STRONG LINEARIZATIONS OF RATIONAL MATRICES*

A. AMPARAN†, F. M. DOPICO‡, S. MARCAIDA†, AND I. ZABALLA†

Abstract. This paper defines for the first time strong linearizations of arbitrary rational matrices, studies in depth properties and characterizations of such linear matrix pencils, and develops infinitely many examples of strong linearizations that can be explicitly and easily constructed from a minimal state-space realization of the strictly proper part of the considered rational matrix and the coefficients of the polynomial part. As a consequence, the results in this paper establish a rigorous foundation for the numerical computation of the complete structure of zeros and poles, both finite and at infinity, of any rational matrix by applying any well known backward stable algorithm for generalized eigenvalue problems to any of the strong linearizations constructed in this work.

Key words. linearization, minimal polynomial system matrix, nonlinear eigenvalue problem, rational matrix, strong block minimal bases linearization, strong linearization

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1. Introduction.

Given a nonsingular rational matrix $G(\lambda)$ (i.e., a matrix whose entries are rational functions) the rational eigenvalue problem (REP) is to find scalars λ and nonzero vectors x satisfying $G(\lambda)x = 0$. The scalars λ and the vectors x are called, respectively, eigenvalues and eigenvectors of the rational matrix $G(\lambda)$. The REP arises in many applications, either directly [30] or as approximation of other nonlinear eigenvalue problems [18], and several approaches can be used to solve it. Actually, in [30] a new method for solving numerically the REP is given based on the fact that any rational matrix $G(\lambda)$ can be uniquely written as the sum of a polynomial matrix and a strictly proper one. The method consists in applying any well established algorithm for computing the eigenvalues of a linear pencil [17] to a pencil constructed out of a linearization of the polynomial part of $G(\lambda)$ and a realization of its strictly proper part, which preserves the finite zeros of $G(\lambda)$. This method has been formalized and generalized in [1] where a precise definition of linearization of a square rational matrix is given.

The linearizations defined in [1] reflect the finite structure of rational matrices but no evidence is given that they preserve also the infinite structure. The main goal of the present paper is to provide a new definition of linearization of rational matrices that both: preserves the finite as well as the infinite poles and zeros of the original matrix and generalizes in a natural way the notion of strong linearization of matrix polynomials. We emphasize that this goal will be achieved in the general context of arbitrary rational matrices, i.e., square or rectangular, regular or singular, in contrast to the references [1, 30] which only consider square matrices. In addition, infinitely many of such linearizations will be explicitly constructed.

The definition of strong linearization is based on the following property of the minimal polynomial system matrices of a rational matrix $G(\lambda)$ established by Rosen-

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[†] Departamento de Matemática Aplicada y Estadística e Investigación Operativa, Universidad del País Vasco UPV/EHU, Apdo. Correos 644, Bilbao 4808, Spain (agurtzane.amparan@ehu.eus, silvia.marcaida@ehu.eus, ion.zaballa@ehu.eus).

[‡] Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (dopico@math.uc3m.es).

brock [28] (see Theorem 2.3): under very mild conditions if

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}$$

is a minimal polynomial system matrix giving rise to $G(\lambda)$ (i.e., $G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda)$) then the finite poles and zeros of $G(\lambda)$, counting with multiplicities, are the finite zeros of $A(\lambda)$ and $P(\lambda)$, respectively, counting with multiplicities. With this property in mind, a linear pencil $L(\lambda)$ is said to be a linearization of $G(\lambda)$ if it is a minimal polynomial system matrix of a rational matrix $\widehat{G}(\lambda)$ such that, for some nonnegative integers s_1, s_2, $\operatorname{Diag}(\widehat{G}(\lambda), I_{s_1})$ and $\operatorname{Diag}(G(\lambda), I_{s_2})$ are equivalent via unimodular polynomial matrices. This is Definition 3.2 which looks very much like the standard definition of linearization of polynomial matrices [16, 22, 24, 10]. It can be extended to preserve also the poles and zeros of $G(\lambda)$ at infinity leading to the concept of strong linearization (Definition 3.4). We will see that when $G(\lambda)$ is polynomial Definition 3.4 reduces to the definition of strong linearization of polynomial matrices.

Notions well-established in the theory of linear systems like polynomial system matrices, realizations, Smith–McMillan forms (finite and at infinity), strict system equivalence, and transfer function matrices play an important role in this paper. They will be reviewed in Section 2. In contrast with [1], polynomial system matrices of least order, or minimal, are relevant in our developments because the eigenvalues of the REP $G(\lambda)x = 0$ are the finite zeros of $G(\lambda)$ that are not finite poles, which leads to look for linearizations that preserve the poles of $G(\lambda)$ (with their partial multiplicities) but that do not incorporate spurious ones. See item 2 of Remark 3.3.

The definitions of linearization, weak and strong, are formally given in Section 3 where, among other things, it is proved that for polynomial matrices these definitions reduce to the usual ones. A spectral characterization of strong linearizations in the spirit of [10, Thm. 4.1] is provided in Subsection 3.1.

In view of the definition of strong linearization, it is important to determine when two polynomial system matrices give rise to rational matrices that are equivalent via unimodular matrices and also equivalent at infinity. This motivates to introduce the concepts of transfer system equivalence and of transfer system equivalence at infinity in Section 4. For a given rational matrix $G(\lambda)$, these equivalence relations give us the precise amount of freedom to obtain strong linearizations out of any polynomial system matrix whose transfer function matrix is $G(\lambda)$. Moreover, these equivalence relations allow us to obtain the practical characterization of strong linearization in Corollary 4.12. This corollary is used in Section 5 for constructing explicitly infinitely many strong linearizations of any rational matrix from any strong block minimal bases linearization of its polynomial part [12, Def. 3.1] and any minimal state-space realization of its strictly proper part. There exist infinitely many strong block minimal bases linearizations of any matrix polynomial, including Frobenius companion forms, all Fiedler linearizations [13, 6, 9] modulo permutations, and all block Kronecker linearizations [12, Def. 5.1]. Thus, we construct in this way a very wide class of strong linearizations of arbitrary rational matrices. Examples are provided in Subsection 5.3. The main conclusion of this work and possible lines of future research are discussed in Section 6.

We emphasize that, although the general definition and theory developed in this work are new to the best of our knowledge, the idea of constructing pencils that contain the complete finite and infinite structure of poles and zeros of rational matrices, as well as their minimal indices, has been considered before in the literature. Some examples can be found, for instance, in the classical reference [32].

2. Preliminaries. In this section we review the basic notions of linear system theory that we will use in the subsequent sections. Our basic references are [28, 21, 33].

Although for practical purposes the rational matrices of interest are those whose elements have real or complex coefficients, the results in this paper are of algebraic nature and apply for matrices with coefficients in arbitrary fields. Thus $\mathbb{F}$ will denote any arbitrary field, $\mathbb{F}[\lambda]$ the ring of polynomials with coefficients in $\mathbb{F}$ and $\mathbb{F}(\lambda)$ the field of *rational functions*, i.e., quotients of coprime polynomials of $\mathbb{F}[\lambda]$. A rational function $r(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ is said to be *proper* if $\deg(n(\lambda)) \leq \deg(d(\lambda))$, where $\deg(\cdot)$ stands for degree. If $\deg(n(\lambda)) < \deg(d(\lambda))$ then $r(\lambda)$ is called *strictly proper.* Let $\mathbb{F}(\lambda)^{p\times m}$ be the set of $p \times m$ matrices with elements in $\mathbb{F}(\lambda)$. Any rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be written as

$$G(\lambda) = D(\lambda) + C(\lambda)A(\lambda)^{-1}B(\lambda), \tag{1}$$

for some nonsingular matrix polynomial $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and matrix polynomials $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with $n \geq \deg(\det A(\lambda))$ (see [28]). The matrix polynomial

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \tag{2}$$

is called a *polynomial system matrix* of (or giving rise to) $G(\lambda)$. Then $G(\lambda)$ is called the *transfer function matrix* of $P(\lambda)$ and $\deg(\det A(\lambda))$ is its *order.* When $A(\lambda)$ is a monic linear matrix polynomial, say $A(\lambda) = \lambda I_n - A$, $B(\lambda) = B$ and $C(\lambda) = C$ are constant matrices, $P(\lambda)$ is said to be a polynomial system matrix of $G(\lambda)$ in *state-space form.*

The integer n or the polynomial matrices of (1) are not uniquely determined by $G(\lambda)$. It turns out that different polynomial system matrices may exist with different orders giving rise to the same transfer function matrix. For example, for any nonsingular polynomial matrix $\widehat{A}(\lambda)$, the rational matrix (1) can be written as follows:

$$G(\lambda) = D(\lambda) + \begin{bmatrix} C(\lambda) & 0 \end{bmatrix} \begin{bmatrix} A(\lambda) & 0 \\ 0 & \widehat{A}(\lambda) \end{bmatrix}^{-1} \begin{bmatrix} B(\lambda) \\ 0 \end{bmatrix}.$$

A polynomial system matrix of $G(\lambda)$ is said to have *least order*, or to be *minimal*, if its order is the smallest integer for which matrix polynomials $A(\lambda)$ (nonsingular, with size $n \times n$, $n \geq \deg(\det A(\lambda))$), $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ satisfying (1) exist. The least order is uniquely determined by $G(\lambda)$ and is denoted by $\nu(G(\lambda))$. It is called the *least order* of $G(\lambda)$ ([28, Ch. 3, Sec. 5.1] or [33, Sec. 1.10]). Let us recall two equivalent conditions that characterize when the polynomial system matrix in (2) has least order:

(i) $A(\lambda)$ and $B(\lambda)$ are left coprime and $A(\lambda)$ and $C(\lambda)$ are right coprime.
(ii) (A, B) is controllable and (A, C) is observable assuming that $P(\lambda)$ is in state-space form.

The meaning of these conditions is well-known in the theory of linear control systems. Only property (i) will be analyzed: Two polynomial matrices $A(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{q\times n}$ are called *right coprime* if their only right common divisors are *unimodular* matrices (polynomial matrices with nonzero constant determinant). That is to say, if there exist $\bar{A}(\lambda) \in \mathbb{F}[\lambda]^{p\times n}, \bar{B}(\lambda) \in \mathbb{F}[\lambda]^{q\times n}, X(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ such that $A(\lambda) =$

$\bar{A}(\lambda)X(\lambda)$ and $B(\lambda) = \bar{B}(\lambda)X(\lambda)$ then $X(\lambda)$ is unimodular. On the other hand, $A(\lambda) \in \mathbb{F}[\lambda]^{n\times p}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times q}$ are *left coprime* if their transposes $A(\lambda)^T$ and $B(\lambda)^T$ are right coprime.

Any rational function matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ can be uniquely written as

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda) \tag{3}$$

where $D(\lambda)$ is a polynomial matrix and $G_{sp}(\lambda)$ is a *strictly proper rational matrix*, i.e., the entries of $G_{sp}(\lambda)$ are strictly proper rational functions. Now, it is a well-known fact that any strictly proper rational matrix admits *realizations* (see, for example, [28, Ch. 3, Sec. 5.2] or [21, Sec. 6.4]). This means that for some positive integer n there exist matrices $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $B(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ and $C(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ such that $G_{sp}(\lambda) = C(\lambda)A(\lambda)^{-1}B(\lambda)$ and $\begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix}$ is a polynomial system matrix of $G(\lambda)$. Furthermore, any strictly proper rational matrix admits state-space realizations. Therefore any rational matrix $G(\lambda)$ can be written as $G(\lambda) = D(\lambda) + C(\lambda I_n - A)^{-1}B$ where $D(\lambda)$ is polynomial and $C(\lambda I_n - A)^{-1}B$ is strictly proper. Moreover, the realization may always be taken of least order (i.e., such that the corresponding polynomial system matrix in state-space form is of least order). Such realizations are called *minimal*.

We will see in Theorem 2.3 that minimal polynomial system matrices convey precise information about the finite poles and zeros of their transfer function matrices. Before stating that theorem and analyzing its consequences, we revise some definitions and related results. Recall (see, for example, [28, Ch. 3, Sec. 4] or [21, Sec. 6.5.2]) that any rational matrix is *(finite) equivalent*[1] to its *(finite) Smith–McMillan form.* That is to say, if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ then there are unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ such that

$$M(\lambda) = U(\lambda)G(\lambda)V(\lambda) = \operatorname{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{p-r,m-r}\right) \tag{4}$$

where $r = \operatorname{rank} G(\lambda)$, $\epsilon_1(\lambda), \dots, \epsilon_r(\lambda), \psi_1(\lambda), \dots, \psi_r(\lambda)$ are nonzero monic polynomials, $\epsilon_i(\lambda), \psi_i(\lambda)$ are coprime for all $i = 1, \dots, r$, and $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ while $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$, where $\mid$ stands for divisibility. The irreducible fractions $\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}$ are called the *(finite) invariant rational functions* of $G(\lambda)$. In addition, $\psi_1(\lambda)$ is the monic least common denominator of the entries in $G(\lambda)$ and so, $G(\lambda)$ is polynomial if and only if $\psi_1(\lambda) = 1$. In this case (i.e., if $G(\lambda)$ is a polynomial matrix), $M(\lambda)$ is called the *(finite) Smith normal form* of $G(\lambda)$ and the monic polynomials $\epsilon_1(\lambda) \mid \cdots \mid \epsilon_r(\lambda)$ are called the *invariant polynomials* of $G(\lambda)$.

The *(finite) poles* of $G(\lambda)$ are the roots in $\overline{\mathbb{F}}$ (the algebraic closure of $\mathbb{F}$) of $\psi_1(\lambda)$ and its *(finite) zeros* are the roots in $\overline{\mathbb{F}}$ of $\epsilon_r(\lambda)$. If $\lambda_0 \in \overline{\mathbb{F}}$ is a zero of $G(\lambda)$ then, for $i = 1, \dots, r$, we can write $\epsilon_i(\lambda) = (\lambda - \lambda_0)^{m_i}\widehat{\epsilon}_i(\lambda)$ with $\widehat{\epsilon}_i(\lambda_0) \neq 0$ and $m_i \geq 0$. The nonzero elements in $(m_1, \dots, m_r)$ are called the *partial multiplicities of* λ_0 *as a zero* of $G(\lambda)$. The *partial multiplicities of the poles* of $G(\lambda)$ are defined similarly. Notice that although $\epsilon_i(\lambda)$ and $\psi_i(\lambda)$ are coprime polynomials for all $i = 1, \dots, r$, $G(\lambda)$ may have zeros and poles at the same points. When $G(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is a polynomial matrix then $\psi_i(\lambda) = 1$ for $i = 1, \dots, r$ and the polynomials $(\lambda - \lambda_0)^{m_i}$

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[1] In this manuscript, two rational matrices $G_1(\lambda)$ and $G_2(\lambda)$ are said to be equivalent if there exist two unimodular polynomial matrices $U(\lambda)$ and $V(\lambda)$ such that $G_1(\lambda) = U(\lambda)G_2(\lambda)V(\lambda)$. Other types of equivalence relations are often used in this paper, but in those cases the corresponding type of equivalence will be always explicitly mentioned.

with $m_i \neq 0$ are the *finite elementary divisors* of $G(\lambda)$ with respect to, or associated to, λ_0. For computing the partial multiplicities of the poles and zeros of $G(\lambda)$ only the numerators and denominators different from 1 in the Smith–McMillan form of $G(\lambda)$ must be taken into account. They will be called *nontrivial invariant numerators* and *denominators* of $G(\lambda)$, respectively. Similarly the *nontrivial invariant polynomials* of a matrix polynomial are those different from 1.

For matrix polynomials, it is clear that the nontrivial invariant polynomials of $P(\lambda)$ and $\mathrm{Diag}(P(\lambda), I_s)$ are the same. However, this fact is not true for the invariant rational functions of a rational matrix $G(\lambda)$ and $\mathrm{Diag}(G(\lambda), I_s)$. In view of the proposed definitions of linearization and strong linearization of a rational matrix (Section 3), it is important to determine how the invariant rational functions of $G(\lambda)$ and $\mathrm{Diag}(G(\lambda), I_s)$ are related. We answer this question in the following lemma.

