk} \, : \, x_1 \in \mathcal{N}(A)\} = \mathcal{N}(Z_k(A,B))$. The converse follows from (23).

Proof of part 2. Assume first that $\dim \mathcal{N}(Z_k(A,B)) = \dim \mathcal{N}(A)$. If $y \in \mathcal{N}(A) \cap \mathcal{C}(B^{-1}A)$, then $Ay = 0$ and $y = -B^{-1}A\,x$ for some $x \in \mathbb{C}^n$, which implies $[x^\top, y^\top, 0, \ldots, 0]^\top \in \mathcal{N}(Z_k(A,B))$ by (24). This, in turn, implies $y = 0$, by part 1, and $\mathcal{N}(A) \cap \mathcal{C}(B^{-1}A) = \{0\}$.

Conversely, assume $\mathcal{N}(A) \cap \mathcal{C}(B^{-1}A) = \{0\}$. If $[x_1^\top, x_2^\top, \ldots, x_k^\top]^\top \in \mathcal{N}(Z_k(A,B))$, where $x_i \in \mathbb{C}^n$, then (24) implies $x_k \in \mathcal{N}(A) \cap \mathcal{C}(B^{-1}A)$. So $x_k = 0$ and again from (24), $x_{k-1} \in \mathcal{N}(A) \cap \mathcal{C}(B^{-1}A)$, which yields $x_{k-1} = 0$. The remaining equalities in (24) give $x_2 = x_3 = \cdots = x_k = 0$ and $Ax_1 = 0$, i.e., $\mathcal{N}(Z_k(A,B)) = \{[x_1^\top, 0, \ldots, 0]^\top \in \mathbb{C}^{nk} \, : \, x_1 \in \mathcal{N}(A)\}$. The result follows from part 1. □

Lemma 7.11. *The matrix β_4' in Table 1 is real-congruent to*

$$\Gamma_k \otimes C\left(\pm\frac{|b|}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+b^2}}\right) \qquad \textit{if } k \textit{ is even,}$$

$$\Gamma_k \otimes C\left(\pm\frac{1}{\sqrt{1+b^2}}, \frac{|b|}{\sqrt{1+b^2}}\right) \qquad \textit{if } k \textit{ is odd.}$$

Proof. Since β_4' in (20) only depends on $|b|$, we assume throughout the proof that $b = |b| > 0$. The proof proceeds by computing the real-KCF of the pair $((\beta_4')^\top, \beta_4')$ and applying Theorem 4.2. For this purpose, we consider the pencil

$$\lambda\left(\beta_4'\right)^\top + \beta_4' = \varepsilon \begin{bmatrix} & & & \lambda R^\top + R \\ & & \lambda R^\top + R & (\lambda-1)S^\top \\ & \iddots & \iddots & \\ \lambda R^\top + R & (\lambda-1)S^\top & & \end{bmatrix}_{2k\times 2k} \tag{25}$$

in the variable λ. Since R is invertible for any $b > 0$, the pencil $\lambda\left(\beta_4'\right)^\top + \beta_4'$ has no minimal indices nor eigenvalues at infinity. So, its KCF is determined by the Jordan blocks associated to its finite eigenvalues. The finite eigenvalues of $\lambda\left(\beta_4'\right)^\top + \beta_4'$ are the two roots of the polynomial

$$\det(\lambda R^\top + R) = (\lambda+1)^2 + b^2(\lambda-1)^2 = (\lambda + 1 + \mathrm{i}b(\lambda-1))(\lambda + 1 - \mathrm{i}b(\lambda-1))$$

each with algebraic multiplicity k. These roots are

$$\lambda_1 = \frac{b+\mathrm{i}}{b-\mathrm{i}} = \frac{b^2 - 1 + \mathrm{i}\,2b}{1+b^2} \quad \text{and} \quad \lambda_2 = \overline{\lambda_1},$$

