A PARAMETRIZATION OF THE GROUP OF SYMPLECTIC MATRICES AND ITS APPLICATIONS

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Symplectic Matrices

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{where} \quad I_n \quad n \times n \text{ identity matrix}$$

Definition: $S \in \mathbb{R}^{2n\times 2n}$ is symplectic if $S^T J S = J$

We will consider often the partition of symplectic S

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \text{with} \quad S_{ij} \in \mathbb{R}^{n\times n}$$

The symplectic group is important in theory and applications (Hamiltonian mechanics, control,....)

The problem to be solved

Symplectic matrices are **implicitly defined** as solutions of the nonlinear matrix equation $S^T J S = J$

This makes difficult to work with them both in theory and in numerical algorithms.

OUR GOAL: To present an **explicit description (parametrization)** of the group of symplectic matrices, i.e., to find the set of solutions of

$$S^T J S = J \text{ where } S \text{ unknown}$$

and to apply this parametrization to different problems.

Outline of the talk

1. Previous results
2. Parametrization
3. Subparametrization problems
4. Description of Doubly structured sets (symplectic and other property):
   - LU factorizations of symplectic
   - Positive definite symplectic
   - Positive elementwise symplectic
   - TN, TP, oscillatory symplectic
   - Symplectic M-Matrices
5. Conclusions and Open problems

Previous I: A result by Mehrmann (SIMAX, 1988)

Theorem: Let $S = \left[\begin{smallmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{smallmatrix}\right]$ be symplectic with S_{11} non-singular. Then

$$S = \begin{bmatrix} I & 0 \\ S_{21}S_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & S_{11}^{-T} \end{bmatrix} \begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}$$

where

1. The three factors are symplectic.

2. $S_{21}S_{11}^{-1}$ and $S_{11}^{-1}S_{12}$ are symmetric.

Let us combine this with three trivial facts...

Parametrization with nonsingular (1,1)-block

1. $\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$ is symplectic if and only if $X = X^T$.
2. $\begin{bmatrix} G & 0 \\ 0 & Y \end{bmatrix}$ is symplectic if and only if $Y = G^{-T}$.
3. Products and transposes of symplectic are symplectic.

Theorem: The set of symplectic matrices with non-singular S_{11} is

$$\mathcal{S}^{(1,1)} = \left\{ \underbrace{\begin{bmatrix} G & GE \\ CG & G^{-T} + CGE \end{bmatrix}}_{\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & G^{-T} \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix}} : \begin{array}{c} G \text{ nonsingular} \\ C = C^T,\ E = E^T \end{array} \right\}$$

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$2n^2 + n$ free parameters in this parametrization. This is precisely the dimension of the symplectic group.

What happens if S_{11} is singular?

Definition: Symplectic interchange matrices

$$
\Pi_j = \begin{array}{cc}
 & \begin{array}{ccccccccc} & & j & & & & j+n & & \end{array} \\
\begin{array}{c} \\ \\ j \\ \\ \\ \\ \\ j+n \\ \\ \\ \end{array} &
\left[\begin{array}{ccccc|ccccc}
1 & & & & & & & & & \\
 & \ddots & & & & & & & & \\
 & & 0 & & & & & 1 & & \\
 & & & \ddots & & & & & & \\
 & & & & 1 & 0 & & & & \\
\hline
 & & & & & 1 & & & & \\
 & & & & & & \ddots & & & \\
 & & -1 & & & & & 0 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 1
\end{array}\right]
\end{array}
\in \mathbb{R}^{2n \times 2n}
$$

Theorem (FD, Johnson, LAA 2006): If $S \in \mathbb{R}^{2n \times 2n}$ is symplectic with singular (11)-block then there exist matrices Q and Q' that are products of at most n symplectic interchange matrices such that:

$Q\,S$ and $S\,Q'$ are symplectic with nonsingular $(1,1)$ block.

Theorem: The group of symplectic matrices is

$$\mathcal{S} = \left\{ Q \begin{bmatrix} G & GE \\ CG & G^{-T} + CGE \end{bmatrix} : \begin{array}{c} G \text{ nonsingular} \\ C = C^T, \; E = E^T \\ Q \text{ product of symplectic interchanges} \end{array} \right\}$$

REMARK: Given a symplectic matrix, Q may be not unique, then the previous description is not a *rigurous parametrization*. The nonuniqueness of Q can be useful for numerical purposes.

