LOW RANK PERTURBATIONS OF SPECTRAL CANONICAL FORMS

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- Low rank perturbations of matrices arise frequently in applications and in theory.

- They appear when a system with many degrees of freedom is controlled with actions on a small subset of the degrees of freedom

- Well-known example: Sherman-Morrison-Woodbury formula.

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}$$

$$A \;\; n \times n, \; U, V \;\; n \times k, \; \text{rank}(U) = \text{rank}(V) = k.$$

Our Goal: How are typically modified spectral canonical forms by low rank perturbations?

- Jordan canonical form (JCF) of $A \in \mathbb{C}^{n\times n}$.

- Weierstrass canonical form (WCF) of regular matrix pencils

$$A + \lambda B,$$

$A, B \in \mathbb{C}^{n\times n}$ and $\det(A + \lambda B)$ does not vanish identically. Generalized eigenvalue problem

$$(A + \lambda B)v = 0$$

- Kronecker canonical form (KCF) of singular matrix pencils

$$A + \lambda B,$$

$A, B \in \mathbb{C}^{m\times n}$ or $A, B \in \mathbb{C}^{n\times n}$ and $\det(A + \lambda B) = 0$ for all λ.

Many different things may happen

$$A+E_1 = \left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$A+E_2 = \left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Our goal is to describe the GENERIC or most frequent behavior. This will be a behavior that holds for all perturbations E except those in a set of zero Lebesgue measure. We are able to describe explicitly this set (HARD AND NOT EASY).

Notation

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda \end{bmatrix} \in \mathbb{C}^{k \times k}$$

$$J_{k_1}(\lambda_1) \oplus \ldots \oplus J_{k_p}(\lambda_p) = \begin{bmatrix} J_{k_1}(\lambda_1) & & & \\ & \cdot & & \\ & & \cdot & \\ & & & J_{k_p}(\lambda_p) \end{bmatrix}$$

Direct sum or block diagonal matrix of Jordan blocks.

Perturbation of Jordan canonical form: an example (I)

$$\text{JCF of } A = J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)$$

Notice that A has only two different eigenvalues 9 and -3.

Let E be such that $\text{rank}(E) = 2$. Then generically

$$\text{JCF of } A + E = * \oplus \ldots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus * \oplus \ldots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)$$

In the $* \oplus \ldots \oplus *$ of the JCF of $A + E$ there are no Jordan blocks associated to the eigenvalues 9 and -3. Besides, in general, it contains only 1×1 Jordan blocks.

Perturbation of Jordan canonical form: an example (II)

$$\text{JCF of } A = J_5(9) \oplus J_5(9) \oplus J_5(9) \oplus J_3(9) \oplus J_7(-3) \oplus J_6(-3) \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)$$

$\text{rank}(E) = 2.$

$$\text{JCF of } A + E = * \oplus \ldots \oplus * \oplus J_5(9) \oplus J_3(9) \oplus * \oplus \ldots \oplus * \oplus J_4(-3) \oplus J_3(-3) \oplus J_1(-3)$$

For every eigenvalue of A the perturbation E destroys the $2 = \text{rank}(E)$ largest Jordan blocks. The other Jordan blocks of A remains as Jordan blocks of $A+E$.

Perturbation of Jordan canonical form: The Theorem

Theorem: *Let* $A \in \mathbb{C}^{n\times n}$ *and* λ_0 *be an eigenvalue of* A *with* g_0 *Jordan blocks in the JCF of* A. *Let* $E \in \mathbb{C}^{n\times n}$ *with* $\text{rank}(E) \le g_0$.

Then the Jordan blocks in the JCF of $A + E$ *with eigenvalue* λ_0 *are just the* $g_0 - \text{rank}(E)$ *smallest Jordan blocks of* A *with eigenvalue* λ_0 *if and only if* E *does not belong to a certain algebraic manifold of codimension one in the matrix space* $\mathbb{C}^{n\times n}$.

JCF: Comments on the Theorem

1. The perturbations E are not small.

2. A perturbation matrix E can satisfy the assumptions of the Theorem for one eigenvalue but not for others.

3. Condition $\text{rank}(E) \leq g_0$ defines what we understand by "low rank" in this context. It depends on the eigenvalue we consider.

4. Let $A = PJP^{-1}$ be a Jordan Canonical factorization. If J and P are given then we are able to give an explicit equation for the algebraic manifold mentioned in the theorem in terms of some minors of $P^{-1}EP$. We have a different equation for each eigenvalue λ_0 of A.