Lemma 2.1. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *be a rational matrix with Smith–McMillan form*

$$\mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 0_{p-r,m-r}\right) \in \mathbb{F}(\lambda)^{p\times m}.$$

Then the Smith–McMillan form of $\mathrm{Diag}(G(\lambda), I_s)$ *is*

$$\mathrm{Diag}\left(\frac{\widetilde{\epsilon}_1(\lambda)}{\widetilde{\psi}_1(\lambda)}, \dots, \frac{\widetilde{\epsilon}_{r+s}(\lambda)}{\widetilde{\psi}_{r+s}(\lambda)}, 0_{p-r,m-r}\right) \tag{5}$$

where

$$\begin{array}{ll} \widetilde{\epsilon}_1(\lambda) = \cdots = \widetilde{\epsilon}_s(\lambda) = 1, & \widetilde{\epsilon}_{s+i}(\lambda) = \epsilon_i(\lambda),\ i = 1, \dots, r, \\ \widetilde{\psi}_i(\lambda) = \psi_i(\lambda),\ i = 1, \dots, r, & \widetilde{\psi}_{r+1}(\lambda) = \cdots = \widetilde{\psi}_{r+s}(\lambda) = 1. \end{array}$$

Proof. Observe that if $s = 0$ then there is nothing to prove. Moreover, if $s > 0$, we only need to prove the result for $s = 1$ because the result for $s > 1$ follows from the result for $s = 1$ applied to $\mathrm{Diag}(G(\lambda), I_{s-1})$ instead of $G(\lambda)$. Note also that from the divisibility relations of the Smith–McMillan form of $G(\lambda)$ it follows that $\epsilon_i(\lambda)$ and $\psi_j(\lambda)$ are coprime if $i < j$ and, so, (5) indeed defines a Smith–McMillan form, i.e., the fractions in (5) are irreducible. Obvious unimodular transformations allow to see that $\mathrm{Diag}(G(\lambda), 1)$ is equivalent to $\mathrm{Diag}\left(\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}, 1, 0_{p-r,m-r}\right) = \frac{Q(\lambda)}{\psi_1(\lambda)}$ with $Q(\lambda) = \mathrm{Diag}\left(\epsilon_1(\lambda), \epsilon_2(\lambda)\frac{\psi_1(\lambda)}{\psi_2(\lambda)}, \dots, \epsilon_r(\lambda)\frac{\psi_1(\lambda)}{\psi_r(\lambda)}, \psi_1(\lambda), 0_{p-r,m-r}\right)$. Thus if $\alpha_i(\lambda) = \frac{\psi_1(\lambda)}{\psi_i(\lambda)}$ for $i = 1, \dots, r$ then

$$Q(\lambda) = \mathrm{Diag}\left(\epsilon_1(\lambda), \epsilon_2(\lambda)\alpha_2(\lambda), \dots, \epsilon_r(\lambda)\alpha_r(\lambda), \psi_1(\lambda), 0_{p-r,m-r}\right).$$

Note that $\epsilon_1(\lambda) \mid \epsilon_2(\lambda)\alpha_2(\lambda) \mid \cdots \mid \epsilon_r(\lambda)\alpha_r(\lambda)$ and

$$\gcd\left(\epsilon_j(\lambda)\alpha_j(\lambda), \psi_1(\lambda)\right) = \gcd(\epsilon_j(\lambda)\alpha_j(\lambda), \psi_j(\lambda)\alpha_j(\lambda)) = \alpha_j(\lambda),\ j = 2, \dots, r,$$

where gcd stands for greatest common divisor. This implies that if for $j = 2, \dots, r$ $D_j(\lambda)$ is the determinantal divisor of order j (i.e., the greatest common divisor of all $j \times j$ minors) of $Q(\lambda)$, then $D_j(\lambda) = \gcd(\epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\psi_1(\lambda), \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_j(\lambda)\alpha_j(\lambda))$. Thus

$$\begin{aligned} D_j(\lambda) &= \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\gcd(\psi_1(\lambda), \epsilon_j(\lambda)\alpha_j(\lambda)) \\ &= \epsilon_1(\lambda)\epsilon_2(\lambda)\alpha_2(\lambda)\cdots\epsilon_{j-1}(\lambda)\alpha_{j-1}(\lambda)\alpha_j(\lambda), \end{aligned}$$

as already seen above. In addition $D_1(\lambda) = \gcd(\epsilon_1(\lambda), \psi_1(\lambda)) = 1$. Hence the invariant polynomials of $Q(\lambda)$ are $1,\ \epsilon_1(\lambda)\ \left(\frac{\psi_1(\lambda)}{\psi_2(\lambda)}\right), \dots, \epsilon_{r-1}(\lambda)\ \left(\frac{\psi_1(\lambda)}{\psi_r(\lambda)}\right),\ \epsilon_r(\lambda)\,\psi_1(\lambda)$. The result follows by dividing these polynomials by $\psi_1(\lambda)$. ∎

EXAMPLE 2.2. Take $G(\lambda) = \operatorname{Diag}\left(\frac{\lambda}{(\lambda-1)(\lambda-2)}, \frac{\lambda^2}{\lambda-1}\right)$, which is in Smith–McMillan form. According to Lemma 2.1, the Smith–McMillan form of $\operatorname{Diag}(G(\lambda), 1)$ is the diagonal matrix $\operatorname{Diag}\left(\frac{1}{(\lambda-1)(\lambda-2)}, \frac{\lambda}{\lambda-1}, \lambda^2\right)$. This can also be easily computed via the Smith normal form of the polynomial $(\lambda-1)(\lambda-2)\operatorname{Diag}(G(\lambda), 1)$.

In general, if two rational matrices of the same size have exactly the same nontrivial numerators and denominators in their invariant rational functions, they are not equivalent unless they have the same rank. In this sense, the null-spaces of these matrices will play an important role. Let us denote $\mathcal{N}_\ell(G(\lambda))$ and $\mathcal{N}_r(G(\lambda))$ the *left* and *right null-spaces* over $\mathbb{F}(\lambda)$ of $G(\lambda)$, respectively, i.e., if $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$,

$$\mathcal{N}_\ell(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^{p\times 1} : x(\lambda)^T G(\lambda) = 0\},$$
$$\mathcal{N}_r(G(\lambda)) = \{x(\lambda) \in \mathbb{F}(\lambda)^{m\times 1} : G(\lambda)x(\lambda) = 0\}.$$

These sets are vector subspaces over the field of rational functions of $\mathbb{F}(\lambda)^p$ and $\mathbb{F}(\lambda)^m$, respectively. Recall the rank-nullity theorem: $\dim \mathcal{N}_\ell(G(\lambda)) = p - \operatorname{rank} G(\lambda)$ and $\dim \mathcal{N}_r(G(\lambda)) = m - \operatorname{rank} G(\lambda)$. Notice that for $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $T(\lambda) \in \mathbb{F}(\lambda)^{(p+s)\times(m+s)}$, $s \geq 0$, $\operatorname{rank} T(\lambda) = s + \operatorname{rank} G(\lambda)$ if and only if $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(T(\lambda))$. Moreover, $\operatorname{rank} T(\lambda) = s + \operatorname{rank} G(\lambda)$ if and only if $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(T(\lambda))$. In the sequel we will bear in mind that $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(T(\lambda))$ and $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(T(\lambda))$ are equivalent and, so, exchangeable conditions.

As announced, the finite poles and zeros of any rational matrix can be found through any of its polynomial system matrices of least order. This follows from the following result by Rosenbrock ([28, Ch. 3, Thm. 4.1]).

THEOREM 2.3. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix of rank r and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)} \tag{6}$$

be a polynomial system matrix of least order whose transfer function matrix is $G(\lambda)$ such that $n \geq r = \operatorname{rank} G(\lambda)$. Let the Smith–McMillan form of $G(\lambda)$ be the matrix $M(\lambda)$ in (4). Then, the invariant polynomials of $A(\lambda)$ are $1|\cdots|1|\psi_r(\lambda)|\cdots|\psi_1(\lambda)$ with at least $n-r$ invariant polynomials equal to 1, and the invariant polynomials of $P(\lambda)$ are $1|\cdots|1|\epsilon_1(\lambda)|\cdots|\epsilon_r(\lambda)$ with at least n invariant polynomials equal to 1.

A consequence of Theorem 2.3 is that the order of any polynomial system matrix of least order giving rise to $G(\lambda)$ is the degree of the polynomial $\psi(\lambda) = \psi_1(\lambda)\cdots\psi_r(\lambda)$ (see [28, Ch. 3, Sec. 5.1]). Hence $\nu(G(\lambda)) = \deg(\psi(\lambda))$. Moreover,

$$\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(P(\lambda)). \tag{7}$$

Also, the finite poles of $G(\lambda)$ are the finite zeros of $A(\lambda)$ and the finite zeros of $G(\lambda)$ are the finite zeros of $P(\lambda)$ (counting in all cases the corresponding partial multiplicities). In particular, if $P(\lambda)$ is a minimal polynomial system matrix in state-space form and $D(\lambda)$ is a linear polynomial, then $P(\lambda)$ is a linear pencil, its finite zeros are the finite zeros of $G(\lambda)$ and the finite zeros of $A(\lambda) = \lambda I - A$ are the finite poles of $G(\lambda)$.

$G(\lambda)$ may also have *poles and zeros at infinity*, which are the poles and zeros at $\lambda = 0$ of $G(1/\lambda)$ (see [21]). Let $\mathbb{F}_{pr}(\lambda)$ denote the ring of proper rational functions. Its units are called *biproper rational functions*, that is, rational functions having the same degree of numerator and denominator. $\mathbb{F}_{pr}(\lambda)^{p\times m}$ denotes the set of $p \times m$ *proper matrices*, i.e., matrices with entries in $\mathbb{F}_{pr}(\lambda)$. A *biproper matrix* is a square proper matrix whose determinant is a biproper rational function. Two rational matrices $G_1(\lambda), G_2(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ are *equivalent at infinity* if there exist biproper matrices $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$, $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that $G_2(\lambda) = B_1(\lambda)G_1(\lambda)B_2(\lambda)$. Every rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is equivalent at infinity to its *Smith–McMillan form at infinity*

$$\operatorname{Diag}\left(\frac{1}{\lambda^{q_1}}, \dots, \frac{1}{\lambda^{q_r}}, 0_{p-r,m-r}\right) \in \mathbb{F}(\lambda)^{p\times m}$$

where $r = \operatorname{rank} G(\lambda)$ and $q_1 \le \cdots \le q_r$ are integers (see [4] or [33]). The rational functions $\frac{1}{\lambda^{q_1}}, \dots, \frac{1}{\lambda^{q_r}}$ are called the *invariant rational functions at infinity* of $G(\lambda)$. The integers $q_1, \dots, q_r$ are called the *invariant orders at infinity* of $G(\lambda)$. The invariant orders at infinity form a complete system of invariants for the equivalence at infinity in $\mathbb{F}(\lambda)^{p\times m}$ and they determine the zeros and poles at infinity of $G(\lambda)$ (see [4, Prop. 6.11]). Namely, if $q_1 \le \cdots \le q_k < 0 = q_{k+1} = \cdots = q_{u-1} < q_u \le \cdots \le q_r$ are the invariant orders at infinity of $G(\lambda)$ then $G(\lambda)$ has $r-u+1$ zeros at infinity each one of order $q_u, \dots, q_r$ and k poles at infinity each one of order $-q_1, \dots, -q_k$. The invariant orders at infinity different from zero will be called *nontrivial*.

Notice that $G(\lambda)$ is proper if and only if $q_1, \dots, q_r$ are nonnegative integers, that is, proper matrices do not have poles at infinity (they are analytic at ∞ when $\mathbb{F} = \mathbb{C}$). However, non-constant polynomial matrices always have poles at infinity (they are never analytic at ∞ when $\mathbb{F} = \mathbb{C}$) and they may have zeros at infinity as well. Moreover for any non strictly proper rational matrix $-q_1$ is the degree of the polynomial part of the matrix in the expression (3) ([4, 33]). The *degree* of a polynomial matrix is the degree of the entries of highest degree.

In addition to finite elementary divisors, matrix polynomials may have elementary divisors at infinity as well [16, p. 185]. The *elementary divisors at infinity* or *infinite elementary divisors* of a matrix polynomial $P(\lambda)$ are defined as follows: Consider the *reversal* of $P(\lambda)$, i.e., the matrix polynomial $\operatorname{rev} P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right)$ where $d = \deg(P(\lambda))$. This matrix polynomial may or may not have 0 as an eigenvalue. If it has 0 as an eigenvalue then $P(\lambda)$ is said to have ∞ as an eigenvalue or to have *eigenvalues at infinity*. The infinite elementary divisors of $P(\lambda)$ are the elementary divisors associated to the eigenvalue 0 of the reversal of $P(\lambda)$. Let $q_1, \dots, q_r$ be the invariant orders at infinity of the polynomial matrix $P(\lambda)$ of degree d and rank r and let $\lambda^{e_1}, \dots, \lambda^{e_r}$ be its infinite elementary divisors (including possible exponents equal to zero). Then (see [4]) $d = \deg(P(\lambda)) = -q_1$ and

$$e_i = q_i - q_1 = d + q_i, \quad i = 1, \dots, r. \tag{8}$$

Similarly to the finite case, the zeros and poles at infinity of a rational matrix $G(\lambda)$ can be determined by its polynomial system matrices (2). However, while for the finite case the minimality of the polynomial system matrix is required (Theorem 2.3), for the infinite case the matrices $A(\lambda)^{-1}B(\lambda)$ and $C(\lambda)A(\lambda)^{-1}$ must both be proper.

Lemma 2.4. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let* $P(\lambda) = \left[\begin{smallmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{smallmatrix}\right] \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ *be a polynomial system matrix of* $G(\lambda)$ *such that both* $A(\lambda)^{-1}B(\lambda)$ *and* $C(\lambda)A(\lambda)^{-1}$ *are proper rational matrices. Then* $P(\lambda)$ *is equivalent at infinity to* $\operatorname{Diag}(A(\lambda), G(\lambda))$.

Proof. The desired result is obtained by pre and post multiplying $P(\lambda)$ by the biproper matrices $\left[\begin{smallmatrix} I_n & 0 \\ C(\lambda)A(\lambda)^{-1} & I_p \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} I_n & -A(\lambda)^{-1}B(\lambda) \\ 0 & I_m \end{smallmatrix}\right]$. ☐

Corollary 2.5. *Under the conditions of the previous lemma, if* $q_1^A, \dots, q_n^A$ *and* $q_1^G, \dots, q_r^G$ *are the invariant orders at infinity of* $A(\lambda)$ *and* $G(\lambda)$ *respectively then* *the invariant orders at infinity of* $P(\lambda)$ *are* $q_1^P, \dots, q_{n+r}^P$ *where* $(q_{n+r}^P, \cdots, q_1^P) = (q_n^A, \cdots, q_1^A) \cup (q_r^G, \cdots, q_1^G)$.

In words, the invariant orders at infinity of $P(\lambda)$ are the ordered reunion of the invariant orders at infinity of $A(\lambda)$ and of $G(\lambda)$. This means that the invariant orders at infinity of $G(\lambda)$ are determined by those of $P(\lambda)$ and $A(\lambda)$. Therefore, the infinite poles of $G(\lambda)$ are determined by the infinite poles of $P(\lambda)$ and $A(\lambda)$ while the infinite zeros of $G(\lambda)$ are determined by the infinite zeros of $P(\lambda)$ and $A(\lambda)$.

3. Strong linearizations of rational matrices. Our aim in this section is to provide a definition of strong linearization for any rational matrix. We want it to be a natural extension of the usual definition for matrix polynomials (see [22, 24, 10]). We will rely primarily on that of [22] although we use a different notation. Let $\mathbb{F}_\lambda(\lambda)$ be the local ring of $\mathbb{F}[\lambda]$ at λ, that is, the ring of rational functions with denominators prime with λ: $\mathbb{F}_\lambda(\lambda) = \left\{ \frac{p(\lambda)}{q(\lambda)} \in \mathbb{F}(\lambda) : q(0) \neq 0 \right\}$. A square matrix $U(\lambda)$ is invertible in $\mathbb{F}_\lambda(\lambda)$ if all its entries are in $\mathbb{F}_\lambda(\lambda)$ and both the numerator and denominator of its determinant are prime with λ.

A *strong linearization* of a matrix polynomial $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ is any linear matrix polynomial $L(\lambda) \in \mathbb{F}[\lambda]^{q\times r}$ such that there are integers $s_1, s_2 \geq 0$, unimodular matrices $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$, $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ and invertible matrices in $\mathbb{F}_\lambda(\lambda)$, $E(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(p+s_1)\times(p+s_1)}$ and $F(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(m+s_1)\times(m+s_1)}$ such that $s_1 - s_2 = q - p = r - m$ and

$$U(\lambda)\,\mathrm{Diag}(P(\lambda), I_{s_1})V(\lambda) = \mathrm{Diag}(L(\lambda), I_{s_2}), \tag{9}$$

$$E(\lambda)\,\mathrm{Diag}(\mathrm{rev}\,P(\lambda), I_{s_1})F(\lambda) = \mathrm{Diag}(\mathrm{rev}\,L(\lambda), I_{s_2}). \tag{10}$$

When only condition (9) is fulfilled $L(\lambda)$ is a *linearization* of $P(\lambda)$.

Remark 3.1. Note that, assuming that (9) holds, condition (10) is equivalent to $\widetilde{U}(\lambda)\,\mathrm{Diag}(\mathrm{rev}\,P(\lambda), I_{s_1})\widetilde{V}(\lambda) = \mathrm{Diag}(\mathrm{rev}\,L(\lambda), I_{s_2})$ where $\widetilde{U}(\lambda)$ and $\widetilde{V}(\lambda)$ are unimodular matrices. This is a condition often used in the definition of strong linearizations of matrix polynomials [10, 24]. We emphasize that such condition includes a high level of redundancy while (10) does not, because unimodular matrices are very particular instances of matrices invertible in $\mathbb{F}_\lambda(\lambda)$. The equivalence of these two conditions when (9) holds is a consequence of the effect of Möbius transformations on the elementary divisors of polynomial matrices [4, 25, 31].