which are conjugate and different from each other, because $b > 0$, and $|\lambda_1| = |\lambda_2| = 1$. Since β_4' is real, the Jordan blocks associated to λ_1 and λ_2 in $\mathrm{KCF}((\beta_4')^\top, \beta_4')$ are paired-up. Then, it suffices to determine the Jordan blocks associated to λ_1. For this, we prove that the geometric multiplicity of λ_1 is 1, i.e., $\dim \mathcal{N}(\lambda_1\left(\beta_4'\right)^\top + \beta_4') = 1$, via Lemma 7.10-part 2 with

$$A = \lambda_1 R^\top + R = \frac{2b(b+\mathrm{i})}{1+b^2}\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \quad \text{and} \quad B = (\lambda_1 - 1)S^\top = \frac{2(\mathrm{i}b-1)}{1+b^2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The matrix B is nonsingular,

$$B^{-1}A = b\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{C}(B^{-1}A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix}\right\}.$$

Therefore, $\mathcal{N}(A) \cap \mathcal{C}(B^{-1}A) = \{0\}$ and $\dim \mathcal{N}(\lambda_1\left(\beta_4'\right)^\top + \beta_4') = \dim \mathcal{N}(\lambda_1 R^\top + R) = 1$, by Lemma 7.10-part 2. As a consequence, the $\mathrm{KCF}((\beta_4')^\top, \beta_4')$ has only one Jordan block $J_k(-\lambda_1)$ associated with λ_1. This implies

$$((\beta_4')^\top, \beta_4') \approx (I_k, J_k(-\lambda_1)) \oplus (I_k, J_k(-\overline{\lambda_1})) \approx \left(I_{2k}\,,\, C_{2k}\left(\frac{1-b^2}{1+b^2}\,,\, \frac{2b}{1+b^2}\right)\right),$$

where the last strict equivalence follows from Lemma 2.6. Lemma 2.4 guarantees that the expression above is real-$\mathrm{KCF}((\beta_4')^\top, \beta_4')$. Finally, the result in the statement follows from the third case in the second row of the table in Theorem 4.2 (recall also Remark 4.3). □

Lemma 7.12. *The matrix β_5' in Table 1 is real-congruent to*

$$\widehat{H}_{4k}\left(\frac{1-(a^2+b^2)}{(1+|a|)^2 + b^2}\,,\, \frac{2|b|}{(1+|a|)^2+b^2}\right).$$

Proof. The proof proceeds again by first computing the real-KCF of $((\beta_5')^\top, \beta_5')$ and, then, by applying Theorem 4.2. For simplicity, let us express β_5' in (21) as

$$\beta_5' = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}, \qquad X, Y \in \mathbb{R}^{2k\times 2k}. \tag{26}$$

Observe that X and Y are symmetric and nonsingular, since $b \neq 0$. Thus,

$$\begin{aligned}((\beta_5')^\top, \beta_5') &= \left(\begin{bmatrix} 0 & Y \\ X & 0 \end{bmatrix}, \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right) \approx \left(\begin{bmatrix} Y & 0 \\ 0 & X \end{bmatrix}, \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) \\ &\approx \left(\begin{bmatrix} I_{2k} & 0 \\ 0 & I_{2k} \end{bmatrix}, \begin{bmatrix} Y^{-1}X & 0 \\ 0 & (Y^{-1}X)^{-1} \end{bmatrix} \right).\end{aligned}$$

Thus

$$\mathrm{KCF}((\beta_5')^\top, \beta_5') = \mathrm{KCF}(Y, X) \oplus \mathrm{KCF}(X, Y), \tag{27}$$

and $\mathrm{KCF}(X, Y)$ can be obtained from $\mathrm{KCF}(Y, X)$ changing each involved Jordan block $J_\ell(\mu)$ by $J_\ell(1/\mu)$. Therefore, we focus on computing $\mathrm{KCF}(Y, X)$. For this purpose, we follow an approach similar to that in Lemma 7.11 for computing $\mathrm{KCF}((\beta_4')^\top, \beta_4')$. We consider the pencil

