Block Full Rank Linearizations of Rational Matrices

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ABSTRACT

We introduce a new family of linearizations of rational matrices, which we call block full rank linearizations. The theory of block full rank linearizations is useful as it establishes very simple criteria to determine if a pencil is a linearization of a rational matrix in a target set or in the whole underlying field, by using rank conditions. Block full rank linearizations allow us to recover locally information about zeros and poles. To recover the pole-zero information at infinity, we will define the grade of the new block full rank linearizations as linearizations at infinity and the notion of degree of a rational matrix will be used. Moreover, the eigenvectors of a rational matrix associated with its eigenvalues in a target set can be obtained from the eigenvectors of its block full rank linearizations in that set. This new family of linearizations generalizes and includes the structures appearing in most of the linearizations for rational matrices constructed in the literature. As example, we study the structure and properties of the linearizations in [P. Lietaert et al., Automatic rational approximation and linearization of nonlinear eigenvalue problems, 2021].

KEYWORDS
rational matrix, rational eigenvalue problem, nonlinear eigenvalue problem, linearization, polynomial system matrix, rational approximation, block full rank linearization.

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1. Introduction

A rational matrix $R(\lambda)$ is a matrix whose entries are quotients of polynomials in the scalar variable λ, i.e., rational functions. Zeros and poles are among the most interesting quantities attached to a rational matrix [20], both from theoretical and applied points of view. The Smith–McMillan form of rational matrices [23] is the classical way to define their poles and zeros, together with their partial multiplicities. However, it is well-known that the Smith–McMillan form is not convenient for computing

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numerically poles and zeros [32].

The problem of determining the pole and zero structure of rational matrices appears in several applications. Many classic problems in linear systems and control theory can be posed in terms of rational matrices [20,27,34] and are related to the computation of their zeros and poles [32]. Currently, the computation of the zeros of rational matrices is playing a fundamental role in the very active area of Nonlinear Eigenvalue Problems (NLEPs) [17], either because they appear directly in rational eigenvalue problems (REPs) modeling real-life problems [24] or because other NLEPs are approximated by REPs [15,17,18,21,22,29]. Recall that for a regular rational matrix $R(\lambda)$, the REP consists in computing numbers λ_0 (eigenvalues) and nonzero vectors v (eigenvectors) such that λ_0 is not a pole of any of the entries of $R(\lambda)$ and $R(\lambda_0)v = 0$. This is equivalent to say that λ_0 is a zero of $R(\lambda)$ but not a pole.

One of the most reliable methods for computing the zeros and poles of a rational matrix $R(\lambda)$ is via linearizations. This approach is based on constructing a matrix pencil $L(\lambda)$, i.e., a matrix polynomial of degree at most 1, containing the pole and zero information of $R(\lambda)$ and then applying to $L(\lambda)$ backward stable eigenvalue algorithms, as [25,31] for problems of moderate size, or Krylov methods adapted to the structure of $L(\lambda)$ in the large-scale setting [10,18]. The pencil $L(\lambda)$ is called a linearization of $R(\lambda)$. This classical approach was introduced at least in the late 1970s [32,35] and has been revisited recently, mainly motivated by research on NLEPs and the reference [28]. Thus, a first formal definition of linearization of a rational matrix was proposed in [1]. A different definition was introduced in [4], together with the first formal definition of strong linearization, i.e., a pencil that allows to recover both the finite and infinite pole and zero structure of $R(\lambda)$.

In addition to formal definitions, recent research on linearizations of rational matrices $R(\lambda)$ has produced new classes of strong linearizations that can be easily constructed from the polynomial part of $R(\lambda)$ and a minimal state-space realization of its strictly proper part. Also, the study of the recovery properties of these linearizations has received considerable attention. References in these lines include [2,3,5,7–9,12]. Among the new classes of strong linearizations we mention the strong block minimal bases linearizations introduced in [4, Theorem 5.11] since they form a very wide general family. For instance, they include as particular cases the Fiedler-like linearizations (modulo permutations) [5, Section 8], are closely connected to other classes of linearizations developed in the literature [12] and are valid for general rectangular rational matrices.

Despite the intense activity described in the previous paragraph, there are pencils that have been used in influential references as [18,21] for solving numerically REPs that approximate NLEPs which do not satisfy the definitions of linearization of rational matrices given in [1,4]. The reason is that these definitions focus on pencils that allow to recover the complete pole and zero structure of rational matrices, while in [18,21] only the eigenvalue information in a certain subset of the complex plane is necessary. This has motivated the development in [13] of a new theory of linearizations of rational matrices in a local sense. These linearizations are pencils that preserve the structure of zeros and poles of the corresponding rational matrix in a particular subset of the underlying field and/or at infinity. Thus, such linearizations are useful in applications where only the zero and pole information in a particular region is needed. Apart from a new definition, a specific family of local linearizations of rational matrices is also introduced in [13, Section 5], that are called block full rank pencils. These pencils are particular instances of local linearizations that allow to recover the information about zeros that are not poles of rational matrices.

The main contribution of this work is to extend the block full rank pencils from [13] to a much larger family of pencils, that allow also to recover pole information under minimality conditions. This new family is called block full rank linearizations, where we use a name similar to that in [13] for emphasizing the connection between both concepts. We remark that block full rank linearizations can also be seen as a wide nontrivial generalization of the strong block minimal bases linearizations introduced in [4, Theorem 5.11]. This generalization allows to study the structure and properties of most of the linearizations of rational matrices in the literature in a general framework. In this manuscript, as a relevant example, we apply the theory of block full rank linearizations to study the precise properties of the pencils used in [21]. Moreover, block full rank linearizations may have future applications to backward error analyses of rational eigenvalue problems solved via linearizations by considering more general linearizations than those in [14] or, perhaps, by allowing more flexible analyses than that in [14].

The paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we introduce block full rank linearizations at finite points and at infinity. The notion of degree of a rational matrix will be used to determine the grade of the new linearizations as linearizations at infinity. In Subsection 3.3, we will see that the block full rank linearizations extend the structure of the strong block minimal bases linearizations in [4]. In Section 4, we will study how to recover eigenvectors of regular rational matrices from those of their block full rank linearizations. Finally, in Section 5, we will show that the linearizations in [21] are block full rank linearizations. Then, we will provide sufficient conditions under which these linearizations are minimal, so that pole information can also be recovered from them. Block full rank linearizations allow also to determine the precise properties of other linearizations appearing in the literature as those in [15,18,28]. Conclusions are presented in Section 6.

2. Preliminaries

Let $\mathbb{F}$ be an algebraically closed field that does not include infinity. $\mathbb{F}[\lambda]$ denotes the ring of polynomials with coefficients in $\mathbb{F}$, and $\mathbb{F}(\lambda)$ the field of rational functions over $\mathbb{F}[\lambda]$. $\mathbb{F}^{p\times m}$, $\mathbb{F}[\lambda]^{p\times m}$ and $\mathbb{F}(\lambda)^{p\times m}$ denote the sets of $p \times m$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[\lambda]$ and $\mathbb{F}(\lambda)$, respectively. The elements of $\mathbb{F}[\lambda]^{p\times m}$ are called polynomial matrices or matrix polynomials. A unimodular matrix is a square polynomial matrix with polynomial inverse or, equivalently, a square polynomial matrix with nonzero constant determinant. Moreover, the elements of $\mathbb{F}(\lambda)^{p\times m}$ are called rational matrices.

Let $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let Ω be a nonempty subset of $\mathbb{F}$. $R(\lambda)$ is *defined or bounded in* Ω if $R(\lambda_0) \in \mathbb{F}^{p\times m}$ for all $\lambda_0 \in \Omega$ (by assuming the entries of $R(\lambda)$ to be irreducible rational functions). Moreover, $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ is *invertible in* Ω if it is defined in Ω and $\det R(\lambda_0) \neq 0$ for all $\lambda_0 \in \Omega$. $R(\lambda)$ is said to be *defined (invertible) at* λ_0 if it is defined (invertible) in $\Omega := \{\lambda_0\}$. A rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ is said to be *regular* if it is invertible at some $\lambda_0 \in \mathbb{F}$, that is, $\det R(\lambda) \not\equiv 0$. Let $\lambda_0 \in \mathbb{F}$. Two rational matrices $G(\lambda), H(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are *equivalent at* λ_0 if there exist rational matrices $R_1(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ and $R_2(\lambda) \in \mathbb{F}(\lambda)^{m\times m}$ both invertible at λ_0 such that $R_1(\lambda)G(\lambda)R_2(\lambda) = H(\lambda)$. If $G(\lambda)$ and $H(\lambda)$ are equivalent for all λ_0 in a nonempty set Ω then $G(\lambda)$ and $H(\lambda)$ are said to be *equivalent in* Ω [13, Proposition 2.4]. Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $\lambda_0 \in \mathbb{F}$. Then $G(\lambda)$ is equivalent at λ_0 to a matrix of

the form (see [6,33])

$$\begin{bmatrix} \operatorname{diag}\left((\lambda-\lambda_0)^{\nu_1},\ldots,(\lambda-\lambda_0)^{\nu_r}\right) & 0 \\ 0 & 0_{(p-r)\times(m-r)} \end{bmatrix}, \tag{1}$$

where r is the normal rank of $G(\lambda)$, which will be denoted by $\operatorname{nrank} G(\lambda)$, and $\nu_1 \leq \cdots \leq \nu_r$ are integers. The integers $\nu_1,\ldots,\nu_r$ are uniquely determined by $G(\lambda)$ and λ_0, and are called the *invariant orders at* λ_0 of $G(\lambda)$. The matrix in (1) is called the *local Smith–McMillan form of* $G(\lambda)$ *at* λ_0. In order to define zeros and poles we need to distinguish between positive and negative invariant orders [20,34]. Let $\nu_1 \leq \cdots \leq \nu_k < 0 = \nu_{k+1} = \cdots = \nu_{u-1} < \nu_u \leq \cdots \leq \nu_r$ be the invariant orders at λ_0 of $G(\lambda)$. Then λ_0 is said to be a *pole* of $G(\lambda)$ with *partial multiplicities* $-\nu_k,\cdots,-\nu_1$, and a *zero* of $G(\lambda)$ with *partial multiplicities* $\nu_u,\cdots,\nu_r$. In particular, the integers $-\nu_k,\cdots,-\nu_1$ and $\nu_u,\cdots,\nu_r$ are called the pole and zero partial multiplicities of $G(\lambda)$ at λ_0, respectively. Moreover, $(\lambda-\lambda_0)^{-\nu_i}$ for $i=1,\ldots,k$ are called the *pole elementary divisors of* $G(\lambda)$ *at* λ_0, while $(\lambda-\lambda_0)^{\nu_i}$ for $i=u,\ldots,r$ are called the *zero elementary divisors of* $G(\lambda)$ *at* λ_0. If $G(\lambda)$ is a polynomial matrix then the polynomials $(\lambda-\lambda_0)^{\nu_i}$ with $\nu_i \neq 0$ are simply called *elementary divisors of* $G(\lambda)$ *at* λ_0, and the nonzero integers $\nu_i \neq 0$ are all positive and are called *partial multiplicities of* $G(\lambda)$ *at* λ_0. The (pole/zero) elementary divisors of $G(\lambda)$ in a nonempty subset Ω of $\mathbb{F}$ are the (pole/zero) elementary divisors of $G(\lambda)$ at every $\lambda_0 \in \Omega$. The invariant orders at infinity $q_1 \leq \cdots \leq q_r$ of a rational matrix $G(\lambda)$ are defined as the invariant orders at $\lambda_0 = 0$ of $G(1/\lambda)$. Then, for $q_i < 0$, $-q_i$ are the partial multiplicities of ∞ as pole while, for $q_i > 0$, q_i are the partial multiplicities of ∞ as zero (see [20]).

A finite $\lambda_0 \in \mathbb{F}$ is said to be a *finite eigenvalue* of $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ if it is a zero of $G(\lambda)$ that is not a pole. Then, for a regular $G(\lambda)$, nonzero constant vectors x and y satisfying

$$G(\lambda_0)x = 0 \quad \text{and} \quad y^T G(\lambda_0) = 0,$$

are called, respectively, *right and left eigenvectors associated with* λ_0. Given $\lambda_0 \in \mathbb{F}$, we consider the following vector spaces over $\mathbb{F}$:

$$\begin{aligned} \mathcal{N}_r(G(\lambda_0)) &= \{x \in \mathbb{F}^{m\times 1} : G(\lambda_0)x = 0\}, \text{ and} \\ \mathcal{N}_\ell(G(\lambda_0)) &= \{y^T \in \mathbb{F}^{1\times p} : y^T G(\lambda_0) = 0\}, \end{aligned}$$

which are called, respectively, the *right and left nullspaces over* $\mathbb{F}$ *of* $G(\lambda_0)$. If λ_0 is an eigenvalue of a regular $G(\lambda)$, then $\mathcal{N}_r(G(\lambda_0))$ and $\mathcal{N}_\ell(G(\lambda_0))$ are non trivial and contain, respectively, the right and left eigenvectors of $G(\lambda)$ associated with λ_0.