5. We will explain this manifold at the end of the talk if we have time. Some additional notation is needed.

Some intuitions

- $\#\lambda_0$-Jordan blocks of A $= \dim \mathrm{Nul}(A - \lambda_0 I) \equiv g_0$.

- $\mathrm{rank}(A + E - \lambda_0 I) \le \mathrm{rank}(A - \lambda_0 I) + \mathrm{rank}(E) \le n$

- $\dim \mathrm{Nul}(C) = n - \mathrm{rank}(C)$

Then $\boxed{g_0 - \mathrm{rank}(E) \le \dim \mathrm{Nul}(A + E - \lambda_0 I)}$

Generically

$$\mathrm{rank}(A + E - \lambda_0 I) = \mathrm{rank}(A - \lambda_0 I) + \mathrm{rank}(E)$$

and

$$\boxed{g_0 - \mathrm{rank}(E) = \dim \mathrm{Nul}(A + E - \lambda_0 I)}$$

Why and when the smallest Jordan Blocks?

Weierstrass canonical form: summary (I)

Theorem (Weierstrass) *Let* $A, B \in \mathbb{C}^{n\times n}$ *such that the polynomial* $p(\lambda) = \det(A + \lambda B)$ *does not vanish identically. Then there exist two nonsingular matrices* R *and* S *such that*

$$R(A + \lambda B)S = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} + \lambda \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix},$$

J is in Jordan canonical form, and N is in Jordan canonical form with all its eigenvalues equal to zero. J and N are unique up to permutations of the diagonal Jordan blocks. This is called the Weierstrass canonical form of the pencil $A + \lambda B$.

Weierstrass canonical form: summary (II)

$$R(A+\lambda B)S = \begin{pmatrix} J & 0 \\ 0 & I_q \end{pmatrix} + \lambda \begin{pmatrix} I_p & 0 \\ 0 & N \end{pmatrix},$$

1. The WCF contains all the spectral information of the generalized eigenvalue problem $(A+\lambda B)v = 0$

2. J shows the Jordan structure of the finite eigenvalues of $A+\lambda B$.

3. N shows the Jordan structure of the infinite eigenvalue of $A+\lambda B$.

4. The Jordan structure of the infinite eigenvalue of $A+\lambda B$ is the Jordan structure of the zero eigenvalue of $B+\lambda A$.

5. Related to systems of algebraic-differential equations

$$B\,\frac{d\,x(t)}{dt} = A\,x(t)$$

Perturbation of Weierstrass canonical form: example (I)

Part WCF of $(A + \lambda B)$ for $(\lambda = -5)$ is

$$J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)$$

Let us consider a perturbation $E_A + \lambda E_B$ such that

$$\text{rank}(E_A - 5\,E_B) = 2 \quad \text{and} \quad \text{rank}(E_B) = 1$$

Then generically

Part WCF of $(A + E_A + \lambda(B + E_B))$ for $(\lambda = -5)$ is

$$J_1(5) \oplus J_2(5)$$

Perturbation Weierstrass canonical form: example (II)

$$\text{WCF of } (A + \lambda B) \text{ is } J_5(5) \oplus J_4(5) \oplus J_3(5) \oplus J_2(5)$$

$$\text{rank}(E_A - 5\,E_B) = 2 \quad \text{and} \quad \text{rank}(E_B) = 1$$

$$\text{WCF of } (A + E_A + \lambda(B + E_B)) \text{ is } J_1(5) \oplus J_2(5)$$

The $2 = \text{rank}(E_A - 5E_B)$ largest Jordan blocks are destroyed.

The $1 = \text{rank}(E_B)$ following largest Jordan blocks turn into 1×1 blocks.

Only the $4 - \text{rank}(E_A - 5E_B) - \text{rank}(E_B)$ smallest Jordan blocks remain unchanged.

Perturbation Weierstrass canonical form: The Theorem

Theorem: *Let* λ_0 *be an eigenvalue of the regular pencil* $A+\lambda B$ *with* g_0 *Jordan blocks in the WCF. Let* $E_A + \lambda E_B$ *be another pencil such that* $\text{rank}(E_A + \lambda_0 E_B) < g_0$. *Let us define*

$$\rho_0 = \text{rank}(E_A + \lambda_0 E_B), \qquad \rho_1 = \text{rank}(E_B).$$

Then for all pencils $E_A + \lambda E_B$ *except those in an algebraic manifold of codimension one:*

1. *There are* $g_0 - \rho_0$ *Jordan blocks for* λ_0 *in the WCF of* $A + E_A + \lambda(B + E_B)$, *and*

2. *they are the* $g_0 - \rho_0 - \rho_1$ *smallest Jordan Blocks for* λ_0 *in the WCF of* $A + \lambda B$,

3. *together with* ρ_1 1×1 *Jordan blocks for* λ_0.