$$\lambda Y + X = \begin{bmatrix} & & & R' - \lambda T \\ & & R' - \lambda T & (1-\lambda)S' \\ & \iddots & \iddots & \\ R' - \lambda T & (1-\lambda)S' & & \end{bmatrix} \tag{28}$$

in the variable λ. This pencil has no minimal indices nor eigenvalues at infinity because R' and T are invertible. This implies that its KCF is determined by the Jordan blocks associated to its finite eigenvalues, which are the roots of the polynomial

$$\begin{aligned}\det(R' - \lambda T) &= -\left(b^2(\lambda-1)^2 + (\lambda(1-a) + (a+1))^2 \right) \\ &= -\left(\lambda(1-a) + (a+1) + \mathrm{i}b(\lambda-1)\right) \left(\lambda(1-a) + (a+1) - \mathrm{i}b(\lambda-1)\right),\end{aligned}$$

each with algebraic multiplicity k. These roots are

$$\lambda_1 = \frac{a+1-\mathrm{i}b}{a-1-\mathrm{i}b} = \frac{a^2+b^2-1+\mathrm{i}\,2b}{(a-1)^2+b^2} \qquad \text{and} \qquad \lambda_2 = \overline{\lambda_1}$$

and are complex conjugate to each other, because $b \neq 0$, and $|\lambda_1| = |\lambda_2| \neq 1$, because, $a \neq 0$. Since Y and X are real, the Jordan blocks associated to λ_1 and λ_2 in $\mathrm{KCF}(Y, X)$ are paired-up. So, it suffices to determine the Jordan blocks associated to λ_1. For this, we prove that the geometric multiplicity of λ_1 is 1, i.e., $\dim \mathcal{N}(\lambda_1 Y + X) = 1$, via Lemma 7.10-part 2 with

$$A = R' - \lambda_1 T = -\frac{2b}{a-1-\mathrm{i}b}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{bmatrix} \qquad \text{and} \qquad B = (1-\lambda_1)S'.$$

The matrix B is nonsingular,

$$B^{-1}A = -b\begin{bmatrix} -\mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{bmatrix} \qquad \text{and} \qquad \mathcal{C}(B^{-1}A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix} \right\}.$$

Therefore, $\mathcal{N}(A) \cap \mathcal{C}(B^{-1}A) = \{0\}$ and $\dim \mathcal{N}(\lambda_1 Y + X) = \dim \mathcal{N}(R' - \lambda_1 T) = 1$, by Lemma 7.10-part 2. As a consequence, the $\mathrm{KCF}(Y, X)$ has only one Jordan block $J_k(-\lambda_1)$ associated with λ_1. This implies $\mathrm{KCF}(Y,X) = (I_k, J_k(-\lambda_1)) \oplus (I_k, J_k(-\overline{\lambda_1}))$. Combining this with (27) and Lemma 2.6, we get

$$
\begin{aligned}
((\beta_5')^\top, \beta_5') &\approx (I_k, J_k(-\lambda_1)) \oplus (I_k, J_k(-\overline{\lambda_1})) \oplus (I_k, J_k(-1/\lambda_1)) \oplus \left(I_k, J_k\left(-1/\overline{\lambda_1}\right)\right) \\
&\approx (I_{2k}\,,\, C_{2k}\left(\frac{1-(a^2+b^2)}{(a-1)^2+b^2}\,,\, \frac{2\,|b|}{(a-1)^2+b^2}\right)) \\
&\quad \oplus (I_{2k}\,,\, C_{2k}\left(\frac{1-(a^2+b^2)}{(a+1)^2+b^2}\,,\, \frac{2\,|b|}{(a+1)^2+b^2}\right)).
\end{aligned}
$$

Lemma 2.4 guarantees that the expression above is the real-KCF$((\beta_5')^\top, \beta_5')$. The result follows from the last row in the table of Theorem 4.2, taking into account that from the two $C_{2k}(c,d)$ above, we choose the parameters c and d such that $c^2 + d^2 < 1$, i.e., the ones with largest denominator. $\square$

Appendices

A Proof of Lemma 3.1

Proof. We will first see that the nonzero entries of $(P_kS_k)^\top \Gamma_k (P_kS_k)$ and $\widetilde{\Gamma}_k$ are placed in the same positions. For this, first notice that the nonzero entries of Γ_k are just 1 or -1, and they are placed in the following positions:

- $(k-i+1, i)$, for $i = 1, \ldots, k$ (placed in the main anti-diagonal).
- $(k-i+2, i)$, for $i = 2, \ldots, k$ (placed below the main anti-diagonal).