2.1. Polynomial system matrices

A matrix polynomial of the form

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{2}$$

with $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ regular, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$, and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, is called a *polynomial system matrix*, and the rational matrix

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$$

is called the *transfer function matrix* of $P(\lambda)$ [27]. The matrix $A(\lambda)$ is called the *state matrix* of $P(\lambda)$. We allow n to be 0, that is, we consider $P(\lambda) = D(\lambda)$ as a polynomial system matrix with $A(\lambda), B(\lambda)$ and $C(\lambda)$ as empty matrices. In this extreme case we say that $P(\lambda) = D(\lambda)$ is a polynomial system matrix with empty state matrix. Its transfer function is $D(\lambda)$.

The right and left eigenvectors of a polynomial system matrix and those of its transfer function matrix are related by the following Lemma 2.1, that follows from [12, Proposition 5.1] and [12, Proposition 5.2].

Lemma 2.1. Let $P(\lambda)$ be a polynomial system matrix as in (2), and let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be its transfer function matrix. Let $\lambda_0 \in \mathbb{F}$ such that $\det A(\lambda_0) \neq 0$. Then, the following statements hold:

(a) The linear map

$$\begin{aligned} F_r \,:\, \mathcal{N}_r(G(\lambda_0)) &\longrightarrow \mathcal{N}_r(P(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0) \\ I_m \end{bmatrix} x \end{aligned}$$

is a bijection between the right nullspaces over $\mathbb{F}$ of $G(\lambda_0)$ and $P(\lambda_0)$.

(b) The linear map

$$\begin{aligned} F_\ell \,:\, \mathcal{N}_\ell(G(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(P(\lambda_0)) \\ y^T &\longmapsto y^T \begin{bmatrix} C(\lambda_0)A(\lambda_0)^{-1} & I_p \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces over $\mathbb{F}$ of $G(\lambda_0)$ and $P(\lambda_0)$.

Let Ω be a nonempty subset of $\mathbb{F}$. The polynomial system matrix $P(\lambda)$ in (2), with $n > 0$, is said to be *minimal in* Ω if

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ C(\lambda_0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A(\lambda_0) & B(\lambda_0) \end{bmatrix} = n,$$

for all $\lambda_0 \in \Omega$. In the particular case that $\Omega = \{\lambda_0\}$, $P(\lambda)$ is said to be *minimal at* λ_0 if $P(\lambda)$ is minimal in Ω. If $n = 0$ in (2), we adopt the agreement that $P(\lambda)$ is minimal at every point $\lambda_0 \in \mathbb{F}$.

2.2. Linearizations of rational matrices in a set and at infinity

We can now state the definition of linearization of a rational matrix in a subset of $\mathbb{F}$. Then, in Theorem 2.3, a spectral characterization is given.

Definition 2.2. [13, Definition 4.1] Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $\Omega \subseteq \mathbb{F}$ be nonempty. Let

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{3}$$

be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$, and let

$$\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0) \in \mathbb{F}(\lambda)^{q\times r}$$

be its transfer function matrix. $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in Ω if the following conditions hold:

(a) $\mathcal{L}(\lambda)$ is minimal in Ω, and
(b) there exist nonnegative integers s_1, s_2 satisfying $s_1 - s_2 = q - p = r - m$, and rational matrices $R_1(\lambda) \in \mathbb{F}(\lambda)^{(p+s_1)\times(p+s_1)}$ and $R_2(\lambda) \in \mathbb{F}(\lambda)^{(m+s_1)\times(m+s_1)}$ invertible in Ω such that

$$R_1(\lambda)\operatorname{diag}(G(\lambda), I_{s_1})R_2(\lambda) = \operatorname{diag}(\widehat{G}(\lambda), I_{s_2}). \tag{4}$$

The most usual case is when $q - p = r - m \geq 0$. This implies that the size of the transfer function matrix $\widehat{G}(\lambda)$ of the linearization $\mathcal{L}(\lambda)$ is at least the size of the rational matrix $G(\lambda)$. We assume in the rest of the paper that this holds and take $s_2 = 0$ and $s := s_1 \geq 0$ without loss of generality.

Theorem 2.3. [13, Theorem 4.4][Spectral characterization of linearizations in a subset of $\mathbb{F}$] Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $\Omega \subseteq \mathbb{F}$ be nonempty. Let

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a linear polynomial system matrix, with state matrix $A_1\lambda + A_0$, minimal in Ω. Then $\mathcal{L}(\lambda)$ is a linearization of $G(\lambda)$ in Ω if and only if the following three conditions hold:

(a) $\operatorname{nrank}\mathcal{L}(\lambda) = \operatorname{nrank} G(\lambda) + n + s$,
(b) the pole elementary divisors of $G(\lambda)$ in Ω are the elementary divisors of $A_1\lambda + A_0$ in Ω, and
(c) the zero elementary divisors of $G(\lambda)$ in Ω are the elementary divisors of $\mathcal{L}(\lambda)$ in Ω.

All the concepts and results introduced above are extended to infinity in [13]. For that, the notion of g-reversal of a rational matrix [13, Definition 3.7] is used. That is, given a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and an integer g, the *g-reversal* of $G(\lambda)$ is defined as the rational matrix

$$\operatorname{rev}_g G(\lambda) := \lambda^g G\left(\frac{1}{\lambda}\right).$$

A nonconstant linear polynomial system matrix $\mathcal{L}(\lambda)$ as in (3) is defined to be a *linearization of a rational matrix $G(\lambda)$ at ∞ of grade g* if $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is a linearization of $\operatorname{rev}_g G(\lambda)$ at 0. Linearizations at ∞ of $G(\lambda)$ provide information about the pole/zero structure of $G(\lambda)$ at ∞ as is explained in Proposition 2.4.

Proposition 2.4. [13, Proposition 4.13][Recovery of the spectral information at infinity] Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix with $\operatorname{nrank} G(\lambda) = r$, and let

$$\mathcal{L}(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix}$$

be a nonconstant linearization at infinity of grade g of $G(\lambda)$. If $e_1 \leq \cdots \leq e_t$ are the partial multiplicities of $\operatorname{rev}_1(A_1\lambda + A_0)$ at 0 and $\widetilde{e}_1 \leq \cdots \leq \widetilde{e}_u$ are the partial multiplicities of $\operatorname{rev}_1 \mathcal{L}(\lambda)$ at 0, then the invariant orders at infinity $q_1 \leq \cdots \leq q_r$ of $G(\lambda)$ are

$$(q_1, \ldots, q_r) = (-e_t, \ldots, -e_1, \underbrace{0, \ldots, 0}_{r-t-u}, \widetilde{e}_1, \ldots, \widetilde{e}_u) - (g, \ldots, g).$$

2.3. Rational and minimal bases

Finally, we recall the notions of rational and minimal bases, which are very useful for the construction of linearizations for rational matrices. A rational matrix $R(\lambda)$ is said to be a *rational basis* if its rows form a basis of the rational subspace they span, i.e., if $R(\lambda)$ has full row normal rank. The term "rational subspace" indicates that $\mathbb{F}(\lambda)$ is the underlying field. Two rational matrices $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $H(\lambda) \in \mathbb{F}(\lambda)^{q\times m}$ are *dual rational bases* if both have full row normal rank, $p+q = m$, and $G(\lambda)\, H(\lambda)^T = 0$. Let $\Omega \subseteq \mathbb{F}$ be nonempty. We will say that a rational matrix $R(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ has *full row rank in* Ω if, for all $\lambda_0 \in \Omega$, $R(\lambda_0) \in \mathbb{F}^{p\times m}$, i.e., $R(\lambda)$ is defined or bounded at λ_0, and $\operatorname{rank} R(\lambda_0) = p$. Notice that a rational matrix having full row rank in $\mathbb{F}$ must be polynomial, since it has to be defined in $\mathbb{F}$.

For polynomial matrices, the notion of minimal basis is also considered [16]. One of the most useful characterizations of minimal bases (see [16, Main Theorem] or [11, Theorem 2.2]) is that $K(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is a minimal basis if and only if $K(\lambda_0)$ has full row rank for all $\lambda_0 \in \mathbb{F}$ and $K(\lambda)$ is row reduced, i.e., its highest row degree coefficient matrix has full row rank (see [11, Definition 2.1]). Minimal bases appear in the definition of strong block minimal bases pencils in [11]. A strong block minimal bases pencil is a linear polynomial matrix with the following structure

$$L(\lambda) = \begin{array}{c} \left[\begin{array}{cc} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{array}\right] \\ \underbrace{}_{m+\widehat{m}} \quad \underbrace{}_{\widehat{p}} \end{array} \begin{array}{l} \}\, p+\widehat{p} \\ \}\, \widehat{m} \end{array}, \tag{5}$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ (respectively $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ (respectively $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$) dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$) all equal. In addition, given a polynomial matrix $D(\lambda)$, $L(\lambda)$ is said to be associated with $D(\lambda)$ if

$$D(\lambda) = N_2(\lambda) M(\lambda) N_1(\lambda)^T. \tag{6}$$

In this case, $L(\lambda)$ is a strong linearization of $D(\lambda)$ [11, Theorem 3.3].

3. Block full rank linearizations of rational matrices

In this section, we introduce in Theorems 3.2 and 3.10 a wide family of pencils that give us the information about the zeros and poles of rational matrices locally, at finite points and also at infinity. They will be called block full rank linearizations. Previously, we introduce the notion of block full rank pencils [13], that are an extension of the strong block minimal bases pencils in [11].

Definition 3.1. [13, Definition 5.1] A block full rank pencil is a linear polynomial matrix over $\mathbb{F}$ with the following structure

$$L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \tag{7}$$

where $K_1(\lambda)$ and $K_2(\lambda)$ are pencils with full row normal rank.

Definition 3.1 includes the cases when $K_1(\lambda)$ or $K_2(\lambda)$ are empty matrices, that is, $L(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \end{bmatrix}$, $L(\lambda) = \begin{bmatrix} M(\lambda) \\ K_1(\lambda) \end{bmatrix}$ or $L(\lambda) = M(\lambda)$.

Theorem 5.3 of [13] states that block full rank pencils can be considered as local linearizations of rational matrices that contain information about their zeros. However, Theorem 5.3 of [13] does not provide any information about poles.

3.1. Block full rank linearizations at finite points

In Theorem 3.2, we generalize [13, Theorem 5.3] in order to obtain local linearizations that give us not only information about the zeros but also about the poles of the corresponding rational matrix. Note also that the linear polynomial system matrix in (8) generalizes the structure of the strong block minimal bases linearizations of rational matrices presented in [4, Theorem 5.11] from three perspectives: general pencils $B(\lambda)$ and $C(\lambda)$ are allowed, while those in [4] have a very particular structure; $A(\lambda)$ can be any regular pencil, while in [4] its coefficient in λ must be invertible; and $\begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ is an arbitrary block full rank pencil (7), while in [4] strong block minimal bases pencils (5) are considered. We give more details in Subsection 3.3.

Theorem 3.2. Consider a nonconstant linear polynomial system matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{8}$$

with $n > 0$ and state matrix $A(\lambda)$. Let $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ be a block full rank pencil, and let $N_1(\lambda)$ and $N_2(\lambda)$ be any rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Let Ω be a nonempty subset of $\mathbb{F}$ such that $K_i(\lambda)$ and $N_i(\lambda)$ have full row rank in Ω for $i = 1, 2$. If

$$\operatorname{rank} \begin{bmatrix} A(\lambda_0) \\ -N_2(\lambda_0)C(\lambda_0) \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A(\lambda_0) & B(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} = n \tag{9}$$

for all $\lambda_0 \in \Omega$ then $\mathcal{L}(\lambda)$ is a linearization of the rational matrix

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T \tag{10}$$

in Ω with state matrix $A(\lambda)$.