Linearizations of matrix polynomials preserve the finite elementary divisors and strong linearizations preserve both the finite and infinite elementary divisors. Our definitions of linearization and strong linearization of a rational matrix follow a similar pattern.

Definition 3.2. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. *A linearization of* $G(\lambda)$ *is a linear pencil*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)}, \tag{11}$$

with $n \geq 0$, *such that the following conditions hold:*

- *(a) if* $n > 0$ *then* $\det(A_1\lambda + A_0) \neq 0$, *and*
- *(b) if* $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ *then:*
  - *(i)* $L(\lambda)$ *is a minimal polynomial system matrix of* $\widehat{G}(\lambda)$, *and*
  - *(ii) there are nonnegative integers* s_1, s_2 *and unimodular matrices* $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ *and* $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ *such that* $s_1 - s_2 = q - p = r - m$ *and* $U(\lambda)\,\mathrm{Diag}(G(\lambda), I_{s_1})V(\lambda) = \mathrm{Diag}(\widehat{G}(\lambda), I_{s_2})$.

REMARK 3.3.

1. Definition 3.2 extends the usual definition of linearization of matrix polynomials. In order to check this, assume for simplicity that $q \geq p$ and $r \geq m$. Let $P(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with $\epsilon_1(\lambda), \ldots, \epsilon_r(\lambda)$ as invariant polynomials and let
$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$
be a linearization of $P(\lambda)$ in the sense of Definition 3.2. Let $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$. Then
    - Since $\nu(P(\lambda)) = 0$, $n \geq \deg(\det(A_1\lambda + A_0)) = 0$. This implies $n = 0$ (i.e., $L(\lambda) = \widehat{G}(\lambda) = D_1\lambda + D_0$) or $A_1\lambda + A_0$ is a unimodular matrix. In both cases, $\widehat{G}(\lambda)$ is a matrix polynomial.
    - From Definition 3.2 *(ii)* and Lemma 2.1 the invariant polynomials of $\widehat{G}(\lambda)$ are $1, \ldots, 1, \epsilon_1(\lambda), \ldots, \epsilon_r(\lambda)$ (with at least s invariant polynomials equal to 1).
    - From Definition 3.2 *(i)* and Theorem 2.3 the invariant polynomials of $L(\lambda)$ are $1, \ldots, 1, \epsilon_1(\lambda), \ldots, \epsilon_r(\lambda)$ (with at least $n+s$ invariant polynomials equal to 1). Thus, there are $U(\lambda) \in \mathbb{F}[\lambda]^{(n+p+s)\times(n+p+s)}$ and $V(\lambda) \in \mathbb{F}[\lambda]^{(n+m+s)\times(n+m+s)}$, both unimodular, such that $U(\lambda)L(\lambda)V(\lambda) = \operatorname{Diag}(P(\lambda), I_{n+s})$.
2. The linearizations of Definition 5.3 of [1] are linearizations in the sense of Definition 3.2 above if the linear polynomial system matrix in [1] is required to be of least order (see [5]). One advantage of Definition 3.2 is that it can be extended to preserve the invariant orders at infinity of the rational matrix. This is our next task.

Recall (see Section 2) that the first invariant order at infinity of a rational matrix is minus the degree of its polynomial part, if such part is not zero. Thus, if the matrix is polynomial its first invariant order at infinity is minus its degree. Let $L(\lambda)$ be a strong linearization of a matrix polynomial $P(\lambda)$. Let q_1 and $\widehat{q}_1$ be the first invariant orders at infinity of $P(\lambda)$ and $L(\lambda)$ respectively. Condition (10) can be written as

$$E(\lambda)\operatorname{Diag}\left(\lambda^{-q_1}P\left(\frac{1}{\lambda}\right), I_{s_1}\right)F(\lambda) = \operatorname{Diag}\left(\lambda^{-\widehat{q}_1}L\left(\frac{1}{\lambda}\right), I_{s_2}\right). \tag{12}$$

This condition is equivalent, by [4, Lem. 6.9 and Prop. 6.10], to

$$B_1(\lambda)\operatorname{Diag}\left(\lambda^{q_1}P(\lambda), I_{s_1}\right)B_2(\lambda) = \operatorname{Diag}\left(\lambda^{\widehat{q}_1}L(\lambda), I_{s_2}\right) \tag{13}$$

where $B_1(\lambda) = E\left(\frac{1}{\lambda}\right)$ and $B_2(\lambda) = F\left(\frac{1}{\lambda}\right)$ are biproper matrices. Taking into account these considerations, our definition for strong linearization of a rational matrix is the following.

DEFINITION 3.4. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. Let q_1 be its first invariant order at infinity and $g = \min(0, q_1)$. Let $n = \nu(G(\lambda))$. A strong linearization of $G(\lambda)$ is a linear polynomial matrix*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+q)\times(n+r)} \tag{14}$$

such that the following conditions hold:

(a) *if* $n > 0$ *then* $\det(A_1\lambda + A_0) \neq 0$*, and*

(b) *if* $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$*,* $\widehat{q}_1$ *is its first invariant order at infinity and* $\widehat{g} = \min(0, \widehat{q}_1)$ *then:*

  (i) *there are nonnegative integers* s_1, s_2 *and unimodular matrices* $U_1(\lambda) \in \mathbb{F}[\lambda]^{(p+s_1)\times(p+s_1)}$ *and* $U_2(\lambda) \in \mathbb{F}[\lambda]^{(m+s_1)\times(m+s_1)}$ *so that* $s_1 - s_2 = q - p = r - m$ *and*

$$U_1(\lambda)\,\mathrm{Diag}(G(\lambda), I_{s_1})U_2(\lambda) = \mathrm{Diag}(\widehat{G}(\lambda), I_{s_2}),\ \textit{and}$$

  (ii) *there are biproper matrices* $B_1(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(p+s_1)\times(p+s_1)}$ *and* $B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{(m+s_1)\times(m+s_1)}$ *such that*

$$B_1(\lambda)\,\mathrm{Diag}(\lambda^{g}G(\lambda), I_{s_1})B_2(\lambda) = \mathrm{Diag}(\lambda^{\widehat{g}}\widehat{G}(\lambda), I_{s_2}).$$

The integer $g = \min(0, q_1)$ in this definition can be interpreted as follows: g is equal to 0 if and only if $G(\lambda)$ is proper; otherwise g is minus the degree of the polynomial part of $G(\lambda)$.

REMARK 3.5.

1. Strong linearizations of polynomial matrices are often defined as those linearizations which also satisfy condition (10) (or its redundant version via unimodular matrices). However, Definition 3.4 does not follow that pattern because $L(\lambda)$ is not explicitly required to be a linearization of $G(\lambda)$. This is however a consequence of that definition. In fact, since we require $n = \nu(G(\lambda))$, by Definition 3.4 *(i)* and Lemma 2.1, $n = \nu(\widehat{G}(\lambda))$. Therefore, $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and so $L(\lambda)$ is a linearization of $G(\lambda)$. In summary, we can equivalently define a strong linearization of a rational matrix as a linearization with $n = \nu(G(\lambda))$ that satisfies in addition condition *(ii)* in Definition 3.4. In our opinion, the self-contained Definition 3.4 is more convenient.
2. Comparing Definitions 3.2 and 3.4 the requirement $n = \nu(G(\lambda))$ might seem rather restrictive. It has however the following explanation: If $n = 0$ then $G(\lambda)$ is a polynomial matrix and $L(\lambda) = \widehat{G}(\lambda) = D_1\lambda + D_0$. Thus, from Definition 3.4 *(i)*, $L(\lambda)$ is a linearization of $G(\lambda)$ in the classical sense of matrix polynomials. Also, according to the first item of Remark 3.3, $L(\lambda)$ is a linearization of $G(\lambda)$ in the sense of Definition 3.2. If, in addition, $L(\lambda)$ also satisfies condition *(ii)* then it is a strong linearization of $G(\lambda)$ in the usual sense of matrix polynomials. In fact, $g = q_1 = -\deg(G(\lambda))$, $\widehat{g} = \widehat{q}_1 = -\deg(\widehat{G}(\lambda))$, $\widehat{G}(\lambda) = L(\lambda)$ and condition *(ii)* becomes (compare with (13)) $B_1(\lambda)\,\mathrm{Diag}(\lambda^{q_1}G(\lambda), I_{s_1})B_2(\lambda) = \mathrm{Diag}(\lambda^{\widehat{q}_1}L(\lambda), I_{s_2})$. On the other hand, if $n > 0$ then $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$. Therefore $n = \deg(\det(A_1\lambda + A_0))$, which implies that A_1 is invertible, and $(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ and $(C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}$ are proper rational matrices. By Corollary 2.5, the invariant orders at infinity of $L(\lambda)$ are the ordered reunion of those of $A_1\lambda + A_0$ and of $\widehat{G}(\lambda)$. Since A_1 is invertible, $B(\lambda) = A_1 + \lambda^{-1}A_0$ is a biproper matrix and $(A_1\lambda + A_0)B(\lambda)^{-1} = \lambda I_n$ meaning that the invariant orders at infinity of $A_1\lambda + A_0$ are all equal to -1. This property and Definition 3.4 allows us to easily relate the invariant orders at infinity of $G(\lambda)$ and the infinite elementary divisors of its strong linearizations (see Theorem 3.10 and its consequences).
3. As in the polynomial case (see (12) and (13)) condition *(ii)* of Definition 3.4 is equivalent to ([4, Sect. 6]):

(ii') there are invertible matrices in $\mathbb{F}_\lambda(\lambda)$, $\widetilde{U}_1(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(p+s_1)\times(p+s_1)}$ and $\widetilde{U}_2(\lambda) \in \mathbb{F}_\lambda(\lambda)^{(m+s_1)\times(m+s_1)}$ so that $\widetilde{U}_1(\lambda)\,\mathrm{Diag}\left(\frac{1}{\lambda^g}G\left(\frac{1}{\lambda}\right), I_{s_1}\right)\widetilde{U}_2(\lambda) = \mathrm{Diag}\left(\frac{1}{\lambda^{\widehat{g}}}\widehat{G}\left(\frac{1}{\lambda}\right), I_{s_2}\right)$.

Notice that both $\frac{1}{\lambda^g}G\left(\frac{1}{\lambda}\right)$ and $\frac{1}{\lambda^{\widehat{g}}}\widehat{G}\left(\frac{1}{\lambda}\right)$ are matrices with elements in $\mathbb{F}_\lambda(\lambda)$.

4. The transfer function matrix $\widehat{G}(\lambda)$ of $L(\lambda)$ in (14) can be written as $\widehat{G}(\lambda) = (D_1+C_1A_1^{-1}B_1)\lambda+\widehat{G}_{pr}(\lambda)$, where $\widehat{G}_{pr}(\lambda)$ is proper. Thus, $D_1+C_1A_1^{-1}B_1 \neq 0$ if and only if $\widehat{g} = -1$, and $D_1 + C_1A_1^{-1}B_1 = 0$ if and only if $\widehat{g} = 0$.

Since strong linearizations of matrix polynomials have been extensively and deeply analyzed, we will mainly focus on the case $n = \nu(G(\lambda)) > 0$.

3.1. Spectral characterization of strong linearizations. In this section we present a spectral characterization of strong linearizations of rational matrices. We will see that such linear pencils preserve not only the finite but also the infinite structure of the associated rational matrix.

In the definitions of linearization or strong linearization we can always take $s_1 = 0$ or $s_2 = 0$ according as $p \geq q$ and $m \geq r$ or $q \geq p$ and $r \geq m$, respectively. In the rest of the paper we will assume $s := s_1 \geq 0$ and $s_2 = 0$.

LEMMA 3.6. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$. *Let* q_1 *be its first invariant order at infinity,* $g = \min(0, q_1)$ *and* $n = \nu(G(\lambda))$. *Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \tag{15}$$

be a polynomial system matrix of $\widehat{G}(\lambda)$, $\widehat{q}_1$ *be the first invariant order at infinity of* $\widehat{G}(\lambda)$ *and* $\widehat{g} = \min(0, \widehat{q}_1)$. *Then* $L(\lambda)$ *is a strong linearization of* $G(\lambda)$ *if and only if the following three conditions hold:*

(a) $\dim \mathcal{N}_r(G(\lambda)) = \dim \mathcal{N}_r(\widehat{G}(\lambda))$,

(b) $G(\lambda)$ *and* $\widehat{G}(\lambda)$ *have the same nontrivial numerators and the same nontrivial denominators in their (finite) Smith–McMillan forms, and*

(c) $\lambda^g G(\lambda)$ *and* $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ *have the same nontrivial invariant orders at infinity.*

Proof. Notice that, by the rank-nullity theorem, condition (a) is equivalent to $\operatorname{rank}\widehat{G}(\lambda) = s + \operatorname{rank} G(\lambda)$. The necessity follows from the definition of strong linearization and Lemma 2.1. For the sufficiency, let $\frac{\epsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\epsilon_r(\lambda)}{\psi_r(\lambda)}$ be the invariant rational functions of $G(\lambda)$ where $r = \operatorname{rank} G(\lambda)$. Assume that $\epsilon_{t+1}(\lambda)|\epsilon_{t+2}(\lambda)|\cdots|\epsilon_r(\lambda)$ and $\psi_q(\lambda)|\psi_{q-1}(\lambda)|\cdots|\psi_1(\lambda)$ are the nontrivial invariant numerators and denominators, respectively, of $G(\lambda)$. By hypothesis these polynomials are also the nontrivial invariant numerators and denominators of $\widehat{G}(\lambda)$. Then, the Smith–McMillan form of this matrix is $\mathrm{Diag}\left(\frac{\widetilde{\epsilon}_1(\lambda)}{\widetilde{\psi}_1(\lambda)}, \ldots, \frac{\widetilde{\epsilon}_{r+s}(\lambda)}{\widetilde{\psi}_{r+s}(\lambda)}, 0_{p-r,m-r}\right)$ where $\widetilde{\epsilon}_1(\lambda) = \cdots = \widetilde{\epsilon}_s(\lambda) = 1$, $\widetilde{\epsilon}_{s+i}(\lambda) = \epsilon_i(\lambda)$, $i = 1, \ldots, r$, $\widetilde{\psi}_i(\lambda) = \psi_i(\lambda)$, $i = 1, \ldots, r$, $\widetilde{\psi}_{r+1}(\lambda) = \cdots = \widetilde{\psi}_{r+s}(\lambda) = 1$. It follows from Lemma 2.1 that $\widehat{G}(\lambda)$ and $\mathrm{Diag}(G(\lambda), I_s)$ are equivalent. And condition *(b)* of Definition 3.4 follows from (a) and (c). This completes the proof. ☐

From now on we use symbol $\overset{ei}{\sim}$ for equivalence at infinity.

LEMMA 3.7. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, *let* q_1 *be its first invariant order at infinity,* $g = \min(0, q_1)$ *and* $n = \nu(G(\lambda))$. *Let* $L(\lambda)$ *as in (15) be a strong linearization of* $G(\lambda)$. *If* $D_1 + C_1A_1^{-1}B_1 \neq 0$ *then* $L(\lambda)$ *is equivalent at infinity to* $\mathrm{Diag}(\lambda I_{n+s}, \lambda^{g+1}G(\lambda))$; *otherwise,* $L(\lambda)$ *is equivalent at infinity to* $\mathrm{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda))$.

Proof. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. By Remark 3.5.2,

$$L(\lambda)\overset{ei}{\sim}\operatorname{Diag}(\lambda I_n, \widehat{G}(\lambda)). \tag{16}$$

Moreover, condition *(ii)* of Definition 3.4 is equivalent to $\lambda^{\widehat{g}}\widehat{G}(\lambda)\overset{ei}{\sim}\operatorname{Diag}(\lambda^g G(\lambda), I_s)$. Now, if $D_1 + C_1A_1^{-1}B_1 \neq 0$ then, by Remark 3.5.4, $\widehat{g} = -1$ and so

$$\operatorname{Diag}(\lambda I_n, \widehat{G}(\lambda))\overset{ei}{\sim}\operatorname{Diag}(\lambda I_{n+s}, \lambda^{g+1}G(\lambda)).$$

However, if $D_1 + C_1A_1^{-1}B_1 = 0$ then, by Remark 3.5.4, $\widehat{g} = 0$ and so

$$\operatorname{Diag}(\lambda I_n, \widehat{G}(\lambda))\overset{ei}{\sim}\operatorname{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda)).$$

Hence, by (16), in the first case $L(\lambda)$ is equivalent at infinity to $\operatorname{Diag}(\lambda I_{n+s}, \lambda^{g+1}G(\lambda))$, while in the second case, $L(\lambda)$ is equivalent at infinity to $\operatorname{Diag}(\lambda I_n, I_s, \lambda^g G(\lambda))$. ☐

The following definitions are introduced with the purpose of stating concisely the spectral characterization of strong linearizations proved in Theorem 3.10.