WCF: Comments on the Theorem

1. Similar remarks to those of JCF.

2. In the case $\text{rank}(E_A) + \text{rank}(E_B) \leq n$ and $\lambda_0 \neq 0$ *generically*

$$\text{rank}(E_A + \lambda_0 E_B) = \text{rank}(E_A) + \text{rank}(E_B),$$

and the number of destroyed and preserved Jordan Blocks "does not" depend on the particular λ_0.

3. Again an elementary rank argument show that

$$g_0 - \text{rank}(E_A + \lambda_0 E_B) \leq \dim \text{Nul}(\, A + E_A + \lambda_0 (B + E_B)\,) =$$

Number of λ_0-Jordan blocks in the WCF

Intuition on differences WCF vs. JCF

JCF. Problem: $(A-\lambda_0 I)v = 0$. Jordan chain corresponding to a $k \times k$ Jordan block

$$(A-\lambda_0 I)v_1 = 0, \quad (A-\lambda_0 I)v_j = v_{j-1} \qquad j = 2:k$$

$\uparrow$ E

WCF. Problem: $(A+\lambda_0 B)v = 0$. Jordan chain corresponding to a $k \times k$ Jordan block

$$(A+\lambda_0 B)v_1 = 0, \quad (A+\lambda_0 B)v_j = Bv_{j-1} \qquad j = 2:k$$

$\nearrow$ $E_A + \lambda_0 E_B$ $\qquad$ $\nwarrow$ E_B

Kronecker canonical form: summary (I)

Theorem (Kronecker) *Let* $A, B \in \mathbb{C}^{m \times n}$. *Then there exist two nonsingular matrices* R *and* S *such that*

$$\begin{aligned} R\,(A + \lambda B)\,S \;=\; & L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_p}(\lambda) \oplus \\ & L^T_{\eta_1}(\lambda) \oplus \ldots \oplus L^T_{\eta_q}(\lambda) \oplus \\ & (J + \lambda I) \oplus (I + \lambda N), \end{aligned}$$

J *is square and is in Jordan canonical form,*

N *is square and is in Jordan canonical form with all its eigenvalues equal to zero,*

Kronecker canonical form: summary (II)

Continuation Kronecker's *Let* $A, B \in \mathbb{C}^{m \times n}$

$$R\,(A + \lambda B)\,S \;=\; L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_p}(\lambda) \oplus L_{\eta_1}^T(\lambda) \oplus \ldots \oplus L_{\eta_q}^T(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N),$$

$$L_{\epsilon_i}(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix} \in \mathbb{C}^{\epsilon_i \times (\epsilon_i + 1)}$$

$0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_p$ *are the column minimal indices.*

$0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_q$ *are the row minimal indices.*

$L_{\epsilon_i}(\lambda)$ $(L_{\eta_i}^T(\lambda))$ *are called column (row) singular blocks.*

Kronecker canonical form: summary (III)

Continuation Kronecker's *Let* $A, B \in \mathbb{C}^{m \times n}$

$$R\,(A+\lambda B)\,S = L_{\epsilon_1}(\lambda)\oplus\ldots\oplus L_{\epsilon_p}(\lambda)\oplus L^T_{\eta_1}(\lambda)\oplus\ldots\oplus L^T_{\eta_q}(\lambda)\oplus(J+\lambda I)\oplus(I+\lambda N)$$

1. $0 \leq \epsilon_1 \leq \epsilon_2 \leq \ldots \leq \epsilon_p$ *and* $0 \leq \eta_1 \leq \eta_2 \leq \ldots \leq \eta_q$ *are unique.*

2. J *and* N *are unique up to permutations of the Jordan diagonal blocks.*

3. $(J + \lambda I) \oplus (I + \lambda N)$ *is the regular part of the pencil.*

4. $\text{rank}(A + \lambda B) = n - p = m - q$

5. $R\,(A+\lambda B)\,S = (J+\lambda I)\oplus(I+\lambda N)$ *if and only if* $\det(A+\lambda B) \neq 0$.