A pencil of the form (8) satisfying the hypotheses in Theorem 3.2 is called a *block full rank linearization*. In particular, $\mathcal{L}(\lambda)$ is said to be a block full rank linearization of $R(\lambda)$ in Ω with state matrix $A(\lambda)$.

Remark 3.3. The extreme case of $n = 0$ in the linear polynomial system matrix (8) is studied in [13, Theorem 5.3]. It states that the block full rank pencil $L(\lambda)$ in Theorem 3.2 is a linearization of the rational matrix

$$G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

in Ω with empty state matrix. In this case, $L(\lambda)$ is said to be a block full rank linearization of $G(\lambda)$ in Ω with empty state matrix.

Proof of Theorem 3.2. In order to simplify the notation, throughout this proof we do not specify the sizes of different identity matrices and all of them are denoted by I. Let $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), \widetilde{N}_1(\lambda)$ and $\widetilde{N}_2(\lambda)$ be minimal bases of the row spaces of $K_1(\lambda)$, $K_2(\lambda)$, $N_1(\lambda)$ and $N_2(\lambda)$, respectively. Then, by [13, Lemma 5.2], there exist rational matrices $S_1(\lambda)$, $S_2(\lambda)$, $W_1(\lambda)$ and $W_2(\lambda)$ such that

$$\begin{aligned} K_i(\lambda) &= S_i(\lambda)\widetilde{K}_i(\lambda), \quad \text{and } S_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2, \\ N_i(\lambda) &= W_i(\lambda)\widetilde{N}_i(\lambda), \quad \text{and } W_i(\lambda) \text{ is invertible in } \Omega, \text{ for } i = 1, 2. \end{aligned}$$

Moreover, $\widetilde{K}_1(\lambda), \widetilde{K}_2(\lambda), S_1(\lambda)$ and $S_2(\lambda)$ are all matrix pencils. We consider the linear polynomial system matrix

$$\widetilde{\mathcal{L}}(\lambda) := \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & \widetilde{K}_2(\lambda)^T \\ 0 & \widetilde{K}_1(\lambda) & 0 \end{array}\right], \tag{11}$$

which is equivalent in Ω to $\mathcal{L}(\lambda)$, since $\begin{bmatrix} I & 0 \\ 0 & S_1(\lambda) \end{bmatrix} \widetilde{\mathcal{L}}(\lambda) \begin{bmatrix} I & 0 \\ 0 & S_2(\lambda)^T \end{bmatrix} = \mathcal{L}(\lambda)$. For $i = 1, 2$, there exist unimodular matrices

$$U_i(\lambda) = \begin{bmatrix} \widetilde{K}_i(\lambda) \\ \widehat{K}_i(\lambda) \end{bmatrix}, \text{ and } \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & \widetilde{N}_i(\lambda)^T \end{bmatrix}$$

as in [11, Theorem 2.10]. Consider now the unimodular matrices

$$V_1(\lambda) := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & \widehat{N}_1(\lambda)^T & \widetilde{N}_1(\lambda)^T & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & -X(\lambda) & 0 & I \end{bmatrix}, \text{ and}$$

$$V_2(\lambda) := \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & -Y(\lambda) \\ 0 & 0 & 0 & I \\ 0 & I & 0 & -Z(\lambda) \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \widehat{N}_2(\lambda) & 0 \\ 0 & \widetilde{N}_2(\lambda) & 0 \\ 0 & 0 & I \end{bmatrix},$$

where $Z(\lambda) := \widehat{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T$, $X(\lambda) := \widehat{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T$, $Y(\lambda) := \widetilde{N}_2(\lambda)M(\lambda)\widehat{N}_1(\lambda)^T$. We obtain that

$$V_2(\lambda)\widetilde{\mathcal{L}}(\lambda)V_1(\lambda) = \left[\begin{array}{c|ccc} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T & B(\lambda)\widehat{N}_1(\lambda)^T & 0 \\ \hline -\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 & 0 \\ 0 & 0 & I & 0 \\ -\widehat{N}_2(\lambda)C(\lambda) & 0 & 0 & I \end{array}\right],$$

which is, in addition, unimodularly equivalent to

$$\left[\begin{array}{c|cc} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ \hline -\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T & 0 \\ 0 & 0 & I \end{array}\right] := P(\lambda).$$

By condition (9), the polynomial matrix

$$H(\lambda) := \left[\begin{array}{c|c} A(\lambda) & B(\lambda)\widetilde{N}_1(\lambda)^T \\ \hline -\widetilde{N}_2(\lambda)C(\lambda) & \widetilde{N}_2(\lambda)M(\lambda)\widetilde{N}_1(\lambda)^T \end{array}\right]$$

is minimal in Ω, and its transfer function matrix is

$$\widetilde{R}(\lambda) = \widetilde{N}_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]\widetilde{N}_1(\lambda)^T.$$

Moreover, $W_2(\lambda)\widetilde{R}(\lambda)W_1(\lambda)^T = R(\lambda)$, and, thus, $\widetilde{R}(\lambda)$ and $R(\lambda)$ are equivalent in Ω. Therefore, the zero elementary divisors of $H(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω, and the zero elementary divisors of $A(\lambda)$ in Ω are the pole elementary divisors of $R(\lambda)$ in Ω. In addition, $P(\lambda) = \begin{bmatrix} H(\lambda) & 0 \\ 0 & I \end{bmatrix}$ is unimodularly equivalent to $\widetilde{\mathcal{L}}(\lambda)$, which is equivalent in Ω to $\mathcal{L}(\lambda)$. Therefore, the zero elementary divisors of $\mathcal{L}(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω. By Theorem 2.3, $\mathcal{L}(\lambda)$ is a linearization in Ω of $R(\lambda)$, since it is immediate to check that the rank condition in Theorem 2.3(a) is satisfied. □

Remark 3.4. Notice that $\mathcal{L}(\lambda)$ being minimal in Ω is a necessary condition, but not sufficient, in order the rank condition (9) to be satisfied.

Under the conditions of Theorem 3.2, Theorem 2.3 guarantees that the elementary divisors of $\mathcal{L}(\lambda)$ in Ω are the zero elementary divisors of $R(\lambda)$ in Ω, and that the elementary divisors of $A(\lambda)$ in Ω are the pole elementary divisors of $R(\lambda)$ in Ω.

We would like to emphasize the fact that Theorem 3.2 gives very simple conditions to determine if a pencil is a linearization (in a target set) and, in addition, to determine the rational matrices associated with it by considering rational bases dual to $K_1(\lambda)$ and $K_2(\lambda)$. Notice that this "association" is not one-to-one because there are infinitely many rational bases $N_1(\lambda)$ and $N_2(\lambda)$ dual to $K_1(\lambda)$ and $K_2(\lambda)$. Thus, Theorem 3.2 is useful, for instance, if a linearization gets perturbed and one wants to know which rational matrix is associated with the new perturbed pencil. Then, if the perturbation is small enough that the rank conditions in Theorem 3.2 are still satisfied and after restoring the zero blocks of the block full rank linearization, we will get a perturbed block full rank linearization and one can then reconstruct the new rational matrices associated with it. This type of perturbation problems arise in backward error analyses of rational eigenvalue problems solved via linearizations. Thus, the block full rank linearizations may have future applications in this setting allowing more flexible and/or more general analyses than the one in [14].

Remark 3.5. If in Theorem 3.2, $K_1(\lambda)$ (resp. $K_2(\lambda)$) is an empty matrix, we can take the dual rational basis $N_1(\lambda)$ (resp. $N_2(\lambda)$) as any rational matrix invertible in Ω of size the number of colums (resp. rows) of $M(\lambda)$.

Example 3.6. Let us see a simple example that illustrates Remark 3.5. For instance, for constructing a linearization of the rational matrix

$$R(\lambda) = \frac{\lambda-2}{\lambda+2}\begin{bmatrix} \dfrac{-\lambda+3}{\lambda^2-1} & \dfrac{1}{\lambda(\lambda-1)} \end{bmatrix} \text{ in the set } \Omega := \mathbb{F} - \{-1, 0, 1\},$$

we can consider $K_1(\lambda)$ and $K_2(\lambda)$ as empty matrices and dual rational bases $N_1(\lambda)^T := \operatorname{diag}\left(\frac{1}{\lambda^2-1}, \frac{1}{(\lambda-1)\lambda}\right)$ and $N_2(\lambda) := 1$, both invertible in Ω. Then, by Theorem 3.2, the following linear polynomial system matrix

$$\mathcal{L}(\lambda) := \left[\begin{array}{c|cc} \lambda+2 & -\lambda+3 & 1 \\ \hline -\lambda+2 & 0 & 0 \end{array}\right] := \left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline -C(\lambda) & M(\lambda) \end{array}\right]$$

is a linearization of $R(\lambda)$ in Ω with state matrix $\lambda+2$, since

$$\operatorname{rank}\begin{bmatrix} \lambda+2 \\ -\lambda+2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda+2 & \dfrac{-\lambda+3}{\lambda^2-1} & \dfrac{1}{\lambda(\lambda-1)} \end{bmatrix} = 1$$

for all $\lambda \in \Omega$. Therefore, we can recover from $\mathcal{L}(\lambda)$ the pole and zero structure of $R(\lambda)$ in Ω. More precisely, -2 is the only zero of the state matrix in Ω and, thus, is the only pole of $R(\lambda)$ in Ω. Moreover, 2 is the unique zero of $\mathcal{L}(\lambda)$ in Ω and, thus, is the unique zero of $R(\lambda)$ in Ω.

In Section 5 we will see that the linearizations for rational approximations of nonlinear eigenvalue problems given in [21] are block full rank linearizations.

Remark 3.7. We notice that, although the state matrix $A(\lambda)$ appears in the (1,1) block in (8), in practice, it can be any regular submatrix of $\mathcal{L}(\lambda)$. In particular, in

some applications [18,21] we have found pencils with the structure of block full rank linearizations of the form

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc|c} M(\lambda) & K_2(\lambda)^T & -C(\lambda) \\ K_1(\lambda) & 0 & 0 \\ \hline B(\lambda) & 0 & A(\lambda) \end{array}\right] \in \mathbb{F}[\lambda]^{(q+n)\times(r+n)}. \tag{12}$$

3.2. Block full rank linearizations at infinity

We now study the counterpart of Theorem 3.2 at infinity. First, we define the notion of degree of a rational matrix. For the scalar case, we define the degree of a rational function $r(\lambda) = \dfrac{n(\lambda)}{d(\lambda)}$ as

$$\deg(r(\lambda)) := \deg(n(\lambda)) - \deg(d(\lambda)). \tag{13}$$

Then, for rational matrices we consider the following definition (see, for instance, [34, p.10]).

Definition 3.8. Let $R(\lambda) = [r_{ij}(\lambda)] \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix with entries $r_{ij}(\lambda) = \dfrac{n_{ij}(\lambda)}{d_{ij}(\lambda)}$. The degree of $R(\lambda)$ is then defined as

$$\deg(R(\lambda)) := \max_{\substack{i=1,\dots,p \\ j=1,\dots,m}} \{\deg(r_{ij}(\lambda))\}.$$

Notice that this notion of degree of a rational matrix generalizes the notion of degree of a polynomial matrix. In what follows, we call the degrees of each row of $R(\lambda)$, the row degrees of $R(\lambda)$.

Lemma 3.9. Let $R(\lambda)$ be a rational matrix. Then there exists an integer t such that all the rows of $\operatorname{rev}_t R(\lambda)$ are defined at 0 and are all different from zero at 0 if and only if all the row degrees of $R(\lambda)$ are equal to t.

Proof. First, we consider a rational function $r(\lambda) = \dfrac{a(\lambda)}{b(\lambda)}$ such that the numerator $a(\lambda)$ has degree n, and that the denominator $b(\lambda)$ has degree m. We assume that there exists an integer t for which $\operatorname{rev}_t r(0)$ is defined and is different from 0. We can write

$$\operatorname{rev}_t r(\lambda) = \lambda^{t+m-n} \frac{\operatorname{rev}_n a(\lambda)}{\operatorname{rev}_m b(\lambda)} := \lambda^{t+m-n} h(\lambda), \tag{14}$$

where 0 is not a pole nor a zero of $h(\lambda)$ since $\operatorname{rev}_n a(0) \neq 0$ and $\operatorname{rev}_m b(0) \neq 0$. That is, $h(\lambda)$ is defined and is different from 0 at 0. Therefore, $t + m - n = 0$, so that $\operatorname{rev}_t r(\lambda)$ is also defined and is different from 0 at 0. Then, $t = n - m = \deg(r(\lambda))$. Now, we assume that we have a rational row vector $v(\lambda) = [r_1(\lambda) \quad \cdots \quad r_s(\lambda)]$ such that, for some integer t, $\operatorname{rev}_t v(0)$ is defined and is different from 0. Then, it must be $t = \max\limits_{i=1,\dots,s} \{\deg(r_i(\lambda))\}$. That is, $t = \deg(v(\lambda))$. Finally, consider a rational matrix $R(\lambda) = [v_1(\lambda)^T \quad \cdots \quad v_q(\lambda)^T]^T$ where $v_i(\lambda)$ are rational row vectors. Assume that

there exists an integer t such that all the rows of $\operatorname{rev}_t R(\lambda)$ are defined at 0 and are different from zero at 0. Then, for each row $v_i(\lambda)$, it must hold that $t = \deg(v_i(\lambda))$.