DEFINITION 3.8. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and let $L(\lambda) \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$ be a linear polynomial system matrix as in (15). We will say that $L(\lambda)$ preserves the finite structure of poles and zeros of $G(\lambda)$ if the following condition holds true: For all $\lambda_0 \in \overline{\mathbb{F}}$, $(\lambda-\lambda_0)^w$, with $w > 0$, appears in the prime factorization of exactly k denominators (respectively numerators) $\psi_i(\lambda)$ (respectively $\epsilon_i(\lambda)$) in the (finite) Smith–McMillan form of $G(\lambda)$ if and only if $A_1\lambda+A_0$ (respectively $L(\lambda)$) has exactly k finite elementary divisors equal to $(\lambda - \lambda_0)^w$.*

DEFINITION 3.9. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ with q_1 as first invariant order at infinity and $g = \min(0, q_1)$. Let $L(\lambda) \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$ be a linear polynomial system matrix as in (15). We will say that $L(\lambda)$ preserves the infinite structure of poles and zeros of $G(\lambda)$ if the following condition holds true: A_1 is invertible if $n > 0$ and for any nonzero integer u, u is an invariant order at infinity with multiplicity k of $\lambda^{-1}L(\lambda)$ if and only if u is an invariant order at infinity with multiplicity k of $\lambda^g G(\lambda)$ if $D_1 + C_1A_1^{-1}B_1 \neq 0$ or of $\operatorname{Diag}(\lambda^{-1}I_s, \lambda^{g-1}G(\lambda))$ otherwise.*

Observe that the matrices $\lambda^{-1}L(\lambda), \lambda^g G(\lambda)$ and $\operatorname{Diag}(\lambda^{-1}I_s, \lambda^{g-1}G(\lambda))$ are all proper and, in consequence, they do not have poles at infinity.

Definitions 3.8 and 3.9 can be stated together equivalently as: $L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$ if (and only if) the finite poles of $G(\lambda)$ are the finite zeros of $A_1\lambda + A_0$, with the same partial multiplicities, in both matrices, the finite zeros of $G(\lambda)$ are the finite zeros of $L(\lambda)$, with the same partial multiplicities, and the number and orders of the infinite zeros of $\lambda^{-1}L(\lambda)$ are the same as the number and orders of the infinite zeros of $\lambda^g G(\lambda)$ if $D_1 + C_1A_1^{-1}B_1 \neq 0$ or of $\operatorname{Diag}(\lambda^{-1}I_s, \lambda^{g-1}G(\lambda))$ otherwise.

THEOREM 3.10 (Spectral characterization of strong linearizations). *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and $n = \nu(G(\lambda))$. Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

be a minimal polynomial system matrix. Then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if the following conditions hold:

(I) $\dim\mathcal{N}_r(G(\lambda)) = \dim\mathcal{N}_r(L(\lambda))$,

(II) $L(\lambda)$ *preserves the finite structure of poles and zeros of* $G(\lambda)$*, and*

(III) $L(\lambda)$ *preserves the infinite structure of poles and zeros of* $G(\lambda)$*.*

Proof. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. By using Theorem 2.3, $\dim \mathcal{N}_r(\widehat{G}(\lambda)) = \dim \mathcal{N}_r(L(\lambda))$ (see (7)) and $L(\lambda)$ preserves the finite structure of poles and zeros of $\widehat{G}(\lambda)$. If $L(\lambda)$ is a strong linearization of $G(\lambda)$, then conditions (I) and (II) follow from Lemma 3.6. Condition (III) is a direct consequence of Lemma 3.7. Conversely, conditions (I) and (II) and the minimality of $L(\lambda)$ imply, by Theorem 2.3, conditions (a) and (b) of Lemma 3.6. Moreover, it follows from Remark 3.5.2 that $L(\lambda)$ is equivalent at infinity to $\mathrm{Diag}(\lambda I_n, \widehat{G}(\lambda))$. Therefore,

$$\lambda^{-1}L(\lambda) \overset{ei}{\sim} \mathrm{Diag}(I_n, \lambda^{-1}\widehat{G}(\lambda)). \tag{17}$$

Now, conditions (I) and (III) imply (recall that (I) is equivalent to $\operatorname{rank} L(\lambda) = n + s + \operatorname{rank} G(\lambda)$)

$$\lambda^{-1}L(\lambda) \overset{ei}{\sim} \begin{cases} \mathrm{Diag}(I_{n+s}, \lambda^{g} G(\lambda)), & \text{if } D_1 + C_1A_1^{-1}B_1 \neq 0 \\ \mathrm{Diag}(I_n, \lambda^{-1}I_s, \lambda^{g-1}G(\lambda)), & \text{if } D_1 + C_1A_1^{-1}B_1 = 0 \end{cases}. \tag{18}$$

Recall (Remark 3.5.4) that if $D_1 + C_1A_1^{-1}B_1 \neq 0$ then $\widehat{g} = -1$ and $\widehat{g} = 0$ otherwise. Thus, by (17) and (18),

$$\lambda^{-1}\widehat{G}(\lambda) \overset{ei}{\sim} \begin{cases} \mathrm{Diag}(I_s, \lambda^{g} G(\lambda)), & \text{if } D_1 + C_1A_1^{-1}B_1 \neq 0 \\ \mathrm{Diag}(\lambda^{-1}I_s, \lambda^{g-1}G(\lambda)), & \text{if } D_1 + C_1A_1^{-1}B_1 = 0 \end{cases}.$$

In any case condition *(c)* of Lemma 3.6 holds. By this lemma the result follows. ☐

Remark 3.11. From the proof of Lemma 3.6, it can be seen that a matrix pencil $L(\lambda)$ as in (15) is a linearization of a rational matrix $G(\lambda)$ if and only if conditions (a) and (b) of that lemma hold. As a consequence, if $L(\lambda)$ is minimal then a spectral characterization for linearizations of rational matrices consists of conditions (I) and (II) of the previous theorem.

Remark 3.12. As explained in Section 2 condition (I) in Theorem 3.10 is equivalent to $\dim \mathcal{N}_\ell(G(\lambda)) = \dim \mathcal{N}_\ell(L(\lambda))$. Therefore condition (I) can be equivalently stated as "$G(\lambda)$ and $L(\lambda)$ have the same number of left and the same number of right minimal indices", as it was done in [10, Thm. 4.1] for linearizations of polynomial matrices. Analogously, condition (a) of Lemma 3.6 can be equivalently stated as "$G(\lambda)$ and $\widehat{G}(\lambda)$ have the same number of left and the same number of right minimal indices".

Theorem 3.10 allows us to obtain the infinite structure of a rational matrix from the elementary divisors at infinity of any of its strong linearizations in a very simple form. Namely, let $G(\lambda)$ be a $p \times m$ rational matrix of rank r, let $q_1 \leq \cdots \leq q_r$ be its invariant orders at infinity and let $L(\lambda)$ be a strong linearization of $G(\lambda)$. Define $g = \min(0, q_1)$. By (I), $\operatorname{rank} L(\lambda) = n + s + r$. Let $\lambda^{e_1}, \ldots, \lambda^{e_{n+s+r}}$ be the infinite elementary divisors (including possible exponents equal to zero) of $L(\lambda)$. Thus, $0 \leq e_1 \leq \cdots \leq e_{n+s+r}$. We want to get the invariant orders at infinity of $G(\lambda)$ out of $e_1, \ldots, e_{n+s+r}$. Recall that the degree of the polynomial part of $G(\lambda)$ (if present) is $-q_1$. Suppose that the degree of $L(\lambda)$ is d (d can only be equal to 1 or 0). By (8), the invariant orders at infinity of $L(\lambda)$ are $e_i - d$, $i = 1, \ldots, n + s + r$. We consider three cases:

- $\underline{n \geq 0 \text{ and } D_1 + C_1 A_1^{-1} B_1 \neq 0}$. In this case $d = 1$ and, by Lemma 3.7, $e_i = 0$ for $i = 1, \ldots, n+s$ and $e_{n+s+1} - 1, \ldots, e_{n+s+r} - 1$ are the invariant orders at infinity of $\lambda^{g+1} G(\lambda)$. Thus, $q_i = e_{n+s+i} + g,\ 1 \leq i \leq r$.
- $\underline{n > 0 \text{ and } D_1 + C_1 A_1^{-1} B_1 = 0}$. Then $d = 1$ and, by Lemma 3.7, $e_i = 0$ for $i = 1, \ldots, n$, $e_{n+i} = 1$ for $i = 1, \ldots, s$, and $e_{n+s+1} - 1, \ldots, e_{n+s+r} - 1$ are the invariant orders at infinity of $\lambda^g G(\lambda)$. Thus, $q_i = e_{n+s+i} + g - 1,\ 1 \leq i \leq r$.
- $\underline{n = 0 \text{ and } D_1 = 0}$. In this case $G(\lambda)$ is a matrix polynomial, $L(\lambda) = \widehat{G}(\lambda) = D_0$, $\operatorname{Diag}(G(\lambda), I_s)$ is equivalent to D_0 (i.e., all its invariant factors are equal to 1) and $\operatorname{Diag}(\lambda^{q_1} G(\lambda), I_s)$ is equivalent at infinity to D_0. Since the invariant orders at infinity of D_0 are 0, $q_i = q_1$ for $i = 1, \ldots, r$.

4. Transfer system equivalence.

We analyze deeper the relationship between rational matrices and linearizations. Let us recall at this point the notion of *strict system equivalence* (see [28, Ch. 2, Sec. 3.1]): Two polynomial system matrices

$$P_1(\lambda) = \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \quad \text{and} \quad P_2(\lambda) = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix} \tag{19}$$

($A_i(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ nonsingular, $\deg(\det A_i(\lambda)) \leq n$, $B_i(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$, $C_i(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$ and $D_i(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, $i = 1, 2$) are said to be *strictly system equivalent* if there exist unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ and polynomial matrices $X(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ such that

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & I_p \end{bmatrix} \begin{bmatrix} A_1(\lambda) & B_1(\lambda) \\ -C_1(\lambda) & D_1(\lambda) \end{bmatrix} \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} A_2(\lambda) & B_2(\lambda) \\ -C_2(\lambda) & D_2(\lambda) \end{bmatrix}. \tag{20}$$

An important feature of strict system equivalence is that any two strictly system equivalent polynomial system matrices have the same order and give rise to the same transfer function matrix ([28, Ch. 2, Thm. 3.1]). Bearing in mind Definition 3.2, we are interested in characterizing when two polynomial system matrices have equivalent transfer function matrices. This will give us the exact amount of freedom that we have to construct new linearizations out of previous given ones. We extend the definition of strict system equivalence in an obvious way to reach this goal.

Definition 4.1. *Two polynomial system matrices $P_1(\lambda)$ and $P_2(\lambda)$ as in (19) both of size $(n+p)\times(n+m)$ will be said to be transfer system equivalent if there exist unimodular matrices $U(\lambda), V(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $W(\lambda) \in \mathbb{F}[\lambda]^{p\times p}$, $T(\lambda) \in \mathbb{F}[\lambda]^{m\times m}$ and polynomial matrices $X(\lambda) \in \mathbb{F}[\lambda]^{p\times n}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{n\times m}$ such that*

$$\begin{bmatrix} U(\lambda) & 0 \\ X(\lambda) & W(\lambda) \end{bmatrix} P_1(\lambda) \begin{bmatrix} V(\lambda) & Y(\lambda) \\ 0 & T(\lambda) \end{bmatrix} = P_2(\lambda). \tag{21}$$

Theorem 4.2. *Let $P_1(\lambda)$ and $P_2(\lambda)$ be two $(n+p)\times(n+m)$ polynomial system matrices of least order. Then $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent if and only if their transfer function matrices are equivalent.*

Proof. Let $P_1(\lambda)$ and $P_2(\lambda)$ be as in (19) and let $G_1(\lambda)$ and $G_2(\lambda)$ be their transfer functions matrices. A straightforward computation shows that if (21) holds then $G_2(\lambda) = W(\lambda)G_1(\lambda)T(\lambda)$ and $A_2(\lambda) = U(\lambda)A_1(\lambda)V(\lambda)$. Assume now that $G_2(\lambda) = W(\lambda)G_1(\lambda)T(\lambda)$ for some unimodular matrices $W(\lambda)$ and $T(\lambda)$. Then

$$\begin{aligned} D_2(\lambda) + C_2(\lambda)A_2(\lambda)^{-1}B_2(\lambda) &= W(\lambda)(D_1(\lambda) + C_1(\lambda)A_1(\lambda)^{-1}B_1(\lambda))T(\lambda) \\ &= W(\lambda)D_1(\lambda)T(\lambda) + W(\lambda)C_1(\lambda)A_1(\lambda)^{-1}B_1(\lambda)T(\lambda). \end{aligned}$$

We recall (see Section 2) that $P_1(\lambda)$ is of least order if and only if $A_1(\lambda)$ and $B_1(\lambda)$ are left coprime and $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime. Now since $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime, $A_1(\lambda)$ and $W(\lambda)C_1(\lambda)$ are also right coprime. In fact, if $X(\lambda)$ is a common right factor of $A_1(\lambda)$ and $W(\lambda)C_1(\lambda)$ then $A_1(\lambda) = \widehat{A}_1(\lambda)X(\lambda)$ and $W(\lambda)C_1(\lambda) = \widehat{C}_1(\lambda)X(\lambda)$, with $\widehat{A}_1(\lambda)$ and $\widehat{C}_1(\lambda)$ both matrix polynomials. But since $W(\lambda)$ is unimodular, $C_1(\lambda) = W(\lambda)^{-1}\widehat{C}_1(\lambda)X(\lambda)$. Hence $A_1(\lambda)$ and $C_1(\lambda)$ have also $X(\lambda)$ as a right common factor. It must be a unimodular matrix because $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime. The proof that $A_1(\lambda)$ and $B_1(\lambda)T(\lambda)$ are left coprime is similar. Now, we have two minimal polynomial system matrices of $G_2(\lambda)$: $P_2(\lambda)$ and $\widehat{P}_1(\lambda) = \left[\begin{smallmatrix} A_1(\lambda) & B_1(\lambda)T(\lambda) \\ -W(\lambda)C_1(\lambda) & W(\lambda)D_1(\lambda)T(\lambda) \end{smallmatrix}\right]$. By ([28, Ch. 2, Thm. 3.1]) these two polynomial system matrices are strictly system equivalent, that is, $\left[\begin{smallmatrix} U(\lambda) & 0 \\ X(\lambda) & I_p \end{smallmatrix}\right]\widehat{P}_1(\lambda)\left[\begin{smallmatrix} V(\lambda) & Y(\lambda) \\ 0 & I_m \end{smallmatrix}\right] = P_2(\lambda)$ for some unimodular matrices $U(\lambda)$ and $V(\lambda)$ and matrix polynomials $X(\lambda)$ and $Y(\lambda)$. Therefore $\left[\begin{smallmatrix} U(\lambda) & 0 \\ X(\lambda) & W(\lambda) \end{smallmatrix}\right]P_1(\lambda)\left[\begin{smallmatrix} V(\lambda) & Y(\lambda) \\ 0 & T(\lambda) \end{smallmatrix}\right] = P_2(\lambda)$, and this means that $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent, as desired. ☐

Theorem 4.2 allows us to obtain linearizations of rational matrices out of their minimal polynomial system matrices by means of elementary operations. This will be a consequence of Theorem 4.4 below. We will need the following technical result.

Lemma 4.3. *Let* $P_1(\lambda), P_2(\lambda) \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$ *be polynomial system matrices. If* $P_1(\lambda)$ *and* $P_2(\lambda)$ *are transfer system equivalent then* $P_1(\lambda)$ *is of least order if and only if* $P_2(\lambda)$ *is of least order.*

Proof. Let $P_1(\lambda)$ and $P_2(\lambda)$ be as in (19). According to [28, Thm. 6.1, Ch. 2] two matrix polynomials $R(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$ and $S(\lambda) \in \mathbb{F}[\lambda]^{m\times p}$ are left coprime if and only if the Smith normal form of $\begin{bmatrix} R(\lambda) & S(\lambda) \end{bmatrix}$ is $\begin{bmatrix} I_m & 0 \end{bmatrix}$. It is easy to prove (see [28, p. 55]) that if $P_1(\lambda)$ and $P_2(\lambda)$ are transfer system equivalent then $\begin{bmatrix} A_1(\lambda) & B_1(\lambda) \end{bmatrix}$ and $\begin{bmatrix} A_2(\lambda) & B_2(\lambda) \end{bmatrix}$ have the same Smith normal form. Thus $A_1(\lambda)$ and $B_1(\lambda)$ are left coprime if and only if $A_2(\lambda)$ and $B_2(\lambda)$ are left coprime.