6. *KCF application in control theory.*

Perturbations of Kronecker canonical form: General Remarks

1. We will consider three singular $m \times n$ pencils with rank less than $\min\{m, n\}$:

$$\text{Unperturbed: } P(\lambda) = A + \lambda B;$$
$$\text{Perturbation: } E(\lambda) = E_A + \lambda E_B,$$
$$\text{Perturbed: } (P + E)(\lambda) = (A + E_A) + \lambda(B + E_B).$$

2. We will assume

$$\text{rank}(P{+}E)(\lambda) = \text{rank}\, P(\lambda) + \text{rank}\, E(\lambda) < \min\{m, n\}$$

3. Therefore, we have the global low rank condition

$$\rho \equiv \text{rank}\, E(\lambda) < \min\{p, q\}$$

Number of minimal indices

$$\text{rank}(P+E)(\lambda) = \text{rank}\, P(\lambda) + \text{rank}\, E(\lambda) < \min\{m,n\}$$

and

$$\rho \equiv \text{rank}\, E(\lambda) < \min\{p,q\}$$

implies

Number of column (row) singular blocks of $(P+E)(\lambda)$ is equal to $p - \text{rank}(E)$ $(q - \text{rank}(E))$,

What are their dimensions?

How is the regular part?

Relevant data of unperturbed and perturbation pencil

$P(\lambda) = A + \lambda B$

$$0 \le \epsilon_1 \le \ldots \le \epsilon_p \text{ and } 0 \le \eta_1 \le \ldots \le \eta_q$$

$$(J + \lambda I) \oplus (I + \lambda N)$$

$E(\lambda) = E_A + \lambda E_B$

$$0 \le \tilde{\epsilon}_1 \le \ldots \le \tilde{\epsilon}_{\tilde{p}} \text{ and } 0 \le \tilde{\eta}_1 \le \ldots \le \tilde{\eta}_{\tilde{q}}$$

$$\tilde{\epsilon} \equiv \tilde{\epsilon}_1 + \ldots + \tilde{\epsilon}_{\tilde{p}} \text{ and } \tilde{\eta} \equiv \tilde{\eta}_1 + \ldots + \tilde{\eta}_{\tilde{q}}$$

$$(J_E + \lambda I) \oplus (I + \lambda N_E)$$

Auxiliary Lemma

Definitions:

$$d_k = \left\lfloor \frac{\sum_{i=1}^{k} \epsilon_i + \tilde{\epsilon}}{k-\rho} \right\rfloor \qquad k = (\rho+1) : p$$

$$h_k = \left\lfloor \frac{\sum_{i=1}^{k} \eta_i + \tilde{\eta}}{k-\rho} \right\rfloor \qquad k = (\rho+1) : q$$

$$d_{min} = \min_k d_k \quad \text{and} \quad h_{min} = \min_k h_k$$

Lemma: There exists only one index s (t) such that

1. $d_s = d_{min}$ ($h_t = h_{min}$)
2. $d_s \geq \epsilon_s \geq \ldots \geq \epsilon_1$ ($h_t \geq \eta_t \geq \ldots \geq \eta_1$)
3. If $k > s$ ($k > t$) then $\epsilon_k > d_k \geq d_s$ ($\eta_k > h_k \geq h_t$)

Low rank perturbation of KCF: The Theorem

Theorem: Let γ_s (μ_t) be the remainder of the integer division of $\sum_{i=1}^{s} \epsilon_i + \tilde{\epsilon}$ by $s - \rho$ (of $\sum_{i=1}^{t} \eta_i + \tilde{\eta}$ by $(t - \rho)$), where $\rho = \text{rank}(E(\lambda))$. Then under certain generic conditions the KCF of $(P + E)(\lambda)$ is determined by

1. $s - \rho - \gamma_s$ column minimal indices equal to d_s,
   γ_s column minimal indices equal to $d_s + 1$,
   $p - s$ column minimal indices equal to $\epsilon_{s+1}, \ldots, \epsilon_p$.

2. $t - \rho - \mu_t$ row minimal indices equal to h_t,
   μ_t row minimal indices equal to $h_t + 1$,
   $q - t$ row minimal indices equal to $\eta_{t+1}, \ldots, \eta_q$.

3. The regular part is

$$(J + \lambda I) \oplus (J_E + \lambda I) \oplus (\lambda I + N) \oplus (\lambda I + N_E)$$

Low rank perturbation of KCF: Three Main Ideas

- The blocks of the regular parts of $P(\lambda)$ and $E(\lambda)$ remain unchanged in the sum $(P+E)(\lambda)$ and no more regular blocks appear.

- The largest $p-s$ column and $q-t$ row singular blocks of $P(\lambda)$ remain unchanged as singular blocks of $(P+E)(\lambda)$.

- The smallest s column and t row singular blocks of $P(\lambda)$ are destroyed or transformed into larger blocks (but not larger than the unchanged ones).