The converse is trivial taking into account equation (14) for each entry of $R(\lambda)$. □

Theorem 3.10. Consider a nonconstant linear polynomial system matrix

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{15}$$

with $n > 0$ and state matrix $A(\lambda)$. Let $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ be a block full rank pencil and let $N_1(\lambda)$ (resp. $N_2(\lambda)$) be any rational basis dual to $K_1(\lambda)$ (resp. $K_2(\lambda)$) with its row degrees all equal to an integer t_1 (resp. t_2). If $\operatorname{rev}_1 K_i(\lambda)$ and $\operatorname{rev}_{t_i} N_i(\lambda)$ have full row rank at zero for $i = 1, 2$ and

$$\operatorname{rank}\begin{bmatrix} \operatorname{rev}_1 A(0) \\ -\operatorname{rev}_{t_2} N_2(0) \operatorname{rev}_1 C(0) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \operatorname{rev}_1 A(0) & \operatorname{rev}_1 B(0) \operatorname{rev}_{t_1} N_1(0)^T \end{bmatrix} = n \tag{16}$$

then $\mathcal{L}(\lambda)$ is a linearization of the rational matrix

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$$

at ∞ of grade $1 + t_1 + t_2$ with state matrix $A(\lambda)$.

A pencil of the form (15) satisfying the hypotheses in Theorem 3.10 is called *block full rank linearization at infinity*. In particular, $\mathcal{L}(\lambda)$ is said to be a block full rank linearization of $R(\lambda)$ at ∞ of grade $1 + t_1 + t_2$ with state matrix $A(\lambda)$. In general, a block full rank linearization is said to be *strong* if it is a linearization in $\mathbb{F} \cup \{\infty\}$.

Remark 3.11. The extreme case of $n = 0$ in the linear polynomial system matrix (15) is studied in [13, Theorem 5.5]. It states that the block full rank pencil $L(\lambda)$ in Theorem 3.10 is a linearization of the rational matrix

$$G(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$$

at ∞ of grade $1 + t_1 + t_2$ with empty state matrix. In this case, $L(\lambda)$ is said to be a block full rank linearization of $G(\lambda)$ at ∞ of grade $1+t_1+t_2$ with empty state matrix. We notice that, by Lemma 3.9, the integers t_1 and t_2 appearing in [13, Theorem 5.5] are the row degrees of the dual bases $N_1(\lambda)$ and $N_2(\lambda)$, respectively.

Proof of Theorem 3.10. The result follows by applying Theorem 3.2 to $\operatorname{rev}_1 \mathcal{L}(\lambda)$ at 0. Then $\operatorname{rev}_1 \mathcal{L}(\lambda)$ is a linearization at 0 of

$$\operatorname{rev}_{t_2} N_2(\lambda)[\operatorname{rev}_1 M(\lambda)+\operatorname{rev}_1 C(\lambda)(\operatorname{rev}_1 A(\lambda))^{-1} \operatorname{rev}_1 B(\lambda)] \operatorname{rev}_{t_1} N_1(\lambda)^T = \operatorname{rev}_{1+t_1+t_2} R(\lambda).$$

□

Example 3.12. We now consider the rational matrix

$$R(\lambda) := \sum_{k=0}^{2} A_k \frac{\lambda^k}{(\lambda-\epsilon)^2} + I_n \frac{1}{\lambda} \in \mathbb{F}(\lambda)^{n\times n} \text{ and the set } \Omega := \mathbb{F} - \{\epsilon\},$$

for some $\epsilon \in \mathbb{F}$. Then we define the following linear polynomial system matrix

$$\mathcal{L}(\lambda) := \left[\begin{array}{c|cc|c} -\lambda I_n & 0 & (\lambda-\epsilon)I_n & 0 \\ \hline 0 & A_2 & 0 & -I_n \\ (\lambda-\epsilon)I_n & 0 & \lambda A_1 + A_0 & \lambda I_n \\ \hline 0 & -I_n & \lambda I_n & 0 \end{array}\right] =: \left[\begin{array}{c|c|c} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ \hline 0 & K_1(\lambda) & 0 \end{array}\right],$$

with state matrix $A(\lambda) = \lambda I_n$. We consider the dual rational bases $N_1(\lambda) := N_2(\lambda) := \left[\begin{array}{cc} \lambda I_n & I_n \\ \hline \lambda-\epsilon & \lambda-\epsilon \end{array}\right]$, which have row degrees $t_1 = t_2 = 0$. Then $R(\lambda)$ can be written as $R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T$. Notice that $\operatorname{rev}_{t_i} N_i(\lambda)$ and $\operatorname{rev}_1 K_i(\lambda)$ have both full row rank at 0, for $i = 1, 2$, and that condition (16) is satisfied since $\operatorname{rev}_1 A(0) = I_n$. Thus, by Theorem 3.10, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at ∞ of grade $1 + t_1 + t_2 = 1$ with state matrix $A(\lambda)$. In addition, by Theorem 3.2, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω, since $N_i(\lambda)$ and $K_i(\lambda)$ have both full row rank in Ω, for $i = 1, 2$, and condition (9) is satisfied in Ω. Observe that, if $R(\lambda)$ has symmetric coefficients, $\mathcal{L}(\lambda)$ preserves the symmetry.

Remark 3.13. If we want a linearization as in (15) to be a linearization at all finite and infinite points we need, besides minimality conditions, the matrices $K_1(\lambda)$ and $K_2(\lambda)$ being minimal bases with all their row degrees equal to 1. Notice that if a pencil $K(\lambda)$ has full row rank in $\mathbb{F}$ and, in addition, $\operatorname{rev}_1 K(\lambda)$ has full row rank at 0 then $K(\lambda)$ is a minimal basis. Conversely, if $K(\lambda)$ is a minimal basis with all its row degrees equal to one then $K(\lambda)$ has full row rank in $\mathbb{F}$ and $\operatorname{rev}_1 K(\lambda)$ has full row rank at 0.

3.3. Strong block minimal bases linearizations as block full rank linearizations

By using strong block minimal bases pencils, in [4, Theorem 5.11] (strong) linearizations are constructed that contain the complete spectral information of rational matrices, finite and infinite, as well as the information about their minimal bases and indices [5], when the corresponding rational matrix $R(\lambda)$ is expressed in the form $R(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$, where $D(\lambda)$ is the polynomial part of $R(\lambda)$ with $\deg(D(\lambda)) > 1$, and $C(\lambda I_n - A)^{-1}B$ is a minimal state-space realization of the strictly proper part of $R(\lambda)$. We will see that such linearizations satisfy Theorem 3.2, with $\Omega = \mathbb{F}$, and Theorem 3.10.

For the construction, in [4] $L(\lambda) := \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$ is considered to be a strong block minimal bases pencil as in (5) associated with $D(\lambda)$ with sharp degree, that is, $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$, where $N_1(\lambda)$ (respectively $N_2(\lambda)$) is a minimal basis dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$). Then, constant matrices $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$ and $\widehat{K}_2 \in \mathbb{F}^{p\times(p+\widehat{p})}$ and matrices $\widehat{N}_1(\lambda)^T \in \mathbb{F}[\lambda]^{(m+\widehat{m})\times\widehat{m}}$ and $\widehat{N}_2(\lambda)^T \in$

$\mathbb{F}[\lambda]^{(p+\widehat{p})\times\widehat{p}}$ exist such that, for $i = 1, 2$,

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

are unimodular. Finally, the following linear polynomial matrix is constructed

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right], \tag{17}$$

where $X, Y \in \mathbb{F}^{n\times n}$ are any nonsingular constant matrices. With these assumptions, $\mathcal{L}(\lambda)$ is a strong linearization of $R(\lambda)$ [4, Theorem 5.11].

This result follows as a corollary of Theorems 3.2 and 3.10 as well. First, since $\widehat{K}_i N_i(\lambda)^T = I$, notice that $R(\lambda)$ can be written as in (10):

$$N_2(\lambda)[M(\lambda)+\widehat{K}_2^T CYY^{-1}(\lambda I_n - A)^{-1}X^{-1}XB\widehat{K}_1]N_1(\lambda)^T = D(\lambda)+C(\lambda I_n - A)^{-1}B = R(\lambda),$$

and, in addition, conditions (9), with $\Omega = \mathbb{F}$, and (16) are satisfied. More precisely, we have that

$$\operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y \\ -N_2(\lambda)\widehat{K}_2^T CY \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y \\ -CY \end{bmatrix} = n, \text{and}$$

$$\operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y & XB\widehat{K}_1 N_1(\lambda)^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} X(\lambda I_n - A)Y & XB \end{bmatrix} = n$$

for all $\lambda \in \mathbb{F}$, since X and Y are nonsingular, and the realization $C(\lambda I_n - A)^{-1}B$ is minimal. Therefore, condition (9) is satisfied and, thus, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in $\mathbb{F}$ by Theorem 3.2. Moreover, we have that condition (16) is satisfied since $\operatorname{rev}_1(\lambda I_n - A)$ evaluated at 0 is just I_n. Then, by Theorem 3.10, $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ at infinity of grade $\deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1 = \deg(D(\lambda))$.

4. Recovery of eigenvectors

In this section, we show how to recover right and left eigenvectors of a regular rational matrix $R(\lambda)$, associated with the eigenvalues in a set Ω, from those of a block full rank linearization of $R(\lambda)$ in Ω. For that, in Theorem 4.2 we study the relation between their right and left nullspaces. We will use the following Lemma 4.1.

Lemma 4.1. Let $K(\lambda)$ be a rational basis and let $N(\lambda)$ be any rational basis dual to $K(\lambda)$. Let Ω be a nonempty subset of $\mathbb{F}$ such that $K(\lambda)$ and $N(\lambda)$ have full row rank in Ω. Then there exist rational matrices of the form

$$V(\lambda) = \begin{bmatrix} K(\lambda) \\ \bar{K}(\lambda) \end{bmatrix} \quad \text{and} \quad V(\lambda)^{-1} = \begin{bmatrix} \bar{N}(\lambda)^T & N(\lambda)^T \end{bmatrix} \tag{18}$$

that are invertible in Ω.

Proof. By [13, Lemma 5.2], there exist rational matrices $S(\lambda)$ and $W(\lambda)$, both invertible in Ω, such that $K(\lambda) = S(\lambda)\widetilde{K}(\lambda)$ and $N(\lambda) = W(\lambda)\widetilde{N}(\lambda)$, where $\widetilde{K}(\lambda)$ and $\widetilde{N}(\lambda)$ are minimal bases of the row spaces of $K(\lambda)$ and $N(\lambda)$, respectively. Then, by [11, Theorem 2.10], there exist unimodular matrices of the form

$$U(\lambda) = \begin{bmatrix} \widetilde{K}(\lambda) \\ \widehat{K}(\lambda) \end{bmatrix} \quad \text{and} \quad U(\lambda)^{-1} = \begin{bmatrix} \widehat{N}(\lambda)^T & \widetilde{N}(\lambda)^T \end{bmatrix}.$$

Then we consider the rational matrix

$$V(\lambda) := \begin{bmatrix} S(\lambda) & 0 \\ 0 & W(\lambda)^{-T} \end{bmatrix} U(\lambda) = \begin{bmatrix} K(\lambda) \\ \bar{K}(\lambda) \end{bmatrix},$$

with $\bar{K}(\lambda) := W(\lambda)^{-T}\widehat{K}(\lambda)$. We have that $V(\lambda)$ is invertible in Ω since $S(\lambda)$ and $W(\lambda)$ are invertible in Ω and $U(\lambda)$ is unimodular. Finally, we note that $V(\lambda)^{-1} = U(\lambda)^{-1}\begin{bmatrix} S(\lambda)^{-1} & 0 \\ 0 & W(\lambda)^T \end{bmatrix} = \begin{bmatrix} \widehat{N}(\lambda)^T S(\lambda)^{-1} & \widetilde{N}(\lambda)^T W(\lambda)^T \end{bmatrix} = \begin{bmatrix} \bar{N}(\lambda)^T & N(\lambda)^T \end{bmatrix}$, with $\bar{N}(\lambda)^T := \widehat{N}(\lambda)^T S(\lambda)^{-1}$. □

Let us now consider the rational bases $K_1(\lambda)$ and $K_2(\lambda)$ of the block full rank linearization in Theorem 3.2 and their dual rational bases $N_1(\lambda)$ and $N_2(\lambda)$. It then follows that, since they have full row rank in a nonempty set Ω, there exist rational matrices of the form

$$V_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \bar{K}_i(\lambda) \end{bmatrix} \quad \text{and} \quad V_i(\lambda)^{-1} = \begin{bmatrix} \bar{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix} \tag{19}$$

that are invertible in Ω, for $i = 1, 2$, by Lemma 4.1. We use these matrices in the following Theorem 4.2.