Subparametrization Problems (I)

1. Parametrization of symplectic matrices with rank$(S_{11}) = k$. This set depends on $2n^2 + n - \frac{(n-k)^2+(n-k)}{2}$ parameters.

SIMILAR FOR ANY OTHER BLOCK

2. Any matrix can be S_{11} of a symplectic. **If** S_{11} is fixed and has rank$(S_{11}) = k$ the set of symplectic matrices compatible can be parametrized and depends on $\frac{n^2+n}{2} + \frac{k^2+k}{2} + n(n-k)$ parameters.

3. $\mathcal{S}^{(1,1)}$ is a dense subset of $\mathcal{S}$ and secuences can be explicitly constructed.

Subparametrization Problems (II)

4. Parametrization of the set of $2n \times n$ matrices that can be $\begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$ of a symplectic.

5. Parametrization of the set of symplectic matrices with given $\begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}$. This is an affine subspace in $\mathbb{R}^{2n \times 2n}$ of dimension $\frac{n^2+n}{2}$.

6. Any $A \in \mathbb{R}^{(n+1)\times(n+1)}$ can be $S(1:n+1, 1:n+1)$ of a symplectic S except by the fact that $a_{n+1,n+1}$ is determined by the other entries.

and more...

LU factorizations of Symplectic Matrices (I)

Theorem: Let $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ be symplectic. Then

1. S has LU factorization if and only if

S_{11} and S_{11}^{-T} have LU factorizations.

2. S has LU factorization if and only if

S_{11} is nonsingular and has LU and UL factorizations.

3. S has LU factorization if and only if

$$\det S_{11}(1:k, 1:k) \ \det S_{11}(k:n, k:n) \neq 0 \qquad k = 1:n$$

to be continued....

LU factorizations of Symplectic Matrices (II)

4. If $S_{11} = L_{11}U_{11}$ and $S_{11}^{-T} = L_{22}U_{22}$ are LU factorizations, then the LU factorization of S is

$$S = \begin{bmatrix} L_{11} & 0 \\ S_{21}U_{11}^{-1} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & L_{11}^{-1}S_{12} \\ 0 & U_{22} \end{bmatrix}$$

5. The LU factors of S are symplectic if and only if S_{11} is diagonal.

Symplectic LU-like factorization

$$S = \begin{bmatrix} L_{11} & 0 \\ S_{21}U_{11}^{-1} & \underbrace{\boxed{L_{11}^{-T}}}_{\text{upper triang.}} \end{bmatrix} \begin{bmatrix} U_{11} & L_{11}^{-1}S_{12} \\ 0 & \underbrace{\boxed{U_{11}^{-T}}}_{\text{lower triang.}} \end{bmatrix}$$

Symplectic Positive Definite (PD) Matrices

Theorem: Let $S = \begin{bmatrix} S_{11} & S_{21}^T \\ S_{21} & S_{22} \end{bmatrix}$ be symmetric and symplectic. Then

1. S is PD if and only if S_{11} is PD.

2. The set of PD symplectic matrices is

$$\mathcal{S}^{\mathsf{PD}} = \left\{ \begin{bmatrix} G & GC \\ CG & G^{-1} + CGC \end{bmatrix} : \begin{array}{c} G \text{ positive definite} \\ C = C^T \end{array} \right\}$$

$\mathcal{S}^{\mathsf{PD}}$ depends on $n^2 + n$ parameters.

3. Every PD symplectic matrix $S = HH^T$ with H symplectic.

Symplectic Matrices with positive entries

Set of symplectic matrices with nonsingular S_{11}

$$\mathcal{S}^{(1,1)} = \left\{ \begin{bmatrix} G & GE \\ CG & G^{-T} + CGE \end{bmatrix} : \begin{array}{c} G \text{ nonsingular} \\ C = C^T, \; E = E^T \end{array} \right\}$$

This allows us to generate symplectic matrices with positive entries (contrast with orthogonal matrices).