JCF: generic conditions

Theorem: *Let $A \in \mathbb{C}^{n\times n}$ and λ_0 be an eigenvalue of A with g_0 Jordan blocks in the JCF of A. Let $E \in \mathbb{C}^{n\times n}$ with $\operatorname{rank}(E) \le g_0$.*

Then the Jordan blocks in the JCF of $A+E$ with eigenvalue λ_0 are just the $g_0 - \operatorname{rank}(E)$ smallest Jordan blocks of A with eigenvalue λ_0 if and only if E does not belong to a certain algebraic manifold of codimension one in the matrix space $\mathbb{C}^{n\times n}$.

JCF: generic conditions. Example (I)

$$A+E=\left[\begin{array}{ccc|cc|cc|c}
0 & 1 & 0 & & & & & \\
0 & 0 & 1 & & & & & \\
0 & 0 & 0 & & & & & \\ \hline
 & & & 0 & 1 & & & \\
 & & & 0 & 0 & & & \\ \hline
 & & & & & 0 & 1 & \\
 & & & & & 0 & 0 & \\ \hline
 & & & & & & & \hat{J}
\end{array}\right]+\left[\begin{array}{ccc|cc|cc|c}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
\blacksquare & * & * & \clubsuit & * & \spadesuit & * & * \\ \hline
* & * & * & * & * & * & * & * \\
\clubsuit & * & * & \clubsuit & * & \heartsuit & * & * \\ \hline
* & * & * & * & * & * & * & * \\
\spadesuit & * & * & \heartsuit & * & \spadesuit & * & * \\ \hline
* & * & * & * & * & * & * & *
\end{array}\right]$$

$$\text{rank}(E)=2 \text{ and } \lambda_0=1$$

$$C_0=\det\left[\begin{array}{c|c} \blacksquare & \clubsuit \\ \hline \clubsuit & \clubsuit \end{array}\right]+\det\left[\begin{array}{c|c} \blacksquare & \spadesuit \\ \hline \spadesuit & \spadesuit \end{array}\right]$$

JCF: generic conditions. Example (II)

$$JCF \text{ of } A+E = \left[\begin{array}{ccc|cc|cc|c} * & * & * & & & & & \\ * & * & * & & & & & \\ * & * & * & & & & & \\ \hline & & & * & * & & & \\ & & & * & * & & & \\ \hline & & & & & 0 & 1 & \\ & & & & & 0 & 0 & \\ \hline & & & & & & & * \end{array}\right]$$

if and only if

$$C_0 = \det \left[\begin{array}{c|c} \blacksquare & \clubsuit \\ \hline \clubsuit & \clubsuit \end{array}\right] + \det \left[\begin{array}{c|c} \blacksquare & \spadesuit \\ \hline \spadesuit & \spadesuit \end{array}\right] \neq 0$$

JCF: generic conditions. General Case.

Theorem: Let rank$(E) = \rho$ and the JCF of A be

$$P^{-1}AP = J_{n_1}(\lambda_0) \oplus \ldots \oplus J_{n_\rho}(\lambda_0) \oplus J_{n_{\rho+1}}(\lambda_0) \oplus \ldots J_{g_0}(\lambda_0) \oplus \hat{J},$$

with $n_1 \geq \ldots \geq n_{g_0}$ and $\det(\hat{J} - \lambda_0 I) \neq 0$.

1. If $n_\rho > n_{\rho+1}$ and Φ_ρ is the minor of $P^{-1}EP$ corresponding to the lower left positions of the ρ largest Jordan blocks of $P^{-1}AP$ then

Generic behavior if and only if $\Phi_\rho \neq 0$.

2. If $n_\rho = n_{\rho+1}$ and Φ_ρ is ANY minor of $P^{-1}EP$ corresponding to the lower left positions of ρ largest Jordan blocks of $P^{-1}AP$ then

Generic behavior if and only if $\sum \Phi_\rho \neq 0$.

References and Work in progress

OUR WORK:

- JCF. Moro and FMD, SIMAX 2003.
- WCF. De Terán, FMD, Moro, submitted.
- KCF. De Terán, FMD, in preparation (one month!)
- KCF (Singular goes to Full Rank). De Terán, FMD, still in progress.

RELATED WORK: ONLY JCF, GENERIC CONDITION NOT GIVEN

- Hörmander and Melin. JCF. Compact operators. Math. Scand. 1994.
- Sevchenko. JCF. Rank one perturbations. Mat. Zametki 2003.
- Sevchenko. JCF. Funct. Anal. Appl. 2004.