Theorem 4.2. Let

$$\mathcal{L}(\lambda) = \left[\begin{array}{c|cc} A(\lambda) & B(\lambda) & 0 \\ \hline -C(\lambda) & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}$$

be a block full rank linearization in a nonempty set Ω of the rational matrix

$$R(\lambda) = N_2(\lambda)[M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)]N_1(\lambda)^T,$$

as in Theorem 3.2. Consider the rational matrices $\bar{N}_1(\lambda)$ and $\bar{N}_2(\lambda)$ in (19) and denote $S(\lambda) := M(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$. Let $\lambda_0 \in \Omega$ such that $\det A(\lambda_0) \neq 0$. Then, the following statements hold:

(a) The linear map

$$\begin{aligned} E_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(\mathcal{L}(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0)N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x \end{aligned}$$

is a bijection between the right nullspaces over $\mathbb{F}$ of $R(\lambda_0)$ and $\mathcal{L}(\lambda_0)$.

(b) The linear map

$$\begin{aligned} E_\ell \,:\, \mathcal{N}_\ell(R(\lambda_0)) &\longrightarrow \mathcal{N}_\ell(\mathcal{L}(\lambda_0)) \\ y^T &\longmapsto y^T \begin{bmatrix} N_2(\lambda_0)C(\lambda_0)A(\lambda_0)^{-1} & N_2(\lambda_0) & -N_2(\lambda_0)S(\lambda_0)\bar{N}_1(\lambda_0)^T \end{bmatrix} \end{aligned}$$

is a bijection between the left nullspaces over $\mathbb{F}$ of $R(\lambda_0)$ and $\mathcal{L}(\lambda_0)$.

Proof. We only prove (a) since (b) is analogous. We will see that E_r is a composition of maps, $E_r = G_r \circ H_r$, where G_r and H_r are linear bijections. For that, we consider the transfer function of $\mathcal{L}(\lambda)$. That is, the rational matrix

$$\widehat{R}(\lambda) = \begin{bmatrix} S(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \in \mathbb{F}(\lambda)^{q\times r}. \tag{20}$$

By Lemma 2.1, the linear map

$$\begin{aligned} G_r \,:\, \mathcal{N}_r(\widehat{R}(\lambda_0)) &\longrightarrow \mathcal{N}_r(\mathcal{L}(\lambda_0)) \\ \hat{x} &\longmapsto \begin{bmatrix} -A(\lambda_0)^{-1}\begin{bmatrix} B(\lambda_0) & 0 \end{bmatrix} \\ I_r \end{bmatrix} \hat{x} \end{aligned}$$

is a bijection between the right nullspaces of $\widehat{R}(\lambda_0)$ and $\mathcal{L}(\lambda_0)$, since $\widehat{R}(\lambda)$ is the transfer function of $\mathcal{L}(\lambda)$ and $\det A(\lambda_0) \neq 0$. Now, we consider the linear map

$$\begin{aligned} H_r \,:\, \mathcal{N}_r(R(\lambda_0)) &\longrightarrow \mathcal{N}_r(\widehat{R}(\lambda_0)) \\ x &\longmapsto \begin{bmatrix} N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x. \end{aligned}$$

To see that H_r is well defined, we consider the rational matrix $V_2(\lambda)^{-1} = \begin{bmatrix} \bar{N}_2(\lambda)^T & N_2(\lambda)^T \end{bmatrix}$ in (19). Then, we have that

$$\begin{bmatrix} V_2(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix} \begin{bmatrix} N_1(\lambda)^T \\ -\bar{N}_2(\lambda)S(\lambda)N_1(\lambda)^T \end{bmatrix} = \begin{bmatrix} 0 \\ R(\lambda) \\ 0 \end{bmatrix}. \tag{21}$$

By (21), H_r is well defined since the matrix $\begin{bmatrix} V_2(\lambda)^{-T} & 0 \\ 0 & I \end{bmatrix}$ is invertible in Ω, by Lemma 4.1. In addition, since $\mathcal{L}(\lambda)$ is a linearization of $R(\lambda)$ in Ω, we have that $\widehat{R}(\lambda)$ and $\operatorname{diag}(R(\lambda), I_s)$ are equivalent in Ω for some $s > 0$ (see Definition 2.2). Therefore, $\dim \mathcal{N}_r(R(\lambda_0)) = \dim \mathcal{N}_r(\widehat{R}(\lambda_0))$. Thus, to see that H_r is a bijection we only have to prove that H_r is injective. Assume that $\begin{bmatrix} N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x = 0$ for some $x \in \mathcal{N}_r(R(\lambda_0))$. In particular, $N_1(\lambda_0)^T x = 0$. Since $N_1(\lambda_0)^T$ has full column rank, $x = 0$. Therefore, H_r is a bijection. Finally, note that $E_r = G_r \circ H_r$. □

Remark 4.3. We recall that, since the linear maps E_r and E_ℓ are bijections, one can recover a basis of the right (resp. left) nullspace of $R(\lambda_0)$ from a basis of the right (resp. left) nullspace of $\mathcal{L}(\lambda_0)$, and conversely. For instance if $\{x_i\}_{i=1}^t$ is a basis

of $\mathcal{N}_r(R(\lambda_0))$ then $\{\begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0)N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x_i\}_{i=1}^t$ is a basis of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$. And, given a basis $\{y_i\}_{i=1}^t$ of $\mathcal{N}_r(\mathcal{L}(\lambda_0))$, we can recover a basis of $\mathcal{N}_r(R(\lambda_0))$ taking into account that the elements will be of the form $y_i = \begin{bmatrix} -A(\lambda_0)^{-1}B(\lambda_0)N_1(\lambda_0)^T \\ N_1(\lambda_0)^T \\ -\bar{N}_2(\lambda_0)S(\lambda_0)N_1(\lambda_0)^T \end{bmatrix} x_i$ for some basis $\{x_i\}_{i=1}^t$ of $\mathcal{N}_r(R(\lambda_0))$. Then, to recover $\{x_i\}_{i=1}^t$ we can use, in general, that $\bar{K}_1(\lambda_0)N_1(\lambda_0)^T x_i = x_i$. However, in practice, it can usually be recovered from the structure of $N_1(\lambda_0)^T x_i$, without computing $\bar{K}_1(\lambda_0)$, since $N_1(\lambda_0)^T$ has often a simple block structure including an identity block or a multiple of an identity block.

5. Block full rank linearizations for "the automatic rational approximations of NLEPs" in [21]

In this section we study the precise properties of the linearizations for rational approximations of nonlinear eigenvalue problems (NLEPs) in [21]. We will see that they satisfy Theorem 3.2 in a particular subset of $\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, and in the whole field $\mathbb{C}$ under mild conditions.

We start by defining a NLEP [17]. Given a non-empty open set $\Omega \subseteq \mathbb{C}$ and an analytic matrix-valued function

$$\begin{array}{rccc} F: & \Omega & \to & \mathbb{C}^{n\times n} \\ & \lambda & \mapsto & F(\lambda), \end{array}$$

the NLEP consists of computing scalars $\lambda_0 \in \Omega$ (eigenvalues) and nonzero vectors $v \in \mathbb{C}^n$ (eigenvectors) such that

$$F(\lambda_0)v = 0,$$

under the regularity assumption $\det(F(\lambda)) \not\equiv 0$. Since a direct solution of NLEPs is usually not possible, they are often solved numerically via rational approximation and by solving the corresponding REP with linearizations adapted to the structure of the obtained rational matrix. An approach to obtain an automatic rational approximation for the NLEP in a region Ω is given in [21]. The authors of [21] consider the $n \times n$ nonlinear matrix $F(\lambda)$ of the NLEP written in the form

$$F(\lambda) = Q(\lambda) + \sum_{i=1}^{s}(C_i - \lambda D_i)g_i(\lambda) \tag{22}$$

with $Q(\lambda)$ a polynomial matrix, C_i and D_i $n\times n$ constant matrices, and $g_i(\lambda)$ nonlinear scalar functions. Then, a CORK linearization of $Q(\lambda)$ of those introduced in [30] is considered, and the functions $g_i(\lambda)$ are approximated by rational functions employing the adaptive Antoulas–Anderson (AAA) algorithm [26], or its set-valued generalization presented in [21]. We recall the definition of CORK linearization as given in [21].

Definition 5.1. Let $G(\lambda)$ be an $n \times n$ rational matrix written as

$$G(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda), \tag{23}$$

where $f_i(\lambda)$ are scalar rational functions with $f_0(\lambda) \equiv 1$, and A_i, B_i are $n \times n$ constant matrices. Define

$$f(\lambda) := [f_0(\lambda) \quad \cdots \quad f_{k-1}(\lambda)]^T,$$

and assume that the rational functions $f_i(\lambda)$ satisfy a linear relation

$$(X - \lambda Y) f(\lambda) = 0, \tag{24}$$

where $\operatorname{rank}(X - \lambda Y) = k - 1$ for all $\lambda \in \mathbb{C}$, and $X - \lambda Y$ has size $(k-1) \times k$. Then the matrix pencil

$$\mathcal{L}_G(\lambda) = \left[\begin{array}{ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \\ \hdashline & (X - \lambda Y) \otimes I_n & \end{array} \right]$$

is called a CORK linearization of $G(\lambda)$.

If the vector $f(\lambda)$ is polynomial then $G(\lambda)$ in (23) is a polynomial matrix and $\mathcal{L}_G(\lambda)$ is a linearization of $G(\lambda)$ in $\mathbb{C}$, in particular, $\mathcal{L}_G(\lambda)$ is a block full rank linearization of $G(\lambda)$ in $\mathbb{C}$ with empty state matrix. However, if $f(\lambda)$ is a rational vector then $\mathcal{L}_G(\lambda)$ is not, in general, a linearization in the sense of [4], that is, in $\mathbb{C}$ but it is a linearization in a local sense [13]. More precisely, it is a block full rank linearization in all the subsets where the rational vector $f(\lambda)$ is defined, i.e., has no poles. Such result is stated in the next theorem.

Theorem 5.2. Let Ω be a nonempty subset of $\mathbb{C}$ where the rational vector $f(\lambda)$ in (24) is defined. Then a CORK linearization $\mathcal{L}_G(\lambda)$ of a rational matrix $G(\lambda)$ as in (23) is a block full rank linearizacion of $G(\lambda)$ in Ω with empty state matrix.