It can be proved in a similar way that $A_1(\lambda)$ and $C_1(\lambda)$ are right coprime if and only if $A_2(\lambda)$ and $C_2(\lambda)$ are right coprime. ☐

Theorem 4.4. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let*

$$P(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(q+p)\times(q+m)}$$

be a polynomial system matrix of least order of $G(\lambda)$. *Let*

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

such that $n, s \geq 0$. *Define* $\widehat{P}(\lambda) = \mathrm{Diag}(I_{n-q}, P(\lambda), I_s)$ *and* $\widehat{L}(\lambda) = L(\lambda)$ *or* $\widehat{P}(\lambda) = \mathrm{Diag}(P(\lambda), I_s)$ *and* $\widehat{L}(\lambda) = \mathrm{Diag}(I_{q-n}, L(\lambda))$ *according as* $n \geq q$ *or* $q \geq n$. *Then* $L(\lambda)$ *is a linearization of* $G(\lambda)$ *if and only if* $\widehat{P}(\lambda)$ *and* $\widehat{L}(\lambda)$ *are transfer system equivalent.*

Proof. Let us assume $q \geq n$. The proof in the other case is similar. Put $\widehat{B}(\lambda) = \begin{bmatrix} B(\lambda) & 0 \end{bmatrix} \in \mathbb{F}[\lambda]^{q\times(m+s)}$, $\widehat{C}(\lambda) = \left[\begin{smallmatrix} C(\lambda) \\ 0 \end{smallmatrix}\right] \in \mathbb{F}[\lambda]^{(p+s)\times q}$ and $\widehat{D}(\lambda) = \mathrm{Diag}(D(\lambda), I_s) \in \mathbb{F}[\lambda]^{(p+s)\times(m+s)}$. Notice also that $\widehat{L}(\lambda) = \left[\begin{smallmatrix} \lambda\widehat{A}_1+\widehat{A}_0 & \lambda\widehat{B}_1+\widehat{B}_0 \\ -(\lambda\widehat{C}_1+\widehat{C}_0) & \lambda D_1+D_0 \end{smallmatrix}\right]$ with $\widehat{A}_0 = \left[\begin{smallmatrix} I_{q-n} & 0 \\ 0 & A_0 \end{smallmatrix}\right]$, $\widehat{A}_1 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & A_1 \end{smallmatrix}\right]$, and for $i = 0, 1$, $\widehat{B}_i = \left[\begin{smallmatrix} 0 \\ B_i \end{smallmatrix}\right], \widehat{C}_i = \begin{bmatrix} 0 & C_i \end{bmatrix}$. So $\widehat{P}(\lambda) = \mathrm{Diag}(P(\lambda), I_s) = \left[\begin{smallmatrix} A(\lambda) & \widehat{B}(\lambda) \\ -\widehat{C}(\lambda) & \widehat{D}(\lambda) \end{smallmatrix}\right]$ is a polynomial system matrix of least order with $\mathrm{Diag}(G(\lambda), I_s)$ as transfer function matrix and $L(\lambda)$ and $\widehat{L}(\lambda)$ have the same transfer function matrix, $\widehat{G}(\lambda)$ say.

Assume that $L(\lambda)$ is a linearization of $G(\lambda)$. This means that $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and this matrix is equivalent to $\operatorname{Diag}(G(\lambda), I_s)$. Since $\det(\lambda A_1 + A_0) = \det(\lambda\widehat{A}_1 + \widehat{A}_0)$, $\widehat{L}(\lambda)$ is also a minimal polynomial system matrix of $\widehat{G}(\lambda)$. Thus we have two minimal polynomial system matrices of the same size, $\widehat{L}(\lambda)$ and $\widehat{P}(\lambda)$, of least order with equivalent transfer functions matrices. By Theorem 4.2 they are transfer system equivalent.

Conversely, assume that $\widehat{P}(\lambda)$ and $\widehat{L}(\lambda)$ are transfer system equivalent and let $\widehat{G}(\lambda)$ be the transfer function matrix of $\widehat{L}(\lambda)$ (and $L(\lambda)$). As $\widehat{P}(\lambda)$ is of least order, it follows from Lemma 4.3 that $\widehat{L}(\lambda)$ (and so $L(\lambda)$) are of least order. By Theorem 4.2, $\widehat{G}(\lambda)$ and $\operatorname{Diag}(G(\lambda), I_s)$ are equivalent. In conclusion, $L(\lambda)$ is a minimal polynomial system matrix of $\widehat{G}(\lambda)$ and this matrix and $\operatorname{Diag}(G(\lambda), I_s)$ are equivalent. By definition, $L(\lambda)$ is a linearization of $G(\lambda)$. ☐

4.1. Transfer system equivalence at infinity. In order to extend the notion of transfer system equivalence to the infinity, it is more convenient for us to work with rational matrices of the following form:

$$R_1(\lambda) = \begin{bmatrix} E_1(\lambda) & F_1(\lambda) \\ -J_1(\lambda) & K_1(\lambda) \end{bmatrix} \quad \text{and} \quad R_2(\lambda) = \begin{bmatrix} E_2(\lambda) & F_2(\lambda) \\ -J_2(\lambda) & K_2(\lambda) \end{bmatrix} \tag{22}$$

with $E_i(\lambda) \in \mathbb{F}(\lambda)^{n\times n}$ nonsingular, $F_i(\lambda) \in \mathbb{F}(\lambda)^{n\times m}$, $J_i(\lambda) \in \mathbb{F}(\lambda)^{p\times n}$, $K_i(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, $i = 1, 2$. $R_1(\lambda)$ and $R_2(\lambda)$ are said to be in *rational form* in [28].

Definition 4.5. *$R_1(\lambda)$ and $R_2(\lambda)$ as in (22) are said to be strictly system equivalent at infinity if there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times n}$ and proper matrices $W(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times n}$, $Z(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times m}$ such that*

$$\begin{bmatrix} B_1(\lambda) & 0 \\ W(\lambda) & I_p \end{bmatrix} R_1(\lambda) \begin{bmatrix} B_2(\lambda) & Z(\lambda) \\ 0 & I_m \end{bmatrix} = R_2(\lambda). \tag{23}$$

This is an equivalence relation since the inverse and product of the block triangular biproper matrices in (23) are biproper matrices with the same block triangular structures (including the identity blocks). Moreover, if two matrices are strictly system equivalent at infinity then they are equivalent at infinity.

Let $G_i(\lambda) = K_i(\lambda) + J_i(\lambda)E_i(\lambda)^{-1}F_i(\lambda)$ be the transfer function matrix of $R_i(\lambda)$, $i = 1, 2$. The next result can be proved straightforwardly.

Proposition 4.6. *If $R_1(\lambda)$ and $R_2(\lambda)$ as in (22) are strictly system equivalent at infinity then they give rise to the same transfer function matrix. Moreover, $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity.*

Definition 4.7. *$R_1(\lambda)$ and $R_2(\lambda)$ as in (22) will be said to be transfer system equivalent at infinity if there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times n}$, $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$, $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ and proper matrices $W(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times n}$, $Z(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times m}$ such that*

$$\begin{bmatrix} B_1(\lambda) & 0 \\ W(\lambda) & B_3(\lambda) \end{bmatrix} R_1(\lambda) \begin{bmatrix} B_2(\lambda) & Z(\lambda) \\ 0 & B_4(\lambda) \end{bmatrix} = R_2(\lambda). \tag{24}$$

This is again an equivalence relation. Furthermore, if two matrices are transfer system equivalent at infinity then they are equivalent at infinity as well.

Proposition 4.8. *If $R_1(\lambda)$ and $R_2(\lambda)$ as in* (22) *are transfer system equivalent at infinity then their transfer function matrices are equivalent at infinity. Moreover, $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity.*

Proof. If (24) holds, $E_2(\lambda) = B_1(\lambda)E_1(\lambda)B_2(\lambda)$ and $G_2(\lambda) = B_3(\lambda)G_1(\lambda)B_4(\lambda)$. ☐

The converse of this result is not true in general, i.e., two matrices of the form (22) that give rise to equivalent transfer function matrices at infinity are not necessarily transfer system equivalent at infinity. However, it does hold true when $J_i(\lambda)E_i(\lambda)^{-1}$ and $E_i(\lambda)^{-1}F_i(\lambda)$ are proper for $i = 1, 2$.

The proof of the following lemma is the same as that of Lemma 2.4.

Lemma 4.9. *Let $R(\lambda) = \left[\begin{smallmatrix} E(\lambda) & F(\lambda) \\ -J(\lambda) & K(\lambda) \end{smallmatrix}\right]$ with $J(\lambda)E(\lambda)^{-1}$ and $E(\lambda)^{-1}F(\lambda)$ proper. Let $G(\lambda) = K(\lambda) + J(\lambda)E(\lambda)^{-1}F(\lambda)$. Then $R(\lambda)$ and $\mathrm{Diag}(E(\lambda), G(\lambda))$ are strictly system equivalent at infinity.*

Theorem 4.10. *Let $R_i(\lambda)$ and $G_i(\lambda)$ be both as in the previous lemma with $J_i(\lambda)E_i(\lambda)^{-1}$ and $E_i(\lambda)^{-1}F_i(\lambda)$ proper, $i = 1, 2$.*

1. *$R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity if and only if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda) = G_2(\lambda)$.*
2. *$R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity if and only if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity.*

Proof. The necessity follows from Propositions 4.6 and 4.8. Suppose that $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda) = G_2(\lambda)$. Then there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times n}$ such that

$$\mathrm{Diag}(B_1(\lambda), I_p)\,\mathrm{Diag}(E_1(\lambda), G_1(\lambda))\,\mathrm{Diag}(B_2(\lambda), I_m) = \mathrm{Diag}(E_2(\lambda), G_2(\lambda)).$$

This means that $\mathrm{Diag}(E_1(\lambda), G_1(\lambda))$ and $\mathrm{Diag}(E_2(\lambda), G_2(\lambda))$ are strictly system equivalent at infinity. By Lemma 4.9, $R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity. Analogously, if $E_1(\lambda)$ and $E_2(\lambda)$ are equivalent at infinity and $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity, there exist biproper matrices $B_1(\lambda), B_2(\lambda) \in \mathbb{F}_{pr}(\lambda)^{n\times n}$, $B_3(\lambda) \in \mathbb{F}_{pr}(\lambda)^{p\times p}$, $B_4(\lambda) \in \mathbb{F}_{pr}(\lambda)^{m\times m}$ such that

$$\mathrm{Diag}(B_1(\lambda), B_3(\lambda))\,\mathrm{Diag}(E_1(\lambda), G_1(\lambda))\,\mathrm{Diag}(B_2(\lambda), B_4(\lambda)) = \mathrm{Diag}(E_2(\lambda), G_2(\lambda)).$$

Then $\mathrm{Diag}(E_1(\lambda), G_1(\lambda))$ and $\mathrm{Diag}(E_2(\lambda), G_2(\lambda))$ are transfer system equivalent at infinity. By Lemma 4.9, since strictly implies transfer system equivalence at infinity, $R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity. ☐

If $A_{11}, A_{21} \in \mathbb{F}^{n\times n}$ are invertible and $A_{10}, A_{20} \in \mathbb{F}^{n\times n}$ then $A_{11}\lambda + A_{10}$ and $A_{21}\lambda + A_{20}$ are equivalent at infinity (see Remark 3.5.2). The next result is a straightforward consequence of Theorem 4.10.

Corollary 4.11. *Let $R_i(\lambda) = \left[\begin{smallmatrix} A_{i1}\lambda + A_{i0} & F_i(\lambda) \\ -J_i(\lambda) & K_i(\lambda) \end{smallmatrix}\right]$ with $A_{i1} \in \mathbb{F}^{n\times n}$ invertible, $A_{i0} \in \mathbb{F}^{n\times n}$, $F_i(\lambda) \in \mathbb{F}(\lambda)^{n\times m}$, $J_i(\lambda) \in \mathbb{F}(\lambda)^{p\times n}$, $K_i(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ such that $J_i(\lambda)(A_{i1}\lambda + A_{i0})^{-1}$ and $(A_{i1}\lambda + A_{i0})^{-1}F_i(\lambda)$ are proper matrices, $i = 1, 2$. Let $G_i(\lambda) = K_i(\lambda) + J_i(\lambda)(A_{i1}\lambda + A_{i0})^{-1}F_i(\lambda)$, $i = 1, 2$.*

1. *$R_1(\lambda)$ and $R_2(\lambda)$ are strictly system equivalent at infinity if and only if $G_1(\lambda) = G_2(\lambda)$.*
2. *$R_1(\lambda)$ and $R_2(\lambda)$ are transfer system equivalent at infinity if and only if $G_1(\lambda)$ and $G_2(\lambda)$ are equivalent at infinity.*

The next corollary provides means to obtain strong linearizations of rational matrices from their minimal polynomial system matrices by performing elementary transformations that preserve both the transfer system equivalence and the transfer system equivalence at infinity.

Corollary 4.12. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, let q_1 be its first invariant order at infinity, $g = \min(0, q_1)$ and $n = \nu(G(\lambda))$. Let*

$$P(\lambda) = \begin{bmatrix} A_{P1}\lambda + A_{P0} & B(\lambda) \\ -C(\lambda) & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)}$$

be a polynomial system matrix of least order giving rise to $G(\lambda)$. Let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \quad (s \geq 0)$$

with A_1 invertible if $n > 0$. Let $\widehat{g} = -1$ if $D_1 + C_1A_1^{-1}B_1 \neq 0$ and $\widehat{g} = 0$ otherwise.

(a) If $C(\lambda)(A_{P1}\lambda + A_{P0})^{-1}$ and $\lambda^g(A_{P1}\lambda + A_{P0})^{-1}B(\lambda)$ are proper then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if

  (i) $\mathrm{Diag}(P(\lambda), I_s)$ and $L(\lambda)$ are transfer system equivalent, and

  (ii) $\left[\begin{array}{c|cc} A_{P1}\lambda + A_{P0} & \lambda^g B(\lambda) & 0 \\ \hline -C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{array}\right]$ *and* $\begin{bmatrix} A_1\lambda + A_0 & \lambda^{\widehat{g}}(B_1\lambda + B_0) \\ -(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix}$ *are transfer system equivalent at infinity.*

(b) If $\lambda^g C(\lambda)(A_{P1}\lambda + A_{P0})^{-1}$ and $(A_{P1}\lambda + A_{P0})^{-1}B(\lambda)$ are proper then $L(\lambda)$ is a strong linearization of $G(\lambda)$ if and only if

  (i) $\mathrm{Diag}(P(\lambda), I_s)$ and $L(\lambda)$ are transfer system equivalent, and

  (ii) $\left[\begin{array}{c|cc} A_{P1}\lambda + A_{P0} & B(\lambda) & 0 \\ \hline -\lambda^g C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{array}\right]$ *and* $\begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -\lambda^{\widehat{g}}(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix}$ *are transfer system equivalent at infinity.*

Proof. We prove item *(a)*. Item *(b)* is proved analogously. By Theorem 4.4, $L(\lambda)$ is a linearization of $G(\lambda)$ if and only if condition *(i)* of *(a)* is satisfied. Let $\widehat{G}(\lambda)$ be the transfer function matrix of $L(\lambda)$. We prove now that $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ and $\mathrm{Diag}(\lambda^g G(\lambda), I_s)$ are equivalent at infinity if and only if condition *(ii)* of *(a)* holds true. Notice that $\mathrm{Diag}(\lambda^g G(\lambda), I_s)$ and $\lambda^{\widehat{g}}\widehat{G}(\lambda)$ are the transfer function matrices of

$$\left[\begin{array}{c|cc} A_{P1}\lambda + A_{P0} & \lambda^g B(\lambda) & 0 \\ \hline -C(\lambda) & \lambda^g D(\lambda) & 0 \\ 0 & 0 & I_s \end{array}\right] \text{ and } \begin{bmatrix} A_1\lambda + A_0 & \lambda^{\widehat{g}}(B_1\lambda + B_0) \\ -(C_1\lambda + C_0) & \lambda^{\widehat{g}}(D_1\lambda + D_0) \end{bmatrix},$$

respectively. The result follows from Corollary 4.11, by taking into account that A_{P1} is invertible, since $P(\lambda)$ is of least order and $n = \nu(G(\lambda))$, and that $(C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}$ and $\lambda^{\widehat{g}}(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ are proper, since A_1 is invertible. ∎

5. Construction of strong linearizations of rational matrices. We show in this section that strong linearizations always exist for every rational matrix by constructing explicitly infinitely many examples. These examples are obtained from Algorithm 5.1 and the formal proof that they are indeed strong linearizations relies on Corollary 4.12. Algorithm 5.1 is based on Corollary 5.1, whose proof is omitted since it is an immediate consequence of Theorem 4.4 for $q = n$.

