Series expansion of the LU and Cholesky factors of a matrix

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Setting the problem

Given a square matrix $A = LU$ having LU factorization and a perturbation $\tilde{A} \equiv A + E = \tilde{L}\tilde{U}$, we look for two convergent series of matrices:

$$\tilde{L} = \sum_{k=0}^{\infty} L_k \quad \text{and} \quad \tilde{U} = \sum_{k=0}^{\infty} U_k$$

with $L_k = O(\|E\|^k)$, $U_k = O(\|E\|^k)$, $L_0 = L$, $U_0 = U$, as well as for domains and rates of convergence.

The same approach is valid for the Cholesky factor of positive definite Hermitian matrices.

Previous work on perturbations of LU

- Barrlund: strict normwise perturbation bounds.
- Sun: strict norm and componentwise perturbation bounds.
- Stewart: first-order perturbation expansion and bounds on the norm of the second-order terms.
- Chang and Paige: optimum first order bounds, i.e. condition numbers.
If \( I + L^{-1}EU^{-1} = LU \) then
\[
\tilde{A} = L(I + L^{-1}EU^{-1})U = (LL)(UU) \equiv \tilde{L}\tilde{U}.
\]
Therefore, we can focus on the series of the LU factors of a perturbation of the identity: \( I + F = LU \).

We will use absolute, or monotone, consistent norms
\[
|A| \leq |B| \Rightarrow \|A\| \leq \|B\| \quad \text{and} \quad \|AB\| \leq \|A\|\|B\|.
\]

Let us introduce for simplicity a parameter \( z \)
\[
I + zF = \mathcal{L}(z)\mathcal{U}(z).
\]
For \( \|zF\| < 1 \), the LU factors of \( I + zF \)
- exist and are unique,
- their entries are rational functions of \( z \) with nonzero denominators, therefore
\[
\mathcal{L}(z) = \sum_{k=0}^{\infty} z^k \mathcal{L}_k \quad \text{and} \quad \mathcal{U}(z) = \sum_{k=0}^{\infty} z^k \mathcal{U}_k.
\]
with \( \mathcal{L}_0 = \mathcal{U}_0 = I \).
Recurrence relations for $L_k$ and $U_k$

\[ I + zF = \mathcal{L}(z)\mathcal{U}(z) = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{k} \mathcal{L}_j U_{k-j} \right) z^k. \]

Consequently

\[ F = \mathcal{U}_1 + \mathcal{L}_1 \]
\[ 0 = \mathcal{U}_k + \mathcal{L}_1 \mathcal{U}_{k-1} + \cdots + \mathcal{L}_{k-1} \mathcal{U}_1 + \mathcal{L}_k \quad \text{for } k \geq 2. \]

Setting $z = 1$, we can state:

**Theorem.** Let $F$ be an $n \times n$ matrix with $\|F\| < 1$ then:

1. $I + F$ has a unique LU factorization: $I + F = \mathcal{L}\mathcal{U}$.

2. \[ \mathcal{L} = \sum_{k=0}^{\infty} \mathcal{L}_k \quad \text{and} \quad \mathcal{U} = \sum_{k=0}^{\infty} \mathcal{U}_k. \]
   where $\mathcal{L}_0 = I$, $\mathcal{U}_0 = I$, $\mathcal{L}_1 = F_L$, $\mathcal{U}_1 = F_U$ and for $k \geq 2$:
   \[ \mathcal{L}_k = (-\mathcal{L}_1 \mathcal{U}_{k-1} - \mathcal{L}_2 \mathcal{U}_{k-2} \cdots - \mathcal{L}_{k-1} \mathcal{U}_1)_L, \]
   \[ \mathcal{U}_k = (-\mathcal{L}_1 \mathcal{U}_{k-1} - \mathcal{L}_2 \mathcal{U}_{k-2} \cdots - \mathcal{L}_{k-1} \mathcal{U}_1)_U. \]

3. $L_k = O(\|F\|^k)$ and $U_k = O(\|F\|^k)$.