These entries 1 or -1 are also the nonzero entries of $(P_kS_k)^\top \Gamma_k (P_kS_k)$, since P_k is a permutation matrix and S_k is just a change of signs matrix, but they are placed in different positions. Now, we are going to identify the positions of these nonzero entries in $(P_kS_k)^\top \Gamma_k (P_kS_k)$. We analyze separately the cases k even and k odd.

► k even: Let us first identify the positions of the nonzero entries in the matrix $P_k^\top \Gamma_k P_k$.

The permutation corresponding to P_k, described in the statement, acts as follows on the ith column-index:

$$
i \quad \longmapsto \quad \begin{cases} k-2i+2 & \text{for } i = 1, \ldots, \frac{k}{2}, \\ 2i-k-1 & \text{for } i = \frac{k}{2}+1, \ldots, k. \end{cases}
$$

As a consquence, when applying the permutation to the indices corresponding to the nonzero entries of Γ_k, these entries go to the following positions:

$$
(k-i+1, i) \quad \longmapsto \quad \begin{cases} (A): \ (k-2i+1, k-2i+2) & \text{for } i = 1, \ldots, \frac{k}{2}, \\ (B): \ (2i-k, 2i-k-1) & \text{for } i = \frac{k}{2}+1, \ldots, k. \end{cases} \tag{29}
$$

$$
(k-i+2, i) \quad \longmapsto \quad \begin{cases} (C): \ (k-2i+3, k-2i+2) & \text{for } i = 2, \ldots, \frac{k}{2}, \\ (E): \ (1,1) & \text{for } i = \frac{k}{2}+1, \\ (D): \ (2i-k-2, 2i-k-1) & \text{for } i = \frac{k}{2}+2, \ldots, k. \end{cases} \tag{30}
$$

The indices (A) and (D) above altogether run through all the entries in the upper diagonal, namely those with indices $(i, i+1)$, for $i = 1, \ldots, k-1$. Similarly, the indices (B) and (C)

correspond to all the entries in the lower diagonal, namely those with indices $(i+1,i)$, for $i=1,\dots,k-1$. These entries, together with $(E)=(1,1)$ are all the nonzero entries of the matrix $P_k^\top \Gamma_k P_k$. Note that these are, precisely, the same positions which contain all the nonzero entries of $\widetilde{\Gamma}_k$.

Now, let us identify the positions with entries -1 in $(P_kS_k)^\top \Gamma_k (P_kS_k)$. Note first that the entries equal to -1 in Γ_k are placed in the positions $(k-2i+1,2i)$, for $i=1,\dots,k/2$ (those in the main anti-diagonal), and $(k-2i+1,2i+1)$, for $i=1,\dots,(k/2)-1$ (those below the main anti-diagonal). Now, we will keep track of these entries when applying the permutation corresponding to P_k. For this, we distinguish the cases $k\equiv 0 \pmod 4$ and $k\equiv 2 \pmod 4$.