Proof. Notice that, by taking $M(\lambda) := \left[\begin{array}{ccc} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \end{array} \right]$, $K_1(\lambda) := (X - \lambda Y) \otimes I_n$, and $K_2(\lambda)$ empty, $\mathcal{L}_G(\lambda)$ is a block full rank pencil. Moreover, $f(\lambda)^T$ is a rational basis dual to $X - \lambda Y$. Then we apply Remark 3.3 with $N_1(\lambda) := f(\lambda)^T \otimes I_n$ and $N_2(\lambda) = I_n$. $\square$

Once CORK linearizations and some of their properties have been revised, we recall the AAA approximation of scalar functions. A given nonlinear function $g : \mathbb{C} \longrightarrow \mathbb{C}$ is approximated in [21] on a set $\Sigma \subset \mathbb{C}$ by a rational function $r(\lambda)$ in barycentric form, that is,

$$r(\lambda) = \sum_{j=1}^{m} \frac{g(z_j) w_j}{\lambda - z_j} \Big/ \sum_{j=1}^{m} \frac{w_j}{\lambda - z_j}, \tag{25}$$

where $z_1, \ldots, z_m$ are distinct support points and $w_1, \ldots, w_m$ are nonzero weights, that can be automatically chosen as explained in [26]. In this case, $\lim_{\lambda \to z_j} r(\lambda) = g(z_j)$. It is

shown in [21, Proposition 2.1] that (25) can be written as

$$r(\lambda) = [g(z_1)w_1 \quad \cdots \quad g(z_m)w_m] \begin{bmatrix} w_1 & w_2 & \cdots & w_{m-1} & w_m \\ \lambda - z_1 & z_2 - \lambda & & & \\ & \lambda - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda & \\ & & & \lambda - z_{m-1} & z_m - \lambda \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where it is easy to check that the inverted pencil is regular whenever $z_1, \ldots, z_m$ are distinct and $w_1, \ldots, w_m$ are nonzero. That is, $r(\lambda)$ can be written as a generalized state-space realization. Then, the pencil

$$P(\lambda) := \left[\begin{array}{ccccc|c} w_1 & w_2 & \cdots & w_{m-1} & w_m & -1 \\ \lambda - z_1 & z_2 - \lambda & & & & 0 \\ & \lambda - z_2 & \ddots & & & \vdots \\ & & \ddots & z_{m-1} - \lambda & & \vdots \\ & & & \lambda - z_{m-1} & z_m - \lambda & 0 \\ \hline g(z_1)w_1 & g(z_2)w_2 & \cdots & g(z_{m-1})w_{m-1} & g(z_m)w_m & 0 \end{array}\right] := \left[\begin{array}{c|c} E - \lambda F & -b \\ \hline a^T & 0 \end{array}\right] \tag{26}$$

is a linear polynomial system matrix of $r(\lambda)$ (i.e., with transfer function $r(\lambda)$) by considering $E - \lambda F$ as state matrix. In order to know what pole and zero information of $r(\lambda)$ we can obtain from this realization, we consider in Proposition 5.4 the polynomial system matrix $P(\lambda)$ and we study its minimality. First, we prove Lemma 5.3. In both Proposition 5.4 and Lemma 5.3, we consider $r(\lambda)$ written as the following quotient of polynomials

$$r(\lambda) := \frac{p(\lambda)}{q(\lambda)}, \tag{27}$$

where

$$p(\lambda) := \left(\prod_{j=1}^{m} (\lambda - z_j)\right) \left(\sum_{j=1}^{m} \frac{g(z_j)w_j}{\lambda - z_j}\right) \text{ and } q(\lambda) := \left(\prod_{j=1}^{m} (\lambda - z_j)\right) \left(\sum_{j=1}^{m} \frac{w_j}{\lambda - z_j}\right).$$

Note that the representation of the rational function (27) might not be irreducible. We will see that the irreducibility of (27) is the key property for the minimality of $P(\lambda)$.

Lemma 5.3. The pencil $E - \lambda F$ in (26) is a strong block minimal bases pencil associated with the polynomial $q(\lambda)$ in (27).

Proof. We set

$$E - \lambda F = \left[\begin{array}{ccccc} w_1 & w_2 & \cdots & w_{m-1} & w_m \\ \hline \lambda - z_1 & z_2 - \lambda & & & \\ & \lambda - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda & \\ & & & \lambda - z_{m-1} & z_m - \lambda \end{array}\right] =: \left[\begin{array}{c} M \\ \hline K(\lambda) \end{array}\right]. \tag{28}$$

Since $z_1, \ldots, z_m$ are distinct, $K(\lambda_0)$ has full row rank for all $\lambda_0 \in \mathbb{C}$. In addition, $K(\lambda)$ is row reduced because its highest row degree coefficient matrix

$$K_{hr} = \begin{bmatrix} 1 & -1 & 0 & & & \\ & 1 & -1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -1 & 0 \\ & & & & 1 & -1 \end{bmatrix}$$

has full row rank. We conclude that $K(\lambda)$ is a minimal basis. Let us denote

$$N(\lambda) := \prod_{j=1}^{m} (\lambda - z_j) \begin{bmatrix} \dfrac{1}{\lambda - z_1} & \cdots & \dfrac{1}{\lambda - z_m} \end{bmatrix}. \tag{29}$$

Then, it is not difficult to prove that $N(\lambda)$ is also a minimal basis, taking again into account that $z_1, \ldots, z_m$ are distinct. Moreover, since $K(\lambda)N(\lambda)^T = 0$ and $\begin{bmatrix} K(\lambda) \\ N(\lambda) \end{bmatrix}$ is a square matrix, we have that $K(\lambda)$ and $N(\lambda)$ are dual minimal bases. In addition, all the row degrees of $K(\lambda)$ are equal to 1 and the row degree of $N(\lambda)$ is equal to $m-1$. Hence, $E - \lambda F$ is a strong block minimal bases pencil associated with the polynomial matrix $MN(\lambda)^T = q(\lambda)$. □

$E - \lambda F$ being a strong block minimal bases pencil associated with the polynomial $q(\lambda)$ implies that $E - \lambda F$ is a (strong) linearization of $q(\lambda)$ and, in particular, that the determinant of $E - \lambda F$ is equal to $q(\lambda)$ up to a scalar multiple. This fact is used to prove the following result.

Proposition 5.4. Consider the rational function $r(\lambda)$ in (27) and its linear polynomial system matrix $P(\lambda)$ in (26). Then, $P(\lambda)$ is not minimal at $\lambda_0 \in \mathbb{C}$ if and only if λ_0 is a zero of both polynomials $p(\lambda)$ and $q(\lambda)$.

Proof. Consider $P(\lambda) = \left[\begin{array}{c|c} E - \lambda F & -b \\ \hline a^T & 0 \end{array}\right]$ as in (26). By Lemma 5.3,

$$\det(E - \lambda F) = \alpha q(\lambda) \text{ with } \alpha \neq 0. \tag{30}$$

In addition, since the Schur complement of $E - \lambda F$ in $P(\lambda)$ is $r(\lambda)$, we have that

$$\det(P(\lambda)) = \det(E - \lambda F) r(\lambda) = \alpha q(\lambda) \frac{p(\lambda)}{q(\lambda)} = \alpha p(\lambda). \tag{31}$$

Now, assume that λ_0 is a zero of both polynomials $p(\lambda)$ and $q(\lambda)$. That is, we can cancel out at least one factor of the form $(\lambda - \lambda_0)$ in both numerator and denominator of $r(\lambda)$. Then, the algebraic multiplicity of λ_0 as a zero of $r(\lambda)$ is not the same as the algebraic multiplicity of λ_0 as a zero of $P(\lambda)$. Therefore, $P(\lambda)$ is not minimal at λ_0. For the converse, assume that $P(\lambda)$ is not minimal at λ_0. Then, rank $\begin{bmatrix} E - \lambda_0 F \\ a^T \end{bmatrix} < m$,

since rank $\begin{bmatrix} E-\lambda_0 F & -b \end{bmatrix} = m$. Actually, rank $\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix} = m-1$, as the sub-matrix

$$K(\lambda_0) = \begin{bmatrix} \lambda_0 - z_1 & z_2 - \lambda_0 & & & \\ & \lambda_0 - z_2 & \ddots & & \\ & & \ddots & z_{m-1} - \lambda_0 & \\ & & & \lambda_0 - z_{m-1} & z_m - \lambda_0 \end{bmatrix}$$

contains a non-zero minor of order $m-1$ for all $\lambda_0 \in \mathbb{C}$, where $K(\lambda)$ is the matrix appearing in (28). Therefore, by using the notation $\mathcal{N}_r(\cdot)$ for the right nullspace, $\dim \mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) = 1$ and $\dim \mathcal{N}_r(K(\lambda_0)) = 1$. Then, $\mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) = \mathcal{N}_r(K(\lambda_0))$, since $\mathcal{N}_r\left(\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix}\right) \subseteq \mathcal{N}_r(K(\lambda_0))$ and both have the same dimension. Actually, $\mathcal{N}_r(K(\lambda_0)) = \mathrm{Span}\{N(\lambda_0)^T\}$, where $N(\lambda)$ is the polynomial matrix in (29). Hence, $\begin{bmatrix} E-\lambda_0 F \\ a^T \end{bmatrix} N(\lambda_0)^T = 0$ and, therefore, $\begin{bmatrix} w_1 & w_2 & \cdots & w_m \end{bmatrix} N(\lambda_0)^T = 0$ and $\begin{bmatrix} g(z_1)w_1 & g(z_2)w_2 & \cdots & g(z_m)w_m \end{bmatrix} N(\lambda_0)^T = 0$. That is, λ_0 is a root of both $q(\lambda)$ and $p(\lambda)$.

□

With these tools at hand, we go back to the original NLEP. Let $F(\lambda)$ be the nonlinear matrix function in (22). Then, each function $g_i(\lambda)$ is approximated in [21] on a set $\Sigma \subset \mathbb{C}$ by a rational function $r_i(\lambda)$ as in (25), i.e.,

$$g_i(\lambda) \approx r_i(\lambda) = \sum_{j=1}^{\ell_i} \frac{g_i(z_j^i) w_j^i}{\lambda - z_j^i} \Big/ \sum_{j=1}^{\ell_i} \frac{w_j^i}{\lambda - z_j^i},$$

where ℓ_i is the number of support points z_j^i and weights w_j^i for each $i = 1, \ldots, s$. For that, one can use the AAA algorithm on each function $g_i(\lambda)$ separately [26], or one can use the set-valued AAA algorithm in [21, Section 2.2], so that the rational approximants $r_i(\lambda)$ are all constructed simultaneously and sharing the same support points $z_j^i := z_j$ and weights $w_j^i := w_j$. By using any of the two approaches above, the following approximation of $F(\lambda)$ on Σ is obtained:

$$F(\lambda) \approx R(\lambda) := Q(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) r_i(\lambda). \tag{32}$$

Next, the polynomial matrix $Q(\lambda)$ is expressed in the form of (23), i.e., $Q(\lambda) := \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda)$, assuming the functions $f_i(\lambda)$ are polynomials, with $f_0(\lambda) = 1$, and each $r_i(\lambda)$ is written in generalized state-space form, that is,

$$R(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i) f_i(\lambda) + \sum_{i=1}^{s} (C_i - \lambda D_i) a_i^T (E_i - \lambda F_i)^{-1} b_i, \tag{33}$$

with $a_i = \begin{bmatrix} g_i(z_1^i)w_1^i & \cdots & g_i(z_{\ell_i}^i)w_{\ell_i}^i \end{bmatrix}^T \in \mathbb{C}^{\ell_i}$, $b_i = [1 \quad 0 \quad \cdots \quad 0]^T \in \mathbb{C}^{\ell_i}$ and $\ell_i \times \ell_i$ matrices

$$E_i = \begin{bmatrix} w_1^i & w_2^i & \cdots & w_{\ell_i-1}^i & w_{\ell_i}^i \\ -z_1^i & z_2^i & & & \\ & -z_2^i & \ddots & & \\ & & \ddots & z_{\ell_i-1}^i & \\ & & & -z_{\ell_i-1}^i & z_{\ell_i}^i \end{bmatrix} \text{ and } F_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & & & \\ & -1 & \ddots & & \\ & & \ddots & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

The linearization constructed in [21] for $R(\lambda)$ is the following.

Definition 5.5. [21, Definition 3.2](CORK linearization for AAA rational approximation) Let $R(\lambda)$ be a rational matrix as in (33). Consider $b := [b_1^T \quad \cdots \quad b_s^T]^T$ and $E - \lambda F := \operatorname{diag}(E_1 - \lambda F_1, \ldots, E_s - \lambda F_s)$. Then a CORK linearization for $R(\lambda)$ is

$$\mathcal{L}_R(\lambda) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1}} & a_1^T \otimes (C_1 - \lambda D_1) \quad \cdots \quad a_s^T \otimes (C_s - \lambda D_s) \\ \hline \multicolumn{2}{c|}{(X - \lambda Y) \otimes I_n} & 0 \\ \hline -b \otimes I_n & 0 & (E - \lambda F) \otimes I_n \end{array}\right]$$

where $\left[\begin{array}{c} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} \\ \hline (X - \lambda Y) \otimes I_n \end{array}\right]$ is any CORK linearization of $Q(\lambda)$.

