FIRST ORDER PERTURBATION EXPANSIONS FOR EIGENVALUES OF SINGULAR MATRIX PENCILS

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BASIC DEFINITIONS (I)

Definition: Let $A_0, A_1 \in \mathbb{C}^{m\times n}$. The matrix pencil

$$A_0 + \lambda A_1$$

is singular if

(i) $m \neq n$, or
(ii) $m = n$ and $\det(A_0 + \lambda A_1) = 0$ for all λ.

Otherwise, the pencil is called regular.

Example:

$$A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \det(A_0 + \lambda A_1) \equiv 0$$

BASIC DEFINITIONS (II)

Definition: The normal rank (nrank) of the matrix pencil

$$A_0 + \lambda A_1$$

is the dimension of its largest minor that is not identically zero.

Example:

$$A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathrm{nrank}(A_0 + \lambda A_1) = 1$$

BASIC DEFINITIONS (III)

Definition: A number μ is an eigenvalue of $A_0 + \lambda A_1$ if

$$\operatorname{rank}(A_0 + \mu A_1) < \operatorname{nrank}(A_0 + \lambda A_1)$$

Example: $\begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ has only one eigenvalue $\mu = 0$, because

$$0 = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} < \operatorname{nrank} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} = 1$$

although for every number μ

$$\begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

BASIC DEFINITIONS (IV)

Definition: The pencil $A_0+\lambda A_1$ has an infinite eigenvalue if zero is an eigenvalue of the dual pencil

$$\lambda A_0 + A_1.$$

Example:

$$A_0+\lambda A_1 = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda A_0+A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvalues of $A_0 + \lambda A_1 = \{\mathbf{0}, \infty\}$.

For brevity, we will only present results for finite eigenvalues.

APPLICATIONS OF SINGULAR PENCILS

Singular pencils do not appear as frequently as regular pencils, but they are used in relevant applications as for instance

- Differential-Algebraic Equations
- System and control theory

EIGENVALUES OF SINGULAR PENCILS ARE DISCONTINUOS FUNCTIONS OF MATRIX ENTRIES

Example: Let us replace $A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$ by the perturbed pencil

$$\begin{aligned} \widehat{A}_0 + \lambda \widehat{A}_1 &= \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \left(\begin{bmatrix} 6 & -3 \\ -10 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\ &= \begin{bmatrix} \lambda + \epsilon 6 & \epsilon(\lambda - 3) \\ \epsilon(\lambda - 10) & 0 \end{bmatrix} \end{aligned}$$

Then $\det(\widehat{A}_0 + \lambda \widehat{A}_1) = -\epsilon^2 (\lambda - 3)(\lambda - 10)$

Eigenvalue of $A_0 + \lambda A_1 = \{\mathbf{0}\}$

Eigenvalues of $\widehat{A}_0 + \lambda \widehat{A}_1 = \{\mathbf{3}, \mathbf{10}\}$ for any $\epsilon \neq 0$!!

…or they may dissapear under arbitrarily small perturbations

Example:

$$A_0+\lambda A_1+\epsilon(B_0+\lambda B_1)=\begin{bmatrix}\lambda & 0 & 0\\ 0 & 0 & 0\end{bmatrix}+\epsilon\begin{bmatrix}1 & 2+\lambda & 3\\ 4 & 5 & 6\end{bmatrix}$$

Eigenvalue de $A_0+\lambda A_1 = \{0\}$

The **perturbed pencil has not eigenvalues** because

$$\text{rank}(A_0+\lambda A_1+\epsilon(B_0+\lambda B_1)) = 2 \text{ for all } \lambda.$$

To see this:

$$\begin{aligned}\text{MINOR}(:,1:2) &= \epsilon((5-4\epsilon)\lambda-3\epsilon)\\ \text{MINOR}(:,1:3) &= 6\epsilon(\lambda-\epsilon)\\ \text{MINOR}(:,2:3) &= 3\epsilon^2(2\lambda-1)\end{aligned}$$

STRUCTURED PERTURBATIONS ARE NEEDED

- Singular pencils have ill-posed eigenvalues since arbitrarily small perturbations may produce arbitrarily large changes in, or may destroy, the eigenvalues.

- We cannot expect a reasonable perturbation theory of eigenvalues for general perturbations.

- We need to restrict the set of allowable perturbations to develop an **eigenvalue perturbation theory.**

PREVIOUS WORK ON SINGULAR PENCILS (I)

1- Wilkinson (1978, LAA 1979): *Kronecker's canonical form and the QZ algorithm.*

- No theory. Only examples.
- Only Square singular pencils.

Wilkinson's Example:

$$\widehat{A}_0 + \lambda \widehat{A}_1 \ = \ \begin{bmatrix} \lambda - 2 & 0 \\ 0 & 0 \end{bmatrix} + \left(\begin{bmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{bmatrix} + \lambda \begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix} \right)$$

Wilkinson's Comment: Although quite respectable eigenvalues may be completely destroyed by arbitrarily small perturbations, for almost all small perturbations ϵ_i and η_i the perturbed pencil $\widehat{A}_0 + \lambda \widehat{A}_1$ has an eigenvalue which is very close to 2.

This holds for several numerical experiments.