- $k\equiv 0 \pmod 4$: Replacing i by $2i$ in (29) and by $2i+1$ in (30), the permutation corresponding to P_k acts as follows on the previous indices:

$$\begin{array}{lcl} (k-2i+1,2i) & \longmapsto & \left\{\begin{array}{ll} (A):\ (k-4i+1,k-4i+2) & \text{for } i=1,\dots,\frac{k}{4},\\ (B):\ (4i-k,4i-k-1) & \text{for } i=\frac{k}{4}+1,\dots,\frac{k}{2}. \end{array}\right. \\ (k-2i+1,2i+1) & \longmapsto & \left\{\begin{array}{ll} (C):\ (k-4i+1,k-4i) & \text{for } i=1,\dots,\frac{k}{4}-1,\\ (E):\ (1,1) & \text{for } i=\frac{k}{4},\\ (D):\ (4i-k,4i-k+1) & \text{for } i=\frac{k}{4}+1,\dots,\frac{k}{2}-1. \end{array}\right. \end{array}$$

  The first indices in $(A),(C)$, and (E) above altogether are $1,5,\dots,k-3$, namely, all indices $i\equiv 1 \pmod 4$ between 1 and k. Therefore, the corresponding entries are all the nonzero entries of $P_k^\top \Gamma_k P_k$ in these rows (two entries per row).

  Similarly, the first indices in (B) and (D) above altogether are $4,8,\dots,k$ (the last one appearing only once in (B) for $i=k/2$). Therefore, the positions (B) and (C) correspond to all the nonzero entries in $P_k^\top \Gamma_k P_k$ in rows with indices $i\equiv 0 \pmod 4$.

  Summarizing, all the entries equal to -1 in the matrix $P_k^\top \Gamma_k P_k$ are in the rows with indices $i\equiv 0,1 \pmod 4$, and all the nonzero entries in these rows are equal to -1.

  The change of signs matrix S_k changes the sign of all rows and columns with indices $i\equiv 3 \pmod 4$. The change of sign in the rows turns into -1 all entries $(i,i-1)$ and $(i,i+1)$ with $i\equiv 3 \pmod 4$, and the change of sign in the columns changes the signs of the entries $(i+1,i)$ and $(i-1,i)$ with $i\equiv 3 \pmod 4$, namely $i+1\equiv 0 \pmod 4$ and $i-1\equiv 2 \pmod 4$. Therefore, in $(P_kS_k)^\top \Gamma_k (P_kS_k)=S_k^\top (P_k^\top \Gamma_k P_k) S_k$ all entries in the upper diagonal, namely those with indices $(i,i+1)$, are equal to -1, whereas only the entries in the positions $(i,i-1)$ with $i\equiv 1,3 \pmod 4$ are equal to -1 (the ones in the positions $(i,i-1)$ with $i\equiv 0,2 \pmod 4$ are all equal to 1 instead). In other words, $(P_kS_k)^\top \Gamma_k (P_kS_k)=-\widetilde{\Gamma}_k$, as claimed.

- $k\equiv 2 \pmod 4$: Replacing, again, i by $2i$ in (29) and by $2i+1$ in (30), the permutation P_k acts on the indices corresponding to the negative entries of Γ_k in a similar way, though the range of the indices now is slightly different, namely:

$$\begin{array}{lcl} (k-2i+1,2i) & \longmapsto & \left\{\begin{array}{ll} (A):\ (k-4i+1,k-4i+2) & \text{for } i=1,\dots,\frac{k-2}{4},\\ (B):\ (4i-k,4i-k-1) & \text{for } i=\frac{k+2}{4},\dots,\frac{k}{2}. \end{array}\right. \\ (k-2i+1,2i+1) & \longmapsto & \left\{\begin{array}{ll} (C):\ (k-4i+1,k-4i) & \text{for } i=1,\dots,\frac{k-2}{4},\\ (D):\ (4i-k,4i-k+1) & \text{for } i=\frac{k+2}{4},\dots,\frac{k}{2}-1. \end{array}\right. \end{array}$$

  Now, the indices in (A) and (C) correspond to the two nonzero entries in the rows $3,7,\dots,k-3$, namely all rows with indices $i\equiv 3 \pmod 4$. Similarly, the entries in (B) and (D) correspond to the nonzero entries in the rows $2,6,\dots,k$ (which are two entries per row, except in row k where there is only one) , namely those rows with indices $i \equiv 2 \pmod 4$. Summarizing, the -1 entries in $P_k^\top \Gamma_k P_k$ are those in rows with indices $i \equiv 2, 3 \pmod 4$.