START by choosing arbitrary $G > 0$, $C > 0$, and $\widetilde{E} > 0$ with positive entries. So $CG\widetilde{E} > 0$.

THEN, a number $\alpha > 0$ is chosen such that

$$\alpha CG\widetilde{E} + G^{-T} > 0.$$

FINALLY: $E \equiv \alpha \widetilde{E}$

Totally Nonnegative (TN) Symplectic Matrices (I)

DEFINITIONS:

Matrices with all minors nonnegative (positive) are called TN (totally positive TP) matrices.

If A is TN and A^k TP for some positive integer k then A is called OSCILLATORY.

Applications in mechanical oscillatory problems

Totally Nonnegative (TN) Symplectic Matrices (II)

I is symplectic and TN.

Are there symplectic TP matrices?
Are there symplectic oscillatory matrices?

What is the set of symplectic TN matrices?

Theorem (the 2×2 case): $S \in \mathbb{R}^{2\times 2}$ is symplectic and TP if and only if $\det S = 1$ and $s_{ij} > 0$ for all (i, j).

If any three positive entries such that $s_{11}s_{22} > 1$ are chosen then the remaining entry is obtained from $\det S = 1$.

Totally Nonnegative (TN) Symplectic Matrices (III)

Theorem: Let $S \in \mathbb{R}^{2n \times 2n}$ with $n > 1$ be symplectic. Then

1. S is not TP.
2. S is not oscillatory.

Sketch of the Proof: LU factorization of S is

$$S = \begin{bmatrix} L_{11} & 0 \\ S_{21}U_{11}^{-1} & L_{22} \end{bmatrix} \begin{bmatrix} U_{11} & L_{11}^{-1}S_{12} \\ 0 & U_{22} \end{bmatrix} \quad \text{where} \quad \begin{array}{l} S_{11} = L_{11}U_{11} \\ S_{11}^{-T} = L_{22}U_{22} \end{array}$$

If S TP then S_{11} is TP and the LU factors of S are triangular TP. Therefore L_{22} and U_{22} are triangular TP and S_{11}^{-T} is TP.

CONTRADICTION!!, S_{11} TP implies that S_{11}^{-T} has negative entries.

Totally Nonnegative (TN) Symplectic Matrices (IV)

Theorem: The set of $2n \times 2n$ $(n > 1)$ symplectic and TN matrices is

$$\mathcal{S}^{\mathsf{TN}} = \left\{ \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} : \; D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \lambda_i > 0 \right\}$$

Symplectic M-Matrices (I)

Definition: $A \in \mathbb{R}^{n \times n}$ is a M-Matrix if $a_{ij} \leq 0$ for $i \neq j$ and $\mathrm{Re}(\lambda) > 0$ for every eigenvalue λ of A.

Theorem: The set of $2n \times 2n$ symplectic M-Matrices is

$$\mathcal{S}^{\mathrm{M}} = \left\{ \begin{bmatrix} D & DK \\ HD & D^{-1} + HDK \end{bmatrix} : \begin{array}{c} D \text{ positive diagonal} \\ H = H^T \leq 0 \\ K = K^T \leq 0 \\ HDK \text{ diagonal} \end{array} \right\}$$

Symplectic M-Matrices (II)

Given an arbitrary $H = H^T \leq 0$, the matrices $K = K^T \leq 0$ such that HDK is *diagonal* can be easily determined. For instance if $h_{12} = h_{21} \neq 0$:

$$H = \begin{bmatrix} & x & & & \\ x & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \longrightarrow K = \begin{bmatrix} 0 & ? & 0 & 0 & 0 \\ ? & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & & \\ 0 & 0 & & & \end{bmatrix}$$

The **?** that remain in K after this process is repeated for all entries $h_{ij} = h_{ji} \neq 0$, are free parameters in $K = K^T \leq 0$ for a given H.

Conclusions and Open Problems

- An explicit description of the group of symplectic matrices has been introduced.
- It allows to solve very easily many theoretical questions.
- Perturbation theory with respect the symplectic parameters? Interesting properties?
- How to compute the parametrization in a stable an efficient way if we are given the entries of a symplectic matrix?
- Is it possible to get a rank revealing factorization?
- Have these symplectic parameters an intrinsic meaning?