In particular, for the set-valued AAA approximation, all the matrices $E_i - \lambda F_i$ in (33) are the same for all i, as well as all the vectors b_i. Then, in [21, Remark 3.3] the following CORK linearization for AAA rational approximation is considered

$$\mathcal{L}_R^{sv}(\lambda) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1}} & \sum_{i=1}^{s} a_i^T \otimes (C_i - \lambda D_i) \\ \hline \multicolumn{2}{c|}{(X - \lambda Y) \otimes I_n} & 0 \\ \hline -b_1 \otimes I_n & 0 & (E_1 - \lambda F_1) \otimes I_n \end{array}\right] \tag{34}$$

Notice that $\mathcal{L}_R^{sv}(\lambda)$ has size $(kn + \ell_1 n) \times (kn + \ell_1 n)$ whereas $\mathcal{L}_R(\lambda)$ in Definition 5.5 has size $(kn + \sum_{i=1}^{s} \ell_i n) \times (kn + \sum_{i=1}^{s} \ell_i n)$.

By using Theorem 3.2, we study in Theorem 5.6 the structure of $\mathcal{L}_R(\lambda)$ and $\mathcal{L}_R^{sv}(\lambda)$ as linearizations of $R(\lambda)$.

Theorem 5.6. Let $R(\lambda)$ be a rational matrix as in (33), and let $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$) be the matrix pencil in Definition 5.5 (resp., in (34)). Let $\Omega \subseteq \mathbb{C}$ be nonempty. If $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$), viewed as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$ (resp., $(E_1 - \lambda F_1) \otimes I_n$), is minimal in Ω then $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$) is a block full rank linearization of $R(\lambda)$ in Ω with state matrix $(E - \lambda F) \otimes I_n$ (resp., $(E_1 - \lambda F_1) \otimes I_n$).

Proof. We only prove the result for $\mathcal{L}_R(\lambda)$, because the proof for $\mathcal{L}_R^{sv}(\lambda)$ is similar. Set $M(\lambda) := \begin{bmatrix} A_0 - \lambda B_0 & \cdots & A_{k-1} - \lambda B_{k-1} \end{bmatrix}$, $C(\lambda) := -[a_1^T \otimes (C_1 - \lambda D_1) \quad \cdots \quad a_s^T \otimes (C_s - \lambda D_s)]$, $B := [-b \otimes I_n \quad 0]$, $A(\lambda) := (E - \lambda F) \otimes I_n$, $K_1(\lambda) := (X - \lambda Y) \otimes I_n$, $N_1(\lambda) := (f(\lambda) \otimes I_n)^T$, and $K_2(\lambda)$ empty. Observe that $K_1(\lambda)$ and $N_1(\lambda)$ have full row rank in $\mathbb{C}$. $\mathcal{L}_R(\lambda)$ being minimal in Ω implies that condition (9) is satisfied in Ω since

$BN_1(\lambda)^T = -b \otimes I_n$ because $f_0(\lambda) = 1$. Then, by Theorem 3.2, $\mathcal{L}_R(\lambda)$ is a linearization of $[M(\lambda) + C(\lambda)A(\lambda)^{-1}B](f(\lambda) \otimes I_n) = R(\lambda)$ in Ω with state matrix $A(\lambda)$. $\square$

Remark 5.7. Theorem 5.6 also holds if $f(\lambda)$ is rational but, in such a case, we need the extra hypothesis of $f(\lambda)$ being defined in Ω.

According to Theorem 5.6, we need minimality on $\mathcal{L}_R(\lambda)$ (resp., $\mathcal{L}_R^{sv}(\lambda)$) to be a linearization of the rational matrix $R(\lambda)$. In the following Theorem 5.8, we give sufficient mild conditions for $\mathcal{L}_R(\lambda)$ to be minimal in $\mathbb{C}$ in the case the rational approximants $r_i(\lambda)$ do not share the same support points and weights. That is, when the functions $g_i(\lambda)$ are approximated employing the adaptive Antoulas–Anderson (AAA) algorithm in [26] separately. For the set-valued AAA approximation [21], i.e., the rational approximants $r_i(\lambda)$ sharing the same support points and weights, the authors in [21] consider the pencil in (34), and we state minimality conditions for it in Theorem 5.9. Recall that the pencils $E_i - \lambda F_i$ are regular.

Theorem 5.8. Assume that, for $i = 1, \ldots, s$, the rational functions $r_i(\lambda)$ in (32) are represented as in (27) and that this representation is irreducible. Let $\mathcal{L}_R(\lambda)$ be the matrix pencil in Definition 5.5. If the pencils $C_i - \lambda D_i$ are regular for $i = 1, \ldots, s$ and the following conditions hold

(a) $C_i - \lambda D_i$ and $E_i - \lambda F_i$ have no finite eigenvalues in common for $i = 1, \ldots, s$, and
(b) $E_i - \lambda F_i$ and $E_j - \lambda F_j$ with $i \neq j$ have no finite eigenvalues in common for $i, j = 1, \ldots, s$,

then $\mathcal{L}_R(\lambda)$, viewed as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$, is minimal in $\mathbb{C}$.

Proof. Assume first that $s = 1$. Then notice that $\mathcal{L}_R(\lambda)$ is minimal in $\mathbb{C}$ if and only if the pencil

$$S(\lambda) := \left[\begin{array}{c|c} 0 & a_1^T \otimes (C_1 - \lambda D_1) \\ \hline -b_1 \otimes I_n & (E_1 - \lambda F_1) \otimes I_n \end{array}\right],$$

considered as a polynomial system matrix with state matrix $(E_1 - \lambda F_1) \otimes I_n$, is minimal in $\mathbb{C}$. Since $r_1(\lambda)$ is irreducible, we have, by Proposition 5.4, that the submatrix $\left[\begin{array}{c|c} -b_1 \otimes I_n & (E_1 - \lambda F_1) \otimes I_n \end{array}\right]$ has full row rank for all $\lambda \in \mathbb{C}$. Then we only have to prove that the submatrix $H(\lambda) := \left[\begin{array}{c} a_1^T \otimes (C_1 - \lambda D_1) \\ \hline (E_1 - \lambda F_1) \otimes I_n \end{array}\right]$ has full column rank for all $\lambda \in \mathbb{C}$. By contradiction, assume that $H(\lambda_0)$ has no full column rank for some $\lambda_0 \in \mathbb{C}$. Notice that, in such a case, λ_0 must be an eigenvalue of $E_1 - \lambda F_1$ since, otherwise, $H(\lambda_0)$ would have full column rank. In addition, there exists a nonzero vector x such that $H(\lambda_0)x = 0$. Now we write

$$H(\lambda_0)x = \begin{bmatrix} C_1 - \lambda_0 D_1 & 0 \\ 0 & I_{\ell_1 n} \end{bmatrix} \left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right] x = 0, \tag{35}$$

and define the vector $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} := \left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right] x$, which is nonzero since $x \neq$

0 and the matrix $\left[\begin{array}{c} a_1^T \otimes I_n \\ \hline (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right]$ has full column rank by Proposition 5.4. Moreover, by (35), we have that $y_2 = 0$ and, thus, $(C_1 - \lambda_0 D_1) y_1 = 0$ with $y_1 \neq 0$. Therefore, λ_0 is an eigenvalue of $C_1 - \lambda D_1$, which is a contradiction by condition (a). Finally, if $s > 1$ we have to take into account condition (b) and the result follows. $\square$

For the set-valued AAA approximation, the minimality conditions we provide on $\mathcal{L}_R^{sv}(\lambda)$ are milder and are necessary and sufficient. We state them in the following Theorem 5.9. This case allows a more complete simple analysis because the denominators of the rational functions $r_i(\lambda)$, for $i = 1, \ldots, s$, when they are represented as in (27), are all equal to the same polynomial $q(\lambda)$. Note also that the roots of $q(\lambda)$ are the finite eigenvalues of $E_1 - \lambda F_1$ according to Lemma 5.3. In Theorem 5.9, we assume that $q(\lambda)$ is not constant, since otherwise $R(\lambda)$ in (32) is a polynomial matrix.

Theorem 5.9. Assume that, for $i = 1, \ldots, s$, the rational functions $r_i(\lambda)$ in (32) share the same support points $z_1, \ldots, z_{\ell_1}$, which are distinct, and the same nonzero weights $w_1, \ldots, w_{\ell_1}$, and that are represented as in (27), i.e., $r_i(\lambda) = p_i(\lambda)/q(\lambda)$, where the denominators are equal for all i. Assume, in addition, that $q(\lambda)$ is not a constant polynomial. Let $\mathcal{L}_R^{sv}(\lambda)$ be the matrix pencil in (34). Then, $\mathcal{L}_R^{sv}(\lambda)$, viewed as a polynomial system matrix with state matrix $(E_1 - \lambda F_1) \otimes I_n$, is minimal in $\mathbb{C}$ if and only if the polynomial matrix $P^{sv}(\lambda) = \sum_{i=1}^{s} p_i(\lambda)\,(C_i - \lambda D_i)$ is regular and has no finite eigenvalues in common with the pencil $E_1 - \lambda F_1$.

Proof. The proof shares several arguments with that of Proposition 5.4. Observe first that $\mathcal{L}_R^{sv}(\lambda)$ is minimal in $\mathbb{C}$ if and only if the pencil

$$S^{sv}(\lambda) := \left[\begin{array}{c|c} 0 & \displaystyle\sum_{i=1}^{s} a_i^T \otimes (C_i - \lambda D_i) \\ \hline -\, b_1 \otimes I_n & (E_1 - \lambda F_1) \otimes I_n \end{array}\right],$$

considered as a polynomial system matrix with state matrix $(E_1 - \lambda F_1) \otimes I_n$, is minimal in $\mathbb{C}$. Thus, we will prove the result for $S^{sv}(\lambda)$, which has as transfer function $(1/q(\lambda))\, P^{sv}(\lambda)$ as a consequence of (26).

Assume that $S^{sv}(\lambda)$ is minimal in $\mathbb{C}$. Then, the finite poles of $(1/q(\lambda))\, P^{sv}(\lambda)$ are exactly the finite zeros (eigenvalues) of $(E_1 - \lambda F_1) \otimes I_n$, with the same partial multiplicities. Since $\det((E_1 - \lambda F_1) \otimes I_n) = \beta\, q(\lambda)^n$ with β a nonzero number (this follows from Lemma 5.3 and [19, p. 249]), the product of the denominators of the nonzero entries of the finite Smith-McMillan form of $(1/q(\lambda))\, P^{sv}(\lambda)$ is a nonzero scalar multiple of $q(\lambda)^n$. Such Smith-McMillan form is a scalar multiple of $\mathrm{diag}(i_1(\lambda)/q(\lambda), \ldots, i_r(\lambda)/q(\lambda), 0, \ldots, 0) \in \mathbb{F}(\lambda)^{n \times n}$, where $\mathrm{diag}(i_1(\lambda), \ldots, i_r(\lambda), 0, \ldots, 0) \in \mathbb{F}[\lambda]^{n \times n}$ is the Smith form of the polynomial matrix $P^{sv}(\lambda)$. Thus, $r = n$, i.e., $P^{sv}(\lambda)$ is regular, and $i_j(\lambda)$ and $q(\lambda)$ have no roots in common for $j = 1, \ldots, n$.