When we multiply on the left and on the right the matrix $P_k^\top \Gamma_k P_k$ by S_k, we introduce a change of sign in the rows and columns with indices $i \equiv 3 \pmod 4$. The change of sign in the rows turn the -1 into 1 in the entries placed in all rows with indices $i \equiv 3 \pmod 4$. Now, the change of sign in the columns changes the sign of the entries with indices $(i-1, i)$ and $(i+1, i)$, with $i \equiv 3 \pmod 4$, namely $i - 1 \equiv 2 \pmod 4$ and $i + 1 \equiv 0 \pmod 4$. Therefore the entries of $(P_k S_k)^\top \Gamma_k (P_k S_k) = S_k^\top (P_k^\top \Gamma_k P_k) S_k$ in the positions $(i, i-1)$, for $i \equiv 0, 2 \pmod 4$, are -1, and these are the only entries which are equal to -1 in this matrix. In other words, $(P_k S_k)^\top \Gamma_k (P_k S_k) = \widetilde{\Gamma}_k$, as claimed.

► k odd: In this case, the permutation corresponding to P_k acts as follows on the ith column-index:

$$i \quad \longmapsto \quad \begin{cases} k - 2i + 2 & \text{for } i = 1, \ldots, \frac{k+1}{2}, \\ 2i - k - 1 & \text{for } i = \frac{k+3}{2}, \ldots, k. \end{cases}$$

As a consequence, the permutation applied to the matrix Γ_k acts as follows on the positions corresponding to the nonzero entries of Γ_k:

$$(k-i+1, i) \quad \longmapsto \quad \begin{cases} (A): \ (k-2i+1, k-2i+2) & \text{for } i = 1, \ldots, \frac{k-1}{2}, \\ (E): \ (1,1) & \text{for } i = \frac{k+1}{2}, \\ (B): \ (2i-k, 2i-k-1) & \text{for } i = \frac{k+3}{2}, \ldots, k. \end{cases} \tag{31}$$

$$(k-i+2, i) \quad \longmapsto \quad \begin{cases} (C): \ (k-2i+3, k-2i+2) & \text{for } i = 2, \ldots, \frac{k+1}{2}, \\ (D): \ (2i-k-2, 2i-k-1) & \text{for } i = \frac{k+3}{2}, \ldots, k. \end{cases} \tag{32}$$

The indices (A) and (D) altogether correspond to the entries in the upper diagonal of $P_k^\top \Gamma_k P_k$, namely those with indices $(i, i+1)$, for $i = 1, \ldots, k-1$. Similarly, the indices (B) and (C) together correspond to $(i, i-1)$, for $i = 2, \ldots, k$, i.e., to the entries in the lower diagonal of $P_k^\top \Gamma_k P_k$. These indices, together with $(E) = (1,1)$ correspond to the nonzero entries of $P_k^\top \Gamma_k P_k$, as well as those of $\widetilde{\Gamma}_k$.

As in the case k even, we are now going to identify the positions of the -1 entries in $(P_k S_k)^\top \Gamma_k (P_k S_k)$. Starting again from the entries equal to -1 in Γ_k, which are placed in the positions $(k - 2i + 1, 2i)$ and $(k - 2i + 1, 2i + 1)$, for $i = 1, \ldots, (k-1)/2$, we keep track of these entries after applying the permutation corresponding to the matrix P_k and then applying the change of signs corresponding to S_k. We analyze separately the cases $k \equiv 1 \pmod 4$ and $k \equiv 3 \pmod 4$.