EXPERIMENT ON WILKINSON EXAMPLE

$$(A_0 + \lambda A_1) + (E_0 + \lambda E_1) = \begin{bmatrix} \lambda - 2 & 0 \\ 0 & 0 \end{bmatrix} + \text{random } \|\cdot\| = 10^{-5}$$

Case	λ_1	λ_2
1	1.999905	1.634357
2	1.999996	2.336862
3	2.000027	1.157266
4	2.000033	3.240760
5	1.999984	2.389794
6	1.999994	-0.300021
$\vdots$	$\vdots$	$\vdots$

PREVIOUS WORK ON SINGULAR PENCILS (II)

2- Sun (1983, Math. Num. Sin. 85)

- Only Square $n \times n$ singular pencils $A_0 + \lambda A_1$.
- A_0 and A_1 simultaneously diagonalizable by equivalence PA_0Q, PA_1Q. RESTRICTIVE.
- nrank$(A_0 + \lambda A_1) = n - 1$. VERY RESTRICTIVE
- Finite bounds of type: Gerschgorin, Hofmann-Wielandt, Bauer-Fike. IN A PROBABILISTIC SENSE.
- **Allowable set of perturbation:** Generic (almost all), uniformly distributed.

PREVIOUS WORK ON SINGULAR PENCILS (III)

3- Demmel and Kågström (LAA 1987)

- Square and Rectangular singular pencils.
- **Allowable set of perturbation**: Non Generic, very specific perturbations.
- They are useful in Algorithms for computing GUPTRI form.

4- Stewart (LAA 1994)

- Only for Rectangular singular pencils.
- **Allowable set of perturbation**: Non Generic, very specific perturbations.

Common strategy in these works: The problem is reduced to an eigenvalue perturbation problem for **regular pencils**.

OUR GOALS

- To characterize generic perturbations of general square singular matrix pencils for which the eigenvalues change continuosly.

- For these perturbations to develop first order perturbation expansions for the variation of eigenvalues (simple or multiple).

- Brief discussion on eigenvectors.

A property is said to be GENERIC if it holds for all pencils except those in an algebraic manifold of codimension larger than or equal to one (then except those in a set of zero Lebesgue measure.)

WHY ONLY SQUARE PENCILS?

- Given a singular square pencil $A_0 + \lambda A_1$, generic perturbations will produce a regular pencil, and we will see that if $A_0 + \lambda A_1$ has eigenvalues then the regular perturbed pencil has some of its eigenvalues close to those of $A_0 + \lambda A_1$.

- Given a singular rectangular pencil $A_0 + \lambda A_1$, generic perturbations will produce a pencil that has not eigenvalues.

EXAMPLE ON DISCONTINUITY REVISITED (I)

Example: Let us change $A_0 + \lambda A_1 = \left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to

$$\begin{aligned} \widehat{A}_0 + \lambda \widehat{A}_1 &= \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \left(\begin{bmatrix} 6 & -3 \\ -10 & c_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & d_{22} \end{bmatrix} \right) \\ &= \begin{bmatrix} \lambda + \epsilon 6 & \epsilon(\lambda - 3) \\ \epsilon(\lambda - 10) & \epsilon(c_{22} + \lambda d_{22}) \end{bmatrix} \end{aligned}$$

$$\det(\widehat{A}_0 + \lambda \widehat{A}_1) = \epsilon(\lambda + \epsilon 6)(c_{22} + \lambda d_{22}) - \epsilon^2 (\lambda - 3)(\lambda - 10)$$

$$\det(\widehat{A}_0 + \lambda \widehat{A}_1) = \epsilon((\lambda + \epsilon 6)(c_{22} + \lambda d_{22}) - \epsilon (\lambda - 3)(\lambda - 10))$$

If $\epsilon \neq 0$ the eigenvalues of $\widehat{A}_0 + \lambda \widehat{A}_1$ are the roots of

$$p_\epsilon(\lambda) = (\lambda + \epsilon 6)(c_{22} + \lambda d_{22}) - \epsilon (\lambda - 3)(\lambda - 10)$$

EXAMPLE ON DISCONTINUITY REVISITED (II)

$$\widehat{A}_0 + \lambda \widehat{A}_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + \epsilon \left(\begin{bmatrix} 6 & -3 \\ -10 & c_{22} \end{bmatrix} + \lambda \begin{bmatrix} 0 & 1 \\ 1 & d_{22} \end{bmatrix} \right)$$

has as eigenvalues, if $\epsilon \neq 0$, the roots of

$$p_\epsilon(\lambda) = (\lambda + \epsilon 6)(c_{22} + \lambda d_{22}) - \epsilon (\lambda - 3)(\lambda - 10)$$

But $p_\epsilon(\lambda)$ is a polynomial in λ whose coefficients are polynomials in ϵ, such that

$$\lim_{\epsilon \to 0} p_\epsilon(\lambda) = \lambda(c_{22} + \lambda d_{22}) \neq 0 \quad \text{if} \quad c_{22} + \lambda d_{22} \neq 0 + \lambda 0$$

Then at least one of the roots of $p_\epsilon(\lambda)$ satisfy:

- $\lim_{\epsilon \to 0} \lambda_1(\epsilon) = 0$, and
- $\lambda_1(\epsilon)$ is a (fractional) power series of ϵ.