Next, we prove the converse. Assume that the polynomial matrix $P^{sv}(\lambda)$ is regular and has no finite eigenvalues in common with the pencil $E_1 - \lambda F_1$. Note that $\left[\begin{array}{c|c} -b_1 \otimes I_n & (E_1 - \lambda_0 F_1) \otimes I_n \end{array}\right]$ has full row rank for all $\lambda_0 \in \mathbb{C}$, as we saw in the

proof of Proposition 5.4. Thus, it only remains to prove that

$$H_0^{sv} := \begin{bmatrix} \sum_{i=1}^{s} a_i^T \otimes (C_i - \lambda_0 D_i) \\ (E_1 - \lambda_0 F_1) \otimes I_n \end{bmatrix}$$

has full column rank for all $\lambda_0 \in \mathbb{C}$. According to [19, Corollary 4.3.10], H_0^{sv} is permutationally equivalent to

$$X_0^{sv} := \begin{bmatrix} \sum_{i=1}^{s} (C_i - \lambda_0 D_i) \otimes a_i^T \\ I_n \otimes (E_1 - \lambda_0 F_1) \end{bmatrix},$$

and we have to prove that X_0^{sv} has full column rank for all $\lambda_0 \in \mathbb{C}$. This is obviously true if λ_0 is not an eigenvalue of $E_1 - \lambda F_1$. Therefore, we assume that λ_0 is an eigenvalue of $E_1 - \lambda F_1$ in the rest of the proof, which implies by hypothesis that λ_0 is not an eigenvalue of $P^{sv}(\lambda)$. Let $y \in \mathbb{C}^{n\ell_1}$ be such that $X_0^{sv} y = 0$ and partition $y = [y_1^T \; y_2^T \; \cdots \; y_n^T]^T$ with $y_i \in \mathbb{C}^{\ell_1}$ for $i = 1, \ldots, n$. Thus,

$$\left(\sum_{i=1}^{s} (C_i - \lambda_0 D_i) \otimes a_i^T \right) y = 0 \quad \text{and} \quad (E_1 - \lambda_0 F_1) y_j = 0, \quad j = 1, \ldots, n. \tag{36}$$

An argument similar to that in the proof of Proposition 5.4 establishes that $\mathcal{N}_r(E_1 - \lambda_0 F_1) = \mathcal{N}_r(K(\lambda_0)) = \mathrm{Span}\{N(\lambda_0)^T\}$, where $N(\lambda)$ is the polynomial vector in (29) for $z_1, \ldots, z_{\ell_1}$ and $K(\lambda)$ is the pencil in (28) corresponding to $E_1 - \lambda F_1$. This implies that $y_j = \alpha_j \, N(\lambda_0)^T$, with $\alpha_j \in \mathbb{C}$, and that $y = v \otimes N(\lambda_0)^T$, with $v = [\alpha_1, \ldots, \alpha_n]^T \in \mathbb{C}^n$. Using this expression for y in the first equation of (36), one gets

$$\sum_{i=1}^{s} ((C_i - \lambda_0 D_i) v) \otimes (a_i^T N(\lambda_0)^T) = \left(\sum_{i=1}^{s} p_i(\lambda_0) \, (C_i - \lambda_0 D_i) \right) v = P^{sv}(\lambda_0) \, v = 0.$$

This implies that $v = 0$, because λ_0 is not an eigenvalue of the regular polynomial matrix $P^{sv}(\lambda)$, and $y = 0$. Therefore, X_0^{sv} has full column rank and H_0^{sv} has also full column rank.

□

Remark 5.10. It is clear that if we consider the set $\Omega := \{\lambda \in \mathbb{C} : E - \lambda F \text{ is invertible}\}$, then $\mathcal{L}_R(\lambda)$ is minimal in Ω and, by Theorem 5.6, $\mathcal{L}_R(\lambda)$ is a block full rank linearization of $R(\lambda)$ in Ω, with state matrix $(E - \lambda F) \otimes I_n$. However, in such a case we do not obtain any information about the poles of $R(\lambda)$ since they do not belong to Ω. For this particular choice of the set Ω, the fact that $\mathcal{L}_R(\lambda)$ is a linearization of $R(\lambda)$ in Ω can also be proved by considering $\mathcal{L}_R(\lambda)$ as a block full rank pencil of the form

$$\mathcal{L}_R(\lambda) := \begin{bmatrix} M(\lambda) \\ \hdashline K_1(\lambda) \end{bmatrix},$$

with

$$M(\lambda) := [\quad A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} \quad | \quad a_1^T \otimes (C_1 - \lambda D_1) \quad \cdots \quad a_s^T \otimes (C_s - \lambda D_s) \quad],$$

and by applying Remark 3.3. For that, write $R(\lambda)$ as

$$R(\lambda) = \sum_{i=0}^{k-1} (A_i - \lambda B_i)(f_i(\lambda) \otimes I_n) + \sum_{i=1}^{s} [a_i^T \otimes (C_i - \lambda D_i)](R_i(\lambda) \otimes I_n),$$

with $R_i(\lambda) := (E_i - \lambda F_i)^{-1} b_i$, and consider the dual rational basis of $K_1(\lambda)$

$$N_1(\lambda) := [f_0(\lambda) \quad \cdots \quad f_{k-1}(\lambda) \quad | \quad R_1(\lambda)^T \quad \cdots \quad R_s(\lambda)^T] \otimes I_n.$$

Then, $\mathcal{L}_R(\lambda)$ is a linearization of $R(\lambda)$ in Ω with empty state matrix. On the other hand, if $\mathcal{L}_R(\lambda)$ (considering the partition with state matrix $(E-\lambda F)\otimes I_n$) were minimal at those $\lambda_0 \in \mathbb{C}$ such that $E - \lambda_0 F$ is singular then $\mathcal{L}_R(\lambda)$ would be a linearization of $R(\lambda)$ in $\mathbb{C}$ with state matrix $(E - \lambda F) \otimes I_n$. That means that the zeros of $\mathcal{L}_R(\lambda)$ would be the zeros of $R(\lambda)$, and the zeros of $(E - \lambda F) \otimes I_n$ would be the poles of $R(\lambda)$, together with their partial multiplicities. This happens, for instance, under the conditions of Theorem 5.8.

Comments analogous to those in Remark 5.10 hold for $\mathcal{L}_R^{sv}(\lambda)$ in (34). They are omitted for brevity.

Remark 5.11. In Remark 5.10, we consider $\mathcal{L}_R(\lambda)$ from two different points of view: as a block full rank pencil, $\left[\begin{array}{c} M(\lambda) \\ \hdashline K_1(\lambda) \end{array}\right]$, and as a polynomial system matrix with state matrix $(E - \lambda F) \otimes I_n$. In the former case, $\mathcal{L}_R(\lambda)$ is not in general a linearization at infinity of $R(\lambda)$ since $\operatorname{rev}_1 K_1(\lambda)$ does not have full row rank at 0. In particular, [13, Theorem 5.5] can not be applied and there is not always an integer g such that $\operatorname{rev}_1 \mathcal{L}_R(\lambda)$ is equivalent at 0 to $\operatorname{diag}(\operatorname{rev}_g R(\lambda), I_{k(n-1)+\sum_{i=1}^{s} \ell_i n})$. It is not difficult to construct examples where such a g does not exist. In the latter case, $\mathcal{L}_R(\lambda)$ is not a linearization at infinity since $\operatorname{rev}_1 \mathcal{L}_R(\lambda)$ is not minimal at 0. Both cases are due to the fact that the matrix $\operatorname{rev}_1(E - \lambda F)$ does not have full row rank at zero since F is singular. Analogous remarks hold for $\mathcal{L}_R^{sv}(\lambda)$ in (34).

5.1. Low-rank structure

Low-rank structures are exploited in [21] for constructing smaller linearizations that allow more efficient computations. We consider this scenario in this subsection, where for the sake of brevity, we study only the case in which each function $g_i(\lambda)$ is approximated via an independent application of the AAA algorithm. More precisely, a trimmed linearization is constructed in [21] if the matrix coefficients $C_i - \lambda D_i$ in (22) have low rank. For this purpose, write

$$C_i - \lambda D_i = (\widetilde{C}_i - \lambda \widetilde{D}_i)\widetilde{Z}_i^*, \tag{37}$$

with $\widetilde{C}_i, \widetilde{D}_i, \widetilde{Z}_i \in \mathbb{C}^{n \times k_i}$, and $\widetilde{Z}_i^* \widetilde{Z}_i = I_{k_i}$. In several applied problems this type of structure appears with $k_i \ll n$ [18,21]. By using the expression (37) for the matrix

coefficients, the matrix $R(\lambda)$ in (33) can be written as:

$$\begin{aligned} R(\lambda) &= \sum_{i=0}^{k-1}(A_i - \lambda B_i)f_i(\lambda) + \sum_{i=1}^{s}(\widetilde{C}_i - \lambda\widetilde{D}_i)\widetilde{Z}_i^* a_i^T (E_i - \lambda F_i)^{-1} b_i \\ &= \sum_{i=0}^{k-1}(A_i - \lambda B_i)(f_i(\lambda)\otimes I_n) + \sum_{i=1}^{s}[a_i^T \otimes (\widetilde{C}_i - \lambda\widetilde{D}_i)]((E_i - \lambda F_i)^{-1} b_i \otimes I_{k_i})\widetilde{Z}_i^*. \end{aligned} \tag{38}$$

Then, the trimmed linearization $\widetilde{\mathcal{L}}_R(\lambda)$ for $R(\lambda)$ constructed in [21] is the following.

Definition 5.12. [21](Trimmed CORK linearization for AAA rational approximation) Let $R(\lambda)$ be a rational matrix as in (38). Consider the matrices

$$\begin{aligned} Z &:= \begin{bmatrix} -\widetilde{Z}_1(b_1^* \otimes I_{k_1}) & \cdots & -\widetilde{Z}_s(b_s^* \otimes I_{k_s}) \end{bmatrix}, \\ E &:= \operatorname{diag}(E_1 \otimes I_{k_1}, \ldots, E_s \otimes I_{k_s}), \text{ and} \\ F &:= \operatorname{diag}(F_1 \otimes I_{k_1}, \ldots, F_s \otimes I_{k_s}). \end{aligned}$$

Then a trimmed CORK linearization for $R(\lambda)$ is

$$\widetilde{\mathcal{L}}_R(\lambda) = \left[\begin{array}{cc|c} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} & & a_1^T \otimes (\widetilde{C}_1 - \lambda\widetilde{D}_1) \quad \cdots \quad a_s^T \otimes (\widetilde{C}_s - \lambda\widetilde{D}_s) \\ \hline (X - \lambda Y)\otimes I_n & & 0 \\ \hline Z^* & 0 & E - \lambda F \end{array}\right],$$

where $\left[\begin{array}{c} A_0 - \lambda B_0 \quad \cdots \quad A_{k-1} - \lambda B_{k-1} \\ \hline (X - \lambda Y)\otimes I_n \end{array}\right]$ is any CORK linearization of $Q(\lambda)$.

Notice that the linearization $\mathcal{L}_R(\lambda)$ in Definition 5.5 has size $(kn + \sum_{i=1}^{s}\ell_i n) \times (kn + \sum_{i=1}^{s}\ell_i n)$ whereas the trimmed pencil $\widetilde{\mathcal{L}}_R(\lambda)$ in Definition 5.12 has size $(kn + \sum_{i=1}^{s}\ell_i k_i) \times (kn + \sum_{i=1}^{s}\ell_i k_i)$ with $k_i \ll n$ in several applications.

Analogous to what we did in Theorem 5.6, we study in Theorem 5.13 the structure of $\widetilde{\mathcal{L}}_R(\lambda)$ as linearization of $R(\lambda)$. The proof is omitted since it is analogous to that of Theorem 5.6.

Theorem 5.13. Let $R(\lambda)$ be a rational matrix as in (38), and let $\widetilde{\mathcal{L}}_R(\lambda)$ be the matrix pencil in Definition 5.12. Let $\Omega \subseteq \mathbb{C}$ be nonempty. If $\widetilde{\mathcal{L}}_R(\lambda)$, viewed as a polynomial system matrix with state matrix $E - \lambda F$, is minimal in Ω then $\widetilde{\mathcal{L}}_R(\lambda)$ is a block full rank linearization of $R(\lambda)$ in Ω with state matrix $E - \lambda F$.

Remark 5.14. As we discussed in Remark 5.11 for the matrix pencil $\mathcal{L}_R(\lambda)$, the trimmed CORK linearization $\widetilde{\mathcal{L}}_R(\lambda)$ is not in general a linearization at infinity of $R(\lambda)$ either. The reason is that, in this case, the matrix F is also singular and $\operatorname{rev}_1(E - \lambda F)$ has not full row rank at zero.

6. Conclusions

Combining the theory in [13] with a nontrivial extension of the structure of the strong block minimal bases linearizations introduced in [4], we have constructed a new wide family of local linearizations of rational matrices that generalizes and includes most of the linearizations for rational matrices appearing in the literature. The linearizations in this family are called block full rank linearizations. Depending on the satisfied minimality conditions, a pencil in this family can be a linearization in a set of finite points and/or at infinity. If the minimality conditions are satisfied in the whole underlying field $\mathbb{F}$ and at infinity, simultaneously, then we can recover from block full rank linearizations the complete pole and zero information, finite and at infinity, of rational matrices. Linearizations at infinity are defined using the notion of grade and, to determine the grade of block full rank linearizations at infinity, we use the notion of degree of rational matrices.

As an application, we use block full rank linearizations to study the structure of the linearizations developed in [21] for solving rational eigenvalue problems coming from rational approximations of nonlinear eigenvalue problems. We provide sufficient mild conditions under which the pencils in [21] are linearizations in $\mathbb{C}$.

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