- $k \equiv 1 \pmod 4$: Replacing i by $2i$ in (31) and by $2i+1$ in (32), the permutation corresponding to P_k acts as follows on the previous indices:

$$(k-2i+1, 2i) \quad \longmapsto \quad \begin{cases} (A): \ (k-4i+1, k-4i+2) & \text{for } i = 1, \ldots, \frac{k-1}{4}, \\ (B): \ (4i-k, 4i-k-1) & \text{for } i = \frac{k+3}{4}, \ldots, \frac{k-1}{2}. \end{cases}$$

$$(k-2i+1, 2i+1) \quad \longmapsto \quad \begin{cases} (C): \ (k-4i+1, k-4i) & \text{for } i = 1, \ldots, \frac{k-1}{4}, \\ (D): \ (4i-k, 4i-k+1) & \text{for } i = \frac{k+3}{4}, \ldots, \frac{k-1}{2}. \end{cases}$$

These are all nonzero entries of the matrix $P_k^\top \Gamma_k P_k$ in the following rows: the entries (A) and (C) together are in the rows $2, 6, \ldots, k-3$, namely all rows with indices $i \equiv 2 \pmod 4$, whereas the entries (B) and (D) together are the nonzero entries in the rows $3, 7, \ldots, k-2$, namely the rows with indices $i \equiv 3 \pmod 4$. Therefore, the entries $(A)-(D)$ correspond to the nonzero entries of $P_k^\top \Gamma_k P_k$ in the rows with indices $i \equiv 2, 3 \pmod 4$. Therefore, we are in the same situation as for $k \equiv 2 \pmod 4$, so $(P_k S_k)^\top \Gamma_k (P_k S_k) = \widetilde{\Gamma}_k$.

- $k \equiv 3 \pmod 4$: Replacing i by $2i$ in (31) and by $2i+1$ in (32), this time we have:

$$\begin{array}{lcl} (k-2i+1, 2i) & \longmapsto & \left\{ \begin{array}{lll} (A): & (k-4i+1, k-4i+2) & \text{for } i = 1, \ldots, \frac{k-3}{4}, \\ (E): & (1,1) & \text{for } i = \frac{k+1}{4}, \\ (B): & (4i-k, 4i-k-1) & \text{for } i = \frac{k+1}{4}+1, \ldots, \frac{k-1}{2}. \end{array} \right. \\ (k-2i+1, 2i+1) & \longmapsto & \left\{ \begin{array}{lll} (C): & (k-4i+1, k-4i) & \text{for } i = 1, \ldots, \frac{k-3}{4}, \\ (D): & (4i-k, 4i-k+1) & \text{for } i = \frac{k+1}{4}, \ldots, \frac{k-1}{2}. \end{array} \right. \end{array}$$

The entries $(A)-(E)$ in this case correspond to all the nonzero entries of $P_k^\top \Gamma_k P_k$ in the rows with indices $i \equiv 0, 1 \pmod 4$. More precisely, the first indices in (A) and (C) together are equal to $4, 8, \ldots, k-3$, and those of (B), (D), and (E) together are equal to $1, 5, \ldots, k-2$ (in particular, the index 1 comes from (E), and from (D) for $i = (k+1)/4$). Therefore, we are in the same situation as in the case $k \equiv 0 \pmod 4$, so, again, $(P_k S_k)^\top \Gamma_k (P_k S_k) = -\widetilde{\Gamma}_k$.

□

B Canonical forms of degenerate blocks β_4' and β_5' in Theorem 7.1

For completeness, the next table presents the canonical forms in Theorem 4.1 of the matrices β_4' and β_5' in (20) and (21) if $a = 0$ or $b = 0$. We omit the proof for brevity.