Corollary 5.1. *Let* $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ *and let*

$$P(\lambda) = \begin{bmatrix} A_{P1}\lambda + A_{P0} & B_P \\ -C_P & D(\lambda) \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+p)\times(n+m)} \tag{25}$$

be a minimal polynomial system matrix of $G(\lambda)$ *with* A_{P1} *nonsingular if* $n > 0$*. Let*

$$L(\lambda) = \begin{bmatrix} A_{L1}\lambda + A_{L0} & B_L \\ -C_L & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \tag{26}$$

be a pencil with A_{L1} *nonsingular if* $n > 0$*. If there exist constant nonsingular matrices* $T, S \in \mathbb{F}^{n\times n}$ *and unimodular matrices* $U(\lambda) \in \mathbb{F}[\lambda]^{(p+s)\times(p+s)}$*,* $V(\lambda) \in \mathbb{F}[\lambda]^{(m+s)\times(m+s)}$ *such that*

$$\begin{bmatrix} T & 0 \\ 0 & U(\lambda) \end{bmatrix} L(\lambda) \begin{bmatrix} S & 0 \\ 0 & V(\lambda) \end{bmatrix} = \mathrm{Diag}(P(\lambda), I_s)\,, \tag{27}$$

then $L(\lambda)$ *is a linearization of* $G(\lambda)$*.*

Note that equation (27) is equivalent to the four equations

$$T\,(A_{L1}\lambda + A_{L0})\,S = (A_{P1}\lambda + A_{P0}), \qquad U(\lambda)\,(D_1\lambda + D_0)\,V(\lambda) = \mathrm{Diag}(D(\lambda), I_s),$$
$$U(\lambda)\,C_L\,S = \begin{bmatrix} C_P \\ 0_{s\times n} \end{bmatrix}, \qquad T\,B_L\,V(\lambda) = \begin{bmatrix} B_P & 0_{n\times s} \end{bmatrix},$$

which reveal that if (27) holds, then $D_1\lambda + D_0$ is a linearization of the polynomial matrix $D(\lambda)$ in the usual sense of matrix polynomials [24, 22, 10]. Thus, Corollary 5.1 suggests the symbolic Algorithm 5.1 for constructing linearizations in (essentially) state-space form of $G(\lambda)$.

Algorithm 5.1 Construct a linearization of a rational matrix

Given a minimal polynomial system matrix $P(\lambda)$ in state-space form as in (25) of a rational matrix $G(\lambda)$, this algorithm constructs a linearization of $G(\lambda)$ in state-space form, when it ends.

`Step 1.` Choose any linearization $D_1\lambda + D_0$ of the polynomial matrix $D(\lambda)$ together with unimodular matrices $U(\lambda), V(\lambda)$ such that $U(\lambda)\,(D_1\lambda + D_0)\,V(\lambda) = \mathrm{Diag}(D(\lambda), I_s)$. We emphasize that there are infinitely many choices available in the literature for constructing linearizations of polynomial matrices (see, for instance, [2, 6, 7, 8, 9, 12, 19, 24, 26] and the references therein).

`Step 2.` Construct $U(\lambda)^{-1}\begin{bmatrix} C_P \\ 0_{s\times n} \end{bmatrix}$ and $\begin{bmatrix} B_P & 0_{n\times s} \end{bmatrix} V(\lambda)^{-1}$ and check whether these matrices are constant matrices. If true, continue; if false, stop.

`Step 3.` Choose any pair of $n \times n$ constant nonsingular matrices T, S and define

$$(A_{L1}\lambda + A_{L0}) := T^{-1}\,(A_{P1}\lambda + A_{P0})S^{-1},$$
$$C_L := U(\lambda)^{-1}\begin{bmatrix} C_P \\ 0_{s\times n} \end{bmatrix} S^{-1}, \text{ and } B_L := T^{-1}\begin{bmatrix} B_P & 0_{n\times s} \end{bmatrix} V(\lambda)^{-1}.$$

`Step 4.` The pencil $L(\lambda)$ constructed as in (26) with all the pencils specified in `Steps 1` and `3` is a linearization of $G(\lambda)$ by Corollary 5.1.

The new class of strong linearizations of rational matrices constructed in this section contains, as very particular cases, the Fiedler-like linearizations (modulo permutations) introduced in [1] only for square rational matrices, and so the extension to rational matrices of the Frobenius companion pencils ([1, Prop. 3.7] and [30]). We emphasize that the strong linearizations introduced in this section are much more general than those in [1] from several important points of view: (1) they are strong linearizations, while [1] does not guarantee that the structure at infinity of the original rational matrix is preserved; (2) they are valid for rational matrices of arbitrary sizes, while [1] only considers square rational matrices, as it also happens in [30]; and (3) the class of linearizations presented here is much wider.

Algorithm 5.1 needs two ingredients: a minimal polynomial system matrix in state-space form of the rational matrix $G(\lambda)$ and a linearization of its polynomial part $D(\lambda)$ (see (3)) when $\deg(D(\lambda)) > 1$, together with the unimodular matrices that transform the linearization of $D(\lambda)$ into $\mathrm{Diag}(D(\lambda), I_s)$. As linearizations of polynomial matrices, we use the recently introduced *strong block minimal bases pencils* [12, Section 3], which include Fiedler-type linearizations, among many others, and have already been used in some applications [23, 27]. Strong block minimal bases linearizations of matrix polynomials and the related unimodular transformations are revised in Section 5.1. Next, we focus on the construction of the starting minimal polynomial system matrix in state-space form via the following two-step approach.

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be any rational matrix.

1. Compute the unique decomposition $G(\lambda) = D(\lambda)+G_{sp}(\lambda)$ with $D(\lambda)$ polynomial and $G_{sp}(\lambda)$ strictly proper. In many applications [30], this decomposition can be obtained (or guessed) without any computational effort.
2. Compute a least order state-space realization (A, B, C) of $G_{sp}(\lambda)$, that is, $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ where $n = \nu(G(\lambda)) = \nu(G_{sp}(\lambda))$. A summary of stable algorithms for constructing minimal state-space realizations for $G_{sp}(\lambda)$ can be found in [29]. In addition, in many applications [30], this realization can be obtained (or guessed) without any computational effort.

The fact that (A, B, C) is a minimal realization is equivalent to the facts that (A, B) is controllable and that (A, C) is observable [28]. That is to say:

$$\mathrm{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B\end{bmatrix} = n, \quad \mathrm{rank}\begin{bmatrix} C^T & (CA)^T & \cdots & (CA^{n-1})^T\end{bmatrix}^T = n.$$

Under these conditions

$$P(\lambda) = \begin{bmatrix} \lambda I_n - A & B \\ -C & D(\lambda)\end{bmatrix} \tag{28}$$

is a polynomial system matrix in state-space form of least order n whose transfer function matrix is $G(\lambda)$. A key observation on (28) is that if $D(\lambda) = 0$ or $\deg(D(\lambda)) \leq 1$, then $P(\lambda)$ is itself a strong linearization of $G(\lambda)$ according to Definition 3.4, since $\widehat{G}(\lambda) = G(\lambda)$ in that definition. Therefore, in Section 5.2 we assume $\deg(D(\lambda)) > 1$.

5.1. Strong block minimal bases linearizations of polynomial matrices.

In this section we briefly review strong block minimal bases linearizations of polynomial matrices and related unimodular transformations. More information can be found in [12, Secs. 3, 4, 5] (we refer in this paper to the extended version of [12] available as MIMS EPrint 2016.34). In addition, some results from [12] are refined in order to use strong block minimal bases linearizations of polynomial matrices in the construction of strong linearizations of rational matrices in Section 5.2. Classical concepts on minimal bases of rational vector spaces are often used in this section. For brevity, we do not review such concepts and refer the reader to the original paper [14] or to [21, Ch. 6]. The summaries in [11, Sec. 2] and [12, Sec. 2] may be of interest since they use the nomenclature employed here. Briefly, we say that a polynomial matrix with more columns than rows is a *minimal basis* when its rows are a minimal basis of the rational subspace they span. Moreover, a minimal basis $N(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ is *dual* to another minimal basis $K(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ if $K(\lambda)N(\lambda)^T = 0$. The Kronecker product of two matrices, denoted $A \otimes B$, is used in this section [20, Ch. 4].

The polynomial matrices (note that the zero entries of a matrix are often omitted)

$$L_k(\lambda) := \begin{bmatrix} -1 & \lambda & & & \\ & -1 & \lambda & & \\ & & \ddots & \ddots & \\ & & & -1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{k\times(k+1)}, \quad \text{and} \tag{29}$$

$$\Lambda_k(\lambda)^T := \begin{bmatrix} \lambda^k & \cdots & \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times(k+1)} \tag{30}$$

are important in this section. Note that $L_k(\lambda)$ and $\Lambda_k(\lambda)^T$ are a pair of dual minimal bases, as well as $L_k(\lambda) \otimes I_t$ and $\Lambda_k(\lambda)^T \otimes I_t$ [12, Ex. 2.6]. With these matrices and the last column of I_{k+1}, denoted by e_{k+1}, we define the unimodular matrix

$$V_k(\lambda) := \left[\begin{array}{c} L_k(\lambda) \\ \hline e_{k+1}^T \end{array}\right] \in \mathbb{F}[\lambda]^{(k+1)\times(k+1)}, \tag{31}$$

whose inverse is

$$V_k(\lambda)^{-1} = \left[\begin{array}{ccccc|c} -1 & -\lambda & -\lambda^2 & \cdots & -\lambda^{k-1} & \lambda^k \\ & -1 & -\lambda & \ddots & \vdots & \lambda^{k-1} \\ & & -1 & \ddots & -\lambda^2 & \vdots \\ & & & \ddots & -\lambda & \lambda^2 \\ & & & & -1 & \lambda \\ & & & & & 1 \end{array}\right] \in \mathbb{F}[\lambda]^{(k+1)\times(k+1)}. \tag{32}$$

Note that the last column of $V_k(\lambda)^{-1}$ is $\Lambda_k(\lambda)$.

The next definition is taken from [12, Def. 3.1 and Thm. 3.3].

Definition 5.2. *Let $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ be a polynomial matrix. A strong block minimal bases pencil associated to $D(\lambda)$ is a linear polynomial matrix with the following structure*

$$\mathcal{L}(\lambda) = \begin{array}{c} \left[\begin{array}{cc} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{array}\right] \begin{array}{l} \}\, p+\widehat{p} \\ \}\, \widehat{m} \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{K_1(\lambda)}}_{m+\widehat{m}} & \underbrace{\hphantom{K_2(\lambda)^T}}_{\widehat{p}} \end{array} \end{array}, \tag{33}$$

where $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ (respectively $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$) is a minimal basis with all its row degrees equal to 1 and with the row degrees of a minimal basis $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ (respectively $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$) dual to $K_1(\lambda)$ (respectively $K_2(\lambda)$) all equal, and such that

$$D(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T. \tag{34}$$

If, in addition, $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ *then $\mathcal{L}(\lambda)$ is said to be a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree.*

The most important property of any strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree is that it is a strong linearization of $D(\lambda)$ [12, Thm. 3.3].

The sizes of the submatrices in (33) are related to the degrees of the dual minimal bases $N_1(\lambda)$ and $N_2(\lambda)$ via [14, Cor. p. 503] as follows. Set

$$\varepsilon := \deg(N_1(\lambda)) \quad \text{and} \quad \eta := \deg(N_2(\lambda)). \tag{35}$$

Since the row degrees of $N_1(\lambda)$ (respectively $N_2(\lambda)$) are all equal, they must be all equal to ε (respectively η) and [14, Cor. p. 503] implies

$$\widehat{m} = m\,\varepsilon \quad \text{and} \quad \widehat{p} = p\,\eta. \tag{36}$$

Definition 5.2 includes the "degenerate" cases $\widehat{m} = 0$, when the second block row in (33) is not present, or $\widehat{p} = 0$, when the second block column in (33) is not present. If $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) then $N_1(\lambda)$ (respectively $N_2(\lambda)$) is taken to be a nonsingular constant matrix of size $m \times m$ (respectively $p \times p$) and the simplest choice is just $N_1(\lambda) = I_m$ (respectively $N_2(\lambda) = I_p$).

A remark on Definition 5.2 is that in [12, Thm. 3.3] the condition $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ defining the strong block minimal bases pencils with sharp degree is not mentioned at all. The reason is that the reversal of $D(\lambda)$ is defined in [12] with respect to the "grade" $\deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$, while here reversals of polynomial matrices are defined in an intrinsic way with respect to the degree (recall Section 2). We emphasize that $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$ is used in the proof of Theorem 5.11 and that this condition and (34) implies $\deg(M(\lambda)) = 1$.

Strong block minimal bases linearizations of polynomial matrices are a very wide set of linearizations which include different types of linearizations (see [12, Secs. 4 and 5], [23], [27]). In Example 5.3, we present a particular class of strong block minimal bases linearizations introduced in [12, Sec. 5], which were called block Kronecker linearizations. They correspond to particular choices of $K_1(\lambda)$ and $K_2(\lambda)$ in (33).

Example 5.3. Consider $D(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 \in \mathbb{F}[\lambda]^{p\times m}$, with $q > 1$ and $D_q \neq 0$, and the matrices in (29) and (30). Then, a *block Kronecker linearization* of $D(\lambda)$ is a pencil

$$\mathcal{L}(\lambda) = \begin{array}{c} \left[\begin{array}{c|c} M(\lambda) & L_\eta(\lambda)^T \otimes I_p \\ \hline L_\varepsilon(\lambda) \otimes I_m & 0 \end{array}\right] \\ \underbrace{\hphantom{L_\varepsilon(\lambda) \otimes I_m}}_{(\varepsilon+1)m} \; \underbrace{\hphantom{L_\eta(\lambda)^T \otimes I_p}}_{\eta p} \end{array} \begin{array}{l} \}\,{\scriptstyle (\eta+1)p} \\ \}\,{\scriptstyle \varepsilon m} \\ \\ \end{array} , \tag{37}$$

such that $D(\lambda) = (\Lambda_\eta(\lambda)^T \otimes I_p)\, M(\lambda)\, (\Lambda_\varepsilon(\lambda) \otimes I_m)$. Theorem 5.4 in [12] explains how to construct all possible $M(\lambda)$ that satisfy the previous equation (there are infinitely many). Observe that in the notation of Definition 5.2, we are taking $N_1(\lambda) = \Lambda_\varepsilon(\lambda)^T \otimes I_m$ and $N_2(\lambda) = \Lambda_\eta(\lambda)^T \otimes I_p$. Particular examples of block Kronecker linearizations of $D(\lambda)$ are the first Frobenius companion form, which corresponds to $M(\lambda) = \begin{bmatrix} D_q\lambda + D_{q-1} & D_{q-2} & \cdots & D_0 \end{bmatrix}$, $\varepsilon = q-1$ and $\eta = 0$, and the second Frobenius companion form, with $M(\lambda) = \begin{bmatrix} D_q^T\lambda + D_{q-1}^T & D_{q-2}^T & \cdots & D_0^T \end{bmatrix}^T$, $\eta = q-1$ and $\varepsilon = 0$. The block Kronecker linearizations corresponding to the remaining (permuted) Fiedler pencils are extremely easy to construct as is discussed in [12, Thm. 4.5]. An interesting block Kronecker linearization for polynomial matrices with odd degrees $q = 2k+1$ is constructed by taking $L_\varepsilon(\lambda) = L_\eta(\lambda) = L_k(\lambda)$ and $M(\lambda) = \mathrm{Diag}(D_{2k+1}\lambda + D_{2k}\,,\, D_{2k-1}\lambda + D_{2k-2}\,,\, \ldots\,,\, D_1\lambda + D_0)$. Such linearization is very simple and is symmetric or Hermitian if $D(\lambda)$ is symmetric or Hermitian. The condition $D_q \neq 0$ guarantees that (37) is a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree [12, Thm. 5.4].

Lemma 5.5 refines [12, Thm. 2.10] for dual minimal bases as those appearing in Definition 5.2. The refinement comes from the fact that $\widehat{K}_1$ and $\widehat{K}_2$ in Lemma 5.5 are constant matrices, a property not guaranteed in [12] and that is essential in Section 5.2. In order to prove Lemma 5.5 we need to prove first Lemma 5.4.

Lemma 5.4. *For $i = 1, 2$, let $K_i(\lambda)$ be a linear pencil as in Definition 5.2 and let $N_i(\lambda)$ be a minimal basis dual to $K_i(\lambda)$. If $Q_i(\lambda)$ is another minimal basis dual to $K_i(\lambda)$, then there exists a nonsingular constant matrix H_i such that $Q_i(\lambda) = H_i\, N_i(\lambda)$.*

Proof. The columns of $Q_i(\lambda)^T$ form a basis of the right null-space $\mathcal{N}_r(K_i(\lambda))$ over $\mathbb{F}(\lambda)$ defined in Section 2 and the columns of $N_i(\lambda)^T$ form another basis of $\mathcal{N}_r(K_i(\lambda))$. Therefore, there exists a nonsingular rational matrix $H_i(\lambda)$ such that $Q_i(\lambda) = H_i(\lambda)\, N_i(\lambda)$. Since $N_i(\lambda)$ and $Q_i(\lambda)$ are both minimal bases, the row degrees of $N_i(\lambda)$ are all equal, the row degrees of $Q_i(\lambda)$ are all equal, and the row degrees of $N_i(\lambda)$ are equal to those of $Q_i(\lambda)$, we get that $H_i(\lambda)$ must be a constant matrix [14].□

Lemma 5.5. *Let $K_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ be a linear pencil as in Definition 5.2 and $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ be any of its minimal dual bases. Then there exist $\widehat{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$ and a constant matrix $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$ such that*

(a) $U_1(\lambda) = \begin{bmatrix} K_1(\lambda) \\ \widehat{K}_1 \end{bmatrix} \in \mathbb{F}[\lambda]^{(\widehat{m}+m)\times(m+\widehat{m})}$ is a unimodular polynomial matrix, and

(b) $U_1(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T \end{bmatrix} \in \mathbb{F}[\lambda]^{(m+\widehat{m})\times(\widehat{m}+m)}$.

An analogous result holds for $K_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$ as in Definition 5.2 just by replacing 1 by 2, $\widehat{m}$ by $\widehat{p}$, and m by p.

Proof. We only prove the result for $K_1(\lambda)$. In the proof, the matrices in (29), (30), (31) and (32) are frequently used. In addition, $V_k(\lambda)^{-1}$ is partitioned as $V_k(\lambda)^{-1} = \begin{bmatrix} W_k(\lambda) & \Lambda_k(\lambda) \end{bmatrix}$. We take $\varepsilon = \deg(N_1(\lambda))$ as in (35) and so $\widehat{m} = m\,\varepsilon$ as in (36). Since all the row degrees of $K_1(\lambda) = \lambda K_1^{(1)} + K_1^{(0)}$ are equal to 1 and $K_1(\lambda)$ is a minimal basis, $K_1^{(1)}$ has full row rank. This fact and [11, Thm. 2.4] implies that $K_1(\lambda)$ has neither infinite nor finite eigenvalues and has no left minimal indices. Therefore the Kronecker canonical form [15, Ch. XII] of $K_1(\lambda)$ has only right singular blocks of size $\varepsilon \times (\varepsilon + 1)$ (because the row degrees of $N_1(\lambda)$ are all equal to ε), i.e., there exist nonsingular constant matrices $R \in \mathbb{F}^{m\varepsilon\times m\varepsilon}$ and $S \in \mathbb{F}^{m(\varepsilon+1)\times m(\varepsilon+1)}$ such that $K_1(\lambda) = R^{-1}\,(I_m \otimes L_\varepsilon(\lambda))\,S^{-1}$. Let $e_{\varepsilon+1}$ be the last column of $I_{\varepsilon+1}$. Define the constant matrix $\widetilde{K}_1 := (I_m \otimes e_{\varepsilon+1}^T)\,S^{-1}$ and the linear polynomial matrix

$$\widetilde{U}_1(\lambda) := \begin{bmatrix} K_1(\lambda) \\ \widetilde{K}_1 \end{bmatrix} = \begin{bmatrix} R^{-1} & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_m \otimes L_\varepsilon(\lambda) \\ I_m \otimes e_{\varepsilon+1}^T \end{bmatrix} S^{-1}.$$

Observe that $\widetilde{U}_1(\lambda)$ is unimodular because via an obvious row permutation Π we get

$$\Pi \begin{bmatrix} I_m \otimes L_\varepsilon(\lambda) \\ I_m \otimes e_{\varepsilon+1}^T \end{bmatrix} = I_m \otimes \begin{bmatrix} L_\varepsilon(\lambda) \\ e_{\varepsilon+1}^T \end{bmatrix} = I_m \otimes V_\epsilon(\lambda),$$

whose determinant is $(\det(V_\epsilon(\lambda)))^m$ and so it is constant. Observe also that $Q_1(\lambda) = (I_m \otimes \Lambda_\varepsilon(\lambda)^T)S^T$ is a minimal basis dual to $K_1(\lambda)$ because it is a minimal basis by

[11, Thm. 2.4] and $K_1(\lambda)Q_1(\lambda)^T = 0$. From Lemma 5.4, we know that $N_1(\lambda) = H_1^{-T}Q_1(\lambda)$ for some nonsingular constant matrix H_1. With this matrix, we finally define

$$U_1(\lambda) := \begin{bmatrix} K_1(\lambda) \\ \widehat{K}_1 \end{bmatrix} = \begin{bmatrix} I_{m\varepsilon} & 0 \\ 0 & H_1 \end{bmatrix} \widetilde{U}_1(\lambda),$$

which is unimodular since $\widetilde{U}_1(\lambda)$ is. Note that $U_1(\lambda)^{-1} = \begin{bmatrix} S(I_m \otimes W_\varepsilon(\lambda))R & N_1(\lambda)^T \end{bmatrix}$, since a direct multiplication proves that $U_1(\lambda)\,U_1(\lambda)^{-1} = I_{m(\varepsilon+1)}$, taking into account (31) and the partition $V_k(\lambda)^{-1} = \begin{bmatrix} W_k(\lambda) & \Lambda_k(\lambda) \end{bmatrix}$. ☐

Example 5.6. In the case of block Kronecker linearizations as in (37) (which include the standard Frobenius companion forms), if we take $N_1(\lambda) = \Lambda_\varepsilon(\lambda)^T \otimes I_m$ and $N_2(\lambda) = \Lambda_\eta(\lambda)^T \otimes I_p$, then the constant matrices $\widehat{K}_1$ and $\widehat{K}_2$ in Lemma 5.5 can be taken to be $\widehat{K}_1 = e_{\varepsilon+1}^T \otimes I_m$ and $\widehat{K}_2 = e_{\eta+1}^T \otimes I_p$. This follows from (31) and (32).

Following the discussion of degenerate cases, if $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) in Lemma 5.5, then $U_1(\lambda) = \widehat{K}_1 = N_1(\lambda)^{-T} \in \mathbb{F}^{m\times m}$ (respectively $U_2(\lambda) = \widehat{K}_2 = N_2(\lambda)^{-T} \in \mathbb{F}^{p\times p}$) is any nonsingular constant matrix and the simplest choice is just $\widehat{K}_1 = I_m$ (respectively $\widehat{K}_2 = I_p$).

Lemma 5.5 allows us to construct in Theorem 5.7 unimodular matrices $U(\lambda)$ and $V(\lambda)$ that will be used in Section 5.2 to develop strong linearizations of rational matrices. The proof of Theorem 5.7 is ommitted, since it is very similar to [12, Proofs of Thm. 3.3 and Lem. 2.14] and proceeds via a direct matrix block product.

Theorem 5.7. *Let $\mathcal{L}(\lambda)$ as in (33) be a strong block minimal bases pencil associated to $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$, let $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ be minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, and for $i = 1, 2$, let*

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

be unimodular matrices with $\widehat{K}_i$ a constant matrix and $\widehat{N}_i(\lambda)$ as in Lemma 5.5. There are matrices $X(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times m}$, $Y(\lambda) \in \mathbb{F}[\lambda]^{p\times \widehat{m}}$ and $Z(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times \widehat{m}}$ such that

$$V(\lambda) := \begin{bmatrix} \widehat{N}_1(\lambda)^T & N_1(\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -X(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix},$$

$$U(\lambda) := \begin{bmatrix} 0 & I_p & -Y(\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -Z(\lambda) \end{bmatrix} \begin{bmatrix} \widehat{N}_2(\lambda) & 0 \\ N_2(\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix}$$

are unimodular matrices and

$$U(\lambda)\,\mathcal{L}(\lambda)\,V(\lambda) = \operatorname{Diag}(D(\lambda), I_{\widehat{m}+\widehat{p}}). \tag{38}$$

Remark 5.8. Note that Theorem 5.7 does not assume that $\mathcal{L}(\lambda)$ is a strong block minimal bases pencil with sharp degree, which is important to obtain Corollary 5.9.

The goal of the rest of this subsection is to construct certain biproper matrices related to strong block minimal bases pencils with sharp degree that will be used in Section 5.2. We revise first some properties of the reversal of a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree as in Definition 5.2. This pencil is

$$\operatorname{rev}\mathcal{L}(\lambda) = \begin{bmatrix} \operatorname{rev} M(\lambda) & \operatorname{rev} K_2(\lambda)^T \\ \operatorname{rev} K_1(\lambda) & 0 \end{bmatrix},$$

since $\deg(M(\lambda)) = 1$ by the condition $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$. From [12, Thm. 2.7 and Proof of Thm. 3.3], $\operatorname{rev}\mathcal{L}(\lambda)$ is a strong block minimal bases pencil associated to $\operatorname{rev} D(\lambda) = (\operatorname{rev} N_2(\lambda))\,(\operatorname{rev} M(\lambda))\,(\operatorname{rev} N_1(\lambda)^T)$, but it is not guaranteed that it has sharp degree. It is proved in [12, Thm. 2.7], that for $i = 1, 2$, $\operatorname{rev} N_i(\lambda)$ is a minimal basis dual to $\operatorname{rev} K_i(\lambda)$ with $\deg(\operatorname{rev} N_i(\lambda)) = \deg(N_i(\lambda))$, which allows us to apply Theorem 5.7 to $\operatorname{rev}\mathcal{L}(\lambda)$ and $\operatorname{rev} D(\lambda)$ and to obtain Corollary 5.9.

Corollary 5.9. *Let $\mathcal{L}(\lambda)$ be a strong block minimal bases pencil associated to $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ with sharp degree as in Definition 5.2 and let $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ be minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively. Then there exist $\widetilde{N}_1(\lambda) \in \mathbb{F}[\lambda]^{\widehat{m}\times(m+\widehat{m})}$, $\widetilde{N}_2(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times(p+\widehat{p})}$, $\widetilde{X}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times m}$, $\widetilde{Y}(\lambda) \in \mathbb{F}[\lambda]^{p\times\widehat{m}}$ and $\widetilde{Z}(\lambda) \in \mathbb{F}[\lambda]^{\widehat{p}\times\widehat{m}}$ such that*

$$\widetilde{V}(\lambda) := \begin{bmatrix} \widetilde{N}_1(\lambda)^T & \operatorname{rev} N_1(\lambda)^T & 0 \\ 0 & 0 & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} 0 & I_{\widehat{m}} & 0 \\ I_m & 0 & 0 \\ -\widetilde{X}(\lambda) & 0 & I_{\widehat{p}} \end{bmatrix}, \tag{39}$$

$$\widetilde{U}(\lambda) := \begin{bmatrix} 0 & I_p & -\widetilde{Y}(\lambda) \\ 0 & 0 & I_{\widehat{m}} \\ I_{\widehat{p}} & 0 & -\widetilde{Z}(\lambda) \end{bmatrix} \begin{bmatrix} \widetilde{N}_2(\lambda) & 0 \\ \operatorname{rev} N_2(\lambda) & 0 \\ 0 & I_{\widehat{m}} \end{bmatrix} \tag{40}$$

are unimodular matrices and $\widetilde{U}(\lambda)\,(\operatorname{rev}\mathcal{L}(\lambda))\,\widetilde{V}(\lambda) = \operatorname{Diag}(\operatorname{rev} D(\lambda), I_{\widehat{m}+\widehat{p}})$. In addition, each of the factors defining $\widetilde{V}(\lambda)$ and $\widetilde{U}(\lambda)$ is unimodular.

Combining Corollary 5.9 and [3, Lem. 4.1] we obtain the last result of this section.

Corollary 5.10. *With the same assumptions and notation as in Corollary 5.9, let $\widetilde{V}(\lambda) \in \mathbb{F}[\lambda]^{(m+\widehat{m}+\widehat{p})\times(m+\widehat{m}+\widehat{p})}$ and $\widetilde{U}(\lambda) \in \mathbb{F}[\lambda]^{(p+\widehat{m}+\widehat{p})\times(p+\widehat{m}+\widehat{p})}$ be the unimodular matrices introduced in Corollary 5.9 and define from them the biproper matrices $\widetilde{V}(1/\lambda)$ and $\widetilde{U}(1/\lambda)$. Then*

$$\widetilde{U}(1/\lambda)\,(\lambda^{-1}\mathcal{L}(\lambda))\,\widetilde{V}(1/\lambda) = \operatorname{Diag}(\lambda^{-q} D(\lambda), I_{\widehat{m}+\widehat{p}}), \tag{41}$$

where $q = \deg(D(\lambda))$. Moreover, each of the factors defining $\widetilde{V}(1/\lambda)$ and $\widetilde{U}(1/\lambda)$ according to (39) and (40) is biproper and any submatrix of these factors is proper.

Proof. The properties of $\widetilde{U}(1/\lambda)$, $\widetilde{V}(1/\lambda)$, their factors, and their submatrices follow from [3, Lem. 4.1]. The equality (41) follows from the previous corollary by replacing λ by $1/\lambda$ and taking into account that $\deg(\mathcal{L}(\lambda)) = 1$ since $\deg(M(\lambda)) = 1$. □

5.2. Strong block minimal bases linearizations of rational matrices. The goal of this section is to state and prove Theorem 5.11, which is the main theorem of this paper on the existence and explicit construction of strong linearizations of any rational matrix $G(\lambda)$. The constructed strong linearizations are presented in equation (42) and the proof relies on Corollary 4.12. We emphasize that such linearizations have been constructed via Algorithm 5.1 with input the minimal polynomial system matrix of $G(\lambda)$ in (28) and choosing in `Step 1` any strong block minimal bases pencil $\mathcal{L}(\lambda)$ as (33) associated to the polynomial part $D(\lambda)$ of $G(\lambda)$, the unimodular matrices

$U(\lambda)$ and $V(\lambda)$ in Theorem 5.7, and taking $s = \widehat{m} + \widehat{p}$. To check this, note that

$$V(\lambda)^{-1} = \begin{bmatrix} 0 & I_m & 0 \\ I_{\widehat{m}} & 0 & 0 \\ 0 & X(\lambda) & I_{\widehat{p}} \end{bmatrix} \begin{bmatrix} K_1(\lambda) & 0 \\ \widehat{K}_1 & 0 \\ 0 & I_{\widehat{p}} \end{bmatrix},$$

$$U(\lambda)^{-1} = \begin{bmatrix} K_2(\lambda)^T & \widehat{K}_2^T & 0 \\ 0 & 0 & I_{\widehat{m}} \end{bmatrix} \begin{bmatrix} 0 & Z(\lambda) & I_{\widehat{p}} \\ I_p & Y(\lambda) & 0 \\ 0 & I_{\widehat{m}} & 0 \end{bmatrix},$$

which in `Step 2` in Algorithm 5.1 yields

$$U(\lambda)^{-1} \begin{bmatrix} C \\ 0_{\widehat{m}\times n} \\ 0_{\widehat{p}\times n} \end{bmatrix} = \begin{bmatrix} \widehat{K}_2^T C \\ 0_{\widehat{m}\times n} \end{bmatrix}, \qquad \begin{bmatrix} B & 0_{n\times\widehat{m}} & 0_{n\times\widehat{p}} \end{bmatrix} V(\lambda)^{-1} = \begin{bmatrix} B\widehat{K}_1 & 0_{n\times\widehat{p}} \end{bmatrix}.$$

Since these matrices are constant, Algorithm 5.1 does not stop. Then taking arbitrary nonsingular matrices T and S in `Step 3`, the linearization in (42) is the output of Algorithm 5.1, where X and Y in (42) are T^{-1} and S^{-1} respectively. These linearizations are called *strong block minimal bases linearizations of rational matrices.*

THEOREM 5.11. *Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ be a rational matrix, let $G(\lambda) = D(\lambda) + G_{sp}(\lambda)$ be its unique decomposition into its polynomial part $D(\lambda) \in \mathbb{F}[\lambda]^{p\times m}$ and its strictly proper part $G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, and let $G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B$ be a minimal state-space realization of $G_{sp}(\lambda)$, where $n = \nu(G(\lambda)) = \nu(G_{sp}(\lambda))$. Assume that $\deg(D(\lambda)) > 1$ and let*

$$\mathcal{L}(\lambda) = \left[\begin{array}{cc} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{array} \right] \begin{array}{l} \}\, p+\widehat{p} \\ \}\, \widehat{m} \end{array}$$
$$\qquad\qquad \underbrace{\qquad\quad}_{m+\widehat{m}} \quad \underbrace{\qquad\quad}_{\widehat{p}}$$

be a strong block minimal bases pencil associated to $D(\lambda)$ with sharp degree, with $N_1(\lambda) \in \mathbb{F}[\lambda]^{m\times(m+\widehat{m})}$ and $N_2(\lambda) \in \mathbb{F}[\lambda]^{p\times(p+\widehat{p})}$ minimal bases dual to $K_1(\lambda)$ and $K_2(\lambda)$, respectively, such that $D(\lambda) = N_2(\lambda)M(\lambda)N_1(\lambda)^T$. Let $\widehat{K}_1 \in \mathbb{F}^{m\times(m+\widehat{m})}$ and $\widehat{K}_2 \in \mathbb{F}^{p\times(p+\widehat{p})}$ be constant matrices such that, for $i = 1, 2$, the matrices

$$U_i(\lambda) = \begin{bmatrix} K_i(\lambda) \\ \widehat{K}_i \end{bmatrix} \quad \text{and} \quad U_i(\lambda)^{-1} = \begin{bmatrix} \widehat{N}_i(\lambda)^T & N_i(\lambda)^T \end{bmatrix}$$

are unimodular. Then, for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n\times n}$ the linear polynomial matrix

$$L(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array} \right] \tag{42}$$

is a strong linearization of $G(\lambda)$.

REMARK 5.12. If $\mathcal{L}(\lambda)$ in Theorem 5.11 is a block Kronecker linearization of $D(\lambda)$ as in (37), Example 5.6 implies that $XB\widehat{K}_1 = e_{\varepsilon+1}^T \otimes XB = \begin{bmatrix} 0_{n\times m} & \cdots & 0_{n\times m} & XB \end{bmatrix}$ and $\widehat{K}_2^T CY = e_{\eta+1} \otimes CY$. Recall also that in the degenerate case $\widehat{m} = 0$ (respectively $\widehat{p} = 0$) $\widehat{K}_1 \in \mathbb{F}^{m\times m}$ (respectively $\widehat{K}_2 \in \mathbb{F}^{p\times p}$) can be any nonsingular matrix with

I_m (respectively I_p) as the simplest choice. There are infinitely many strong block minimal bases pencils $\mathcal{L}(\lambda)$ associated to $D(\lambda)$ with sharp degree and, so, infinitely many strong linearizations of $G(\lambda)$ inside the framework of Theorem 5.11. A subset of these infinitely many can be constructed very easily in the case we restrict ourselves to block Kronecker linearizations $\mathcal{L}(\lambda)$ of $D(\lambda)$.

Proof of Theorem 5.11. The proof is based on Corollary 4.12. In this proof, we adopt the notation in (35) for the degrees of $N_1(\lambda)$ and $N_2(\lambda)$ and $q := \deg(D(\lambda)) > 1$. Therefore, $q = \varepsilon + \eta + 1$ according to Definition 5.2. Note that this condition and (34) imply that $\deg(M(\lambda)) = 1$, therefore $\widehat{g}$ in Corollary 4.12 is in this case $\widehat{g} = -1$ for $L(\lambda)$ in (42), since in the notation of that corollary $D_1 \neq 0$, $C_1 = 0$, and $B_1 = 0$. In addition, g in Corollary 4.12 is $g = -q$ here. A key ingredient in this proof is the minimal polynomial system matrix in state-space form $P(\lambda)$ in (28) giving rise to $G(\lambda)$. Obviously, for this $P(\lambda)$ the matrices $C(\lambda I_n - A)^{-1}$ and $\lambda^{-q}(\lambda I_n - A)^{-1}B$ are both proper, and we are in the scenario of Corollary 4.12 in this respect.

According to Corollary 4.12, Theorem 5.11 is proved if

$$\mathbb{L}(\lambda) = \left[\begin{array}{c|cc} \lambda I_n - A & B\widehat{K}_1 & 0 \\ \hline -\,\widehat{K}_2^T C & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array}\right] \tag{43}$$

is a strong linearization of $G(\lambda)$. The reason is that

$$\mathrm{Diag}(X, I_{p+\widehat{m}+\widehat{p}})\ \mathbb{L}(\lambda)\ \ \mathrm{Diag}(Y, I_{m+\widehat{m}+\widehat{p}}) = L(\lambda),$$

which means that $\mathbb{L}(\lambda)$ and $L(\lambda)$ are transfer system equivalent, and that

$$\begin{aligned}\mathrm{Diag}(X, I_{p+\widehat{m}+\widehat{p}})\,(\mathbb{L}(\lambda)\ \mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}}))\ \mathrm{Diag}(Y, I_{m+\widehat{m}+\widehat{p}})\\ = L(\lambda)\,\mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}}),\end{aligned}$$

which means that $\mathbb{L}(\lambda)\mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}})$ and $L(\lambda)\mathrm{Diag}(I_n, \lambda^{-1}I_{m+\widehat{m}+\widehat{p}})$ are transfer system equivalent at infinity. Thus, in the rest of the proof we focus only on $\mathbb{L}(\lambda)$.

We prove first that $\mathbb{L}(\lambda)$ is transfer system equivalent to $\mathrm{Diag}(P(\lambda), I_{\widehat{m}+\widehat{p}})$, i.e., we prove first *(i)* of *(a)* in Corollary 4.12. To this purpose, the unimodular matrices $U(\lambda)$ and $V(\lambda)$ in Theorem 5.7 and (38) are used to prove that the transfer system equivalence $\mathrm{Diag}(I_n, U(\lambda))\,\mathbb{L}(\lambda)\ \mathrm{Diag}(I_n, V(\lambda)) = \mathrm{Diag}(P(\lambda), I_{\widehat{m}+\widehat{p}})$ holds. This follows from a direct matrix multiplication taking into account that $\widehat{K}_1\widehat{N}_1(\lambda)^T = 0$, $\widehat{K}_1 N_1(\lambda)^T = I_m$, $\widehat{K}_2\widehat{N}_2(\lambda)^T = 0$, $\widehat{K}_2 N_2(\lambda)^T = I_p$.

Next, we prove that *(ii)* of *(a)* in Corollary 4.12 holds for $\mathbb{L}(\lambda)$ in (43) and $P(\lambda)$ in (28). The proof has two steps. The first one uses the biproper matrices $\widetilde{U}(1/\lambda)$ and $\widetilde{V}(1/\lambda)$ in Corollary 5.10 and the submatrices of their factors to define the proper matrices

$$\mathbb{W}(\lambda) := \begin{bmatrix} (\lambda^{-\eta} - 1)I_p \\ 0_{\widehat{m}\times p} \\ \widetilde{N}_2(1/\lambda)\,\widehat{K}_2^T \end{bmatrix} C(\lambda I_n - A)^{-1},$$

$$\mathbb{Z}(\lambda) := \lambda^{-1}(\lambda I_n - A)^{-1}B\left[(\lambda^{-q+1} - \lambda^{-\varepsilon})I_m \qquad -\widehat{K}_1\,\widetilde{N}_1(1/\lambda)^T \quad 0_{m\times\widehat{p}}\right].$$

We have first that (41) holds, $\widehat{K}_1 \operatorname{rev} N_1(1/\lambda)^T = \lambda^{-\varepsilon}I_m$ and $\operatorname{rev} N_2(1/\lambda)\widehat{K}_2^T = \lambda^{-\eta}I_p$. With this and $q = \varepsilon + \eta + 1$, one can prove, after somewhat long but direct algebraic manipulations, the following transfer system equivalence transformation at infinity

$$\begin{bmatrix} I_n & 0 \\ \mathbb{W}(\lambda) & \widetilde{U}(1/\lambda) \end{bmatrix} \left[\begin{array}{c|cc} \lambda I_n - A & \lambda^{-1} B \widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T C & \lambda^{-1} M(\lambda) & \lambda^{-1} K_2(\lambda)^T \\ 0 & \lambda^{-1} K_1(\lambda) & 0 \end{array}\right] \begin{bmatrix} I_n & \mathbb{Z}(\lambda) \\ 0 & \widetilde{V}(1/\lambda) \end{bmatrix}$$

$$= \left[\begin{array}{c|ccc} \lambda I_n - A & \lambda^{-q} B & 0 & 0 \\ \hline -C & \lambda^{-q} D(\lambda) & H_{23}(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & H_{42}(\lambda) & H_{43}(\lambda) & I_{\widehat{p}} \end{array}\right] =: F(\lambda),$$

where $H_{42}(\lambda) = \lambda^{-(\varepsilon+1)}\, \widetilde{N}_2(1/\lambda)\widehat{K}_2^T G_{sp}(\lambda)$, $H_{23}(\lambda) = \lambda^{-(\eta+1)}\, G_{sp}(\lambda)\widehat{K}_1 \widetilde{N}_1(1/\lambda)^T$, and $H_{43}(\lambda) = \lambda^{-1}\, \widetilde{N}_2(1/\lambda)\widehat{K}_2^T G_{sp}(\lambda)\widehat{K}_1 \widetilde{N}_1(1/\lambda)^T$ are strictly proper rational matrices. The second step of our proof of *(ii)* of *(a)* in Corollary 4.12 consists of the following transfer system equivalence transformation at infinity

$$\left[\begin{array}{c|ccc} I_n & 0 & 0 & 0 \\ \hline 0 & I_p & -H_{23}(\lambda) & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & 0 & 0 & I_{\widehat{p}} \end{array}\right] F(\lambda) \left[\begin{array}{c|ccc} I_n & 0 & 0 & 0 \\ \hline 0 & I_m & 0 & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & -H_{42}(\lambda) & -H_{43}(\lambda) & I_{\widehat{p}} \end{array}\right]$$

$$= \left[\begin{array}{c|ccc} \lambda I_n - A & \lambda^{-q} B & 0 & 0 \\ \hline -C & \lambda^{-q} D(\lambda) & 0 & 0 \\ 0 & 0 & I_{\widehat{m}} & 0 \\ 0 & 0 & 0 & I_{\widehat{p}} \end{array}\right],$$

which completes the proof of Theorem 5.11. ☐

5.3. Examples of strong linearizations of symmetric rational matrices.

In the following two examples we implement the schemes developed in this paper to obtain strong linearizations for two symmetric rational matrices discussed in [30, Sec. 4.3 and 4.4] and that appear in applications. In addition, the corresponding strong linearizations will preserve the symmetric structure of the problems.

EXAMPLE 5.13. **Vibration of a fluid-solid structure.** Let $G(\lambda) = A - \lambda B + \sum_{i=1}^{k} \frac{\lambda}{\lambda - \sigma_i} E_i$, with A and B $n \times n$ real nonzero symmetric positive semidefinite matrices, $\sigma_i > 0$, and $E_i = C_i C_i^T$, $C_i \in \mathbb{R}^{n \times r_i}$ and $\operatorname{rank} C_i = r_i$, $i = 1, \ldots, k$. First, we separate the polynomial and strictly proper parts of $G(\lambda)$: $G(\lambda) = A + \sum_{i=1}^{k} C_i C_i^T - \lambda B + G_{sp}(\lambda)$, where $G_{sp}(\lambda) = \sum_{i=1}^{k} \frac{\sigma_i}{\lambda - \sigma_i} C_i C_i^T$. The realization of $G_{sp}(\lambda)$ proposed in [30] is $G_{sp}(\lambda) = C(\lambda I_r - \Sigma)^{-1} \Sigma C^T$ where $\Sigma = \operatorname{Diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_k I_{r_k})$, $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_k \end{bmatrix}$ and $r = r_1 + \cdots + r_k$. Without further assumptions on the matrices C_i, we cannot conclude that this realization is minimal and so, some σ_i may not be poles of $G(\lambda)$. Henceforth the result about the number of eigenvalues of $G(\lambda)$ in a given interval (α, β) given at the end of Section 4.3 of [30] may not be correct. It turns out, however, that under very mild conditions the realization of $G_{sp}(\lambda)$ is controllable and observable. If, for example, $\operatorname{rank} C = r$ and we put $H = \Sigma C^T$ then (Σ, H) is controllable, (Σ, C) is observable and $G_{sp}(\lambda) = C(\lambda I_r - \Sigma)^{-1} H$ is a minimal realization of $G_{sp}(\lambda)$. Hence, if $\operatorname{rank} C = r$,

$$L_1(\lambda) = \begin{bmatrix} \lambda I_r - \Sigma & \Sigma C^T \\ -C & A + CC^T - \lambda B \end{bmatrix}$$

is a strong linearization of $G(\lambda)$, according to the paragraph below (28). Moreover, if

$$L(\lambda) = \begin{bmatrix} -\Sigma^{-1} & 0 \\ 0 & I_n \end{bmatrix} L_1(\lambda) = \begin{bmatrix} -\lambda\Sigma^{-1} + I_r & -C^T \\ -C & -\lambda B + (A + CC^T) \end{bmatrix}$$

then $L_1(\lambda)$ and $L(\lambda)$ are strictly system equivalent. Also,

$$\begin{bmatrix} \lambda I_r - \Sigma & \lambda^{-1}\Sigma C^T \\ -C & \lambda^{-1}(A + CC^T - \lambda B) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\lambda\Sigma^{-1} + I_r & -\lambda^{-1}C^T \\ -C & \lambda^{-1}(-\lambda B + (A + CC^T)) \end{bmatrix}$$

are strictly system equivalent at infinity. Thus, by Corollary 4.12, $L(\lambda)$ is also a strong linearization of $G(\lambda)$. $L(\lambda)$ is a symmetric positive semidefinite (see Proposition 4.1 of [30]) strong linearization of $G(\lambda)$. The eigenvalues (finite and at infinity) of $G(\lambda)$ can be computed via the generalized eigenvalue problem $L(\lambda)z = 0$.

EXAMPLE 5.14. **Damped vibration of a structure.** Let $G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1+b_i\lambda}\Delta G_i$ where M and K are $n \times n$ real symmetric positive definite matrices, $b_i > 0$ and $\Delta G_i = L_i L_i^T$ with $L_i \in \mathbb{R}^{n\times r_i}$ and $\operatorname{rank} L_i = r_i$, $i = 1, \ldots, k$. The goal of this example is to use Theorem 5.11 to construct a strong linearization of $G(\lambda)$ that preserves the symmetric structure of the problem. Let us define $\sigma_i = \frac{1}{b_i}$. Then, the decomposition of $G(\lambda)$ into its polynomial and strictly proper parts is $G(\lambda) = \lambda^2 M + K + G_{sp}(\lambda)$, where $G_{sp}(\lambda) = -\sum_{i=1}^{k} \frac{\sigma_i}{\lambda+\sigma_i} L_i L_i^T$. Let us denote $L = \begin{bmatrix} L_1 & L_2 & \cdots & L_k \end{bmatrix}$, $\Sigma = \operatorname{Diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_k I_{r_k})$, and assume that $\operatorname{rank} L = r = r_1 + r_2 + \cdots + r_k$. Again, if $C = -L$ and $B = \Sigma L^T$ then $G_{sp}(\lambda) = C(\lambda I_r + \Sigma)^{-1}B$ is a minimal state-space realization of $G_{sp}(\lambda)$. In addition, $\mathcal{L}(\lambda) = \left[\begin{smallmatrix} K & \lambda M \\ \lambda M & -M \end{smallmatrix}\right]$ is a strong block minimal bases pencil associated to the polynomial part $D(\lambda) = \lambda^2 M + K$ with sharp degree. To check this, note that in the notation of Definition 5.2, $M(\lambda) = \begin{bmatrix} K & \lambda M \end{bmatrix}$, $K_1(\lambda) = \begin{bmatrix} \lambda M & -M \end{bmatrix}$, which is a minimal basis by [11, Thm. 2.4], $N_1(\lambda) = \begin{bmatrix} I_n & \lambda I_n \end{bmatrix}$ is a minimal bases dual to $K_1(\lambda)$ with all its row degrees equal to 1, and $\widehat{p} = 0$, which allows us to take $N_2(\lambda) = I_n$. So, $D(\lambda) = M(\lambda)N_1(\lambda)^T$ and $\deg(D(\lambda)) = \deg(N_2(\lambda)) + \deg(N_1(\lambda)) + 1$. Now, the use of Theorem 5.11 to construct strong linearizations of $G(\lambda)$ requires to know $\widehat{K}_1$, since $\widehat{K}_2^T$ can be taken to be any nonsingular matrix, and in particular I_n. To this purpose, note that $U_1(\lambda) = \left[\begin{smallmatrix} \lambda M & -M \\ I_n & 0 \end{smallmatrix}\right]$ and $U_1(\lambda)^{-1} = \left[\begin{smallmatrix} 0 & I_n \\ -M^{-1} & \lambda I_n \end{smallmatrix}\right]$ are unimodular, which means that we can choose $\widehat{K}_1 = \begin{bmatrix} I_n & 0 \end{bmatrix}$. With this information, we take $X = \Sigma^{-1}$ and $Y = I_r$ in (42) to obtain, by Theorem 5.11, that

$$L_2(\lambda) = \begin{bmatrix} \lambda\Sigma^{-1} + I_r & L^T & 0 \\ L & K & \lambda M \\ 0 & \lambda M & -M \end{bmatrix}$$

is a strong linearization of $G(\lambda)$. In addition, $L_2(\lambda)$ preserves the symmetry of $G(\lambda)$.

The linearizations constructed in this subsection preserve the finite and infinite poles and zeros of the original rational matrices and also their symmetry. Preserving the symmetry is not always possible, since [10, Sec. 7] shows that there exist real symmetric polynomial matrices with even degree which do not have symmetric strong linearizations. It remains as an open problem to determine whether non-polynomial symmetric rational matrices have always symmetric strong linearizations or not.

6. Conclusions and future work. This paper presents for the first time a definition and a theory of strong linearizations of arbitrary rational matrices, which generalize the existing ones of polynomial matrices. This theory includes a spectral characterization of strong linearizations that shows that these pencils satisfy the expected properties. The concepts of transfer system equivalence and transfer system equivalence at infinity are introduced and used to fully characterize the transformations that allow us to construct strong linearizations of rational matrices. With these transformations, infinitely many explicit examples of strong linearizations of rational matrices are obtained, which can be used to compute the whole set of finite and infinite poles and zeros of any rational matrix via standard algorithms for linear pencils.

In the last years, many researchers have studied strong linearizations of polynomial matrices since they are essential in the numerical solution of polynomial eigenvalue problems. Therefore, we expect that this paper will foster further research on strong linearizations of rational matrices as, for instance, the study of the preservation of structures, the comparison of the conditioning of the zeros in the linearizations and in the rational matrix, the analysis of the backward errors introduced in the original problem by a backward stable eigenvalue algorithm applied on the linearization, the recovery of minimal indices and bases and eigenvectors, etc.

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