Here

- $(\cdot)_L$ stands for the strict lower triangular part.
- $(\cdot)_U$ stands for the upper triangular part.
A few terms:

\[ \mathcal{L}_1 + \mathcal{U}_1 = F \]
\[ \mathcal{L}_2 + \mathcal{U}_2 = -F_L F_U \]
\[ \mathcal{L}_3 + \mathcal{U}_3 = F_L (F_L F_U)_U + (F_L F_U)_L F_U \]
\[ \mathcal{L}_4 + \mathcal{U}_4 = -F_L (F_L (F_L F_U)_U)_U - F_L ((F_L F_U)_L F_U)_U \]
\[ -((F_L F_U)_L F_U)_U - (F_L F_U)_L F_U \]

**Rate of convergence**

**Theorem:** Let \( F \) be an \( n \times n \) matrix with \( \| F \| < 1 \), let \( I + F = \mathcal{L} \mathcal{U} \) be the unique LU factorization. If

\[
\mathcal{L} = \sum_{k=0}^{\infty} \mathcal{L}_k \quad \text{and} \quad \mathcal{U} = \sum_{k=0}^{\infty} \mathcal{U}_k
\]

then

\[ |\mathcal{L}_k + \mathcal{U}_k| \leq |F|^k \quad \text{for} \quad k \geq 1. \]

*Therefore* \( |\mathcal{L}_k| \leq (|F|^k)_L \) and \( |\mathcal{U}_k| \leq (|F|^k)_U. \)
Idea of the proof for $k = 4$: 

\[ |C_L D_U| \leq |C_L| D_U \]

\[ C_4 + U_4 \leq |F_L|(|F_L|(|F_L|F_U)U)U + |F_L|(|F_L|(|F_L|F_U)U)|F_U|U \]
\[ +(|F_L|F_U)|F_L|(|F_L|F_U)U + (|F_L|(|F_L|F_U)U)|F_U| \]
\[ +((|F_L|F_U)|F_U)|F_U|U \]
\[ \leq |F_L|F_L|F_L|F_U|F_U + |F_L|(|F_L|F_U)U|F_U| \]
\[ +|F_L|F_U|F_L|F_U|F_U + |F_L|(|F_L|F_U)U|F_U + |F_L|F_U|F_U|F_U | \]
\[ = |F_L|F_L|F_L|F_U|F_U + |F_L|F_L|F_U|F_U| \]
\[ +|F_L|F_U|F_L|F_U|F_U + |F_L|F_U|F_U|F_U|F_U | \]
\[ \leq (|F_L| + |F_U|)^4 \]
\[ = |F|^4 \]
Bounds on the remainders

**Theorem:** Let $F$ be an $n \times n$ matrix with $\|F\| < 1$, let $I + F = LU$ be the unique LU factorization, then

\[
\left| L - \sum_{k=0}^{N} L_k \right| \leq \left( |F|^{N+1} (I - |F|)^{-1} \right)_L,
\]

\[
\left| U - \sum_{k=0}^{N} U_k \right| \leq \left( |F|^{N+1} (I - |F|)^{-1} \right)_U.
\]

**Proof:**

\[
\left| L - \sum_{k=0}^{N} L_k \right| \leq \sum_{k=N+1}^{\infty} |L_k| \leq \sum_{k=N+1}^{\infty} \left( |F|^k \right)_L
\]

\[
= \left( \sum_{k=N+1}^{\infty} |F|^k \right)_L = \left( |F|^{N+1} \sum_{k=0}^{\infty} |F|^k \right)_L
\]

**Normwise Bounds**

\[
\left\| L - \sum_{k=0}^{N} L_k \right\| \leq \frac{\|F\|^{N+1}}{1 - \|F\|}
\]

**Comparison with $N = 1$ by Stewart.**

Three improvements

1. Smaller bounds.
2. Componentwise bounds.
3. Bounds valid for $\|F\| < 1$ (Stewart’s $\|F\| \leq 1/4$).
Given $A = LU$ and $E$ such that $\|L^{-1}EU^{-1}\| < 1$, then $A + E = \tilde{L}\tilde{U}$ has a unique LU factorization. Denote $F = L^{-1}EU^{-1}$.

\[
\tilde{A} = L(I + F)U = (L\mathcal{L})(UU) = \left( L \sum_{k=0}^{\infty} \mathcal{L}_k \right) \left( \sum_{k=0}^{\infty} \mathcal{U}_k \right) U.
\]

Then

\[
|\tilde{U} - U| = \left| \left( \sum_{k=1}^{\infty} \mathcal{U}_k \right) U \right| \leq \left( \sum_{k=1}^{\infty} |\mathcal{U}_k| \right) |U|
\]

\[
\leq \left( \sum_{k=1}^{\infty} \left( |F|^k \right) \right) |U|
\]

\[
\leq (|F|(I - |F|)^{-1})_U |U|,
\]

and

\[
\|\tilde{U} - U\| \leq \frac{\|L^{-1}EU^{-1}\|||U||}{1 - \|L^{-1}EU^{-1}\|}
\]

Similar results for $L$. 