Degenerate variant of block B in Theorem 7.1	is real-congruent to direct sum of blocks in Theorem 4.1
β_4'	$\Gamma_k \otimes C(0,1)$ if $b=0$ and k even $\left(\Gamma_k \otimes \left(\varepsilon\,(-1)^\ell\right)\right) \oplus \left(\Gamma_k \otimes \left(\varepsilon\,(-1)^\ell\right)\right)$ if $b=0$ and $k=2\ell+1$
β_5'	$(\Gamma_k \otimes C(0,1)) \oplus (\Gamma_k \otimes C(0,1))$ if $b=0$, $a=0$, and k even $\Gamma_k \oplus \Gamma_k \oplus (-\Gamma_k) \oplus (-\Gamma_k)$ if $b=0$, $a=0$, and k odd $J_{2k}(0) \oplus J_{2k}(0)$ if $b=0$, $a=\pm 1$ $H_{2k}\left(\frac{1-\lvert a\rvert}{1+\lvert a\rvert}\right) \oplus H_{2k}\left(\frac{1-\lvert a\rvert}{1+\lvert a\rvert}\right)$ if $b=0$, $a\neq 0$, $a\neq \pm 1$ $(\Gamma_k \otimes C\left(\pm\frac{\lvert b\rvert}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+b^2}}\right)) \oplus (\Gamma_k \otimes C\left(\pm\frac{\lvert b\rvert}{\sqrt{1+b^2}}, \frac{1}{\sqrt{1+b^2}}\right))$ if $b\neq 0$, $a=0$, k even $(\Gamma_k \otimes C\left(\pm\frac{1}{\sqrt{1+b^2}}, \frac{\lvert b\rvert}{\sqrt{1+b^2}}\right)) \oplus (\Gamma_k \otimes C\left(\pm\frac{1}{\sqrt{1+b^2}}, \frac{\lvert b\rvert}{\sqrt{1+b^2}}\right))$ if $b\neq 0$, $a=0$, k odd

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References

[1] F. De Terán. Canonical forms for congruence of matrices and T-palindromic matrix pencils: a tribute to H. W. Turnbull and A. C. Aitken. *SeMA J.*, 73:7–16, 2016.

[2] F. De Terán. A geometric description of the sets of palindromic and alternating matrix pencils with bounded rank. *SIAM J. Matrix Anal. Appl.*, 39:1116–1134, 2018.

[3] F. De Terán, F. M. Dopico, and D. S. Mackey. Spectral equivalence of matrix polynomials and the index sum theorem. *Linear Algebra Appl.*, 459:264–333, 2014.

[4] D. Ž. Doković and K. D. Ikramov. On the congruence of square real matrices. *Linear Algebra Appl.*, 353:149–158, 2002.

[5] V. Futorny, R. A. Horn, and V. V. Sergeichuk. Tridiagonal canonical matrices of bilinear or sesquilinear forms and of pairs of symmetric, skew-symmetric, or Hermitian forms. *J. Algebra*, 319(6):2351–2371, 2008.

[6] P. Gabriel. Appendix: Degenerate bilinear forms. *J. Algebra*, 31:67–72, 1974.

[7] F. R. Gantmacher. *The Theory of Matrices.* Chelsea, New York, 1959.

[8] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Cambridge University Press, New York, 1991.

[9] R. A. Horn and C. R. Johnson. *Matrix Analysis, 2nd Edition.* Cambridge University Press, New York, 2013.

[10] R. A. Horn and V. V. Sergeichuk. Canonical forms for complex matrix congruence and *congruence. *Linear Algebra Appl.*, 416(2-3):1010–1032, 2006.

[11] P. Lancaster and L. Rodman. Canonical forms for Hermitian matrix pairs under strict equivalence and congruence. *SIAM Rev.*, 47(3):407–443, 2005.

[12] J. M. Lee and D. A. Weinberg. A note on canonical forms for matrix congruence. *Linear Algebra Appl.*, 249:207–215, 1996.

[13] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Structured polynomial eigenvalue problems: good vibrations from good linearizations. *SIAM J. Matrix Anal. Appl.*, 28:1029–1051, 2006.

[14] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Smith forms of palindromic matrix polynomials. *Electron. J. Linear Algebra*, 22:53–91, 2011.

[15] C. Riehm. The equivalence of bilinear forms. *J. Algebra*, 31:45–66, 1974.

[16] R. C. Thompson. Pencils of complex and real symmetric and skew matrices. *Linear Algebra Appl.*, 147:323–371, 1991.

[17] H. W. Turnbull and A. C. Aitken. *An Introduction to the Theory of Canonical Matrices.* Dover, New York, 1961. First published by Blackie and Son Ltd. in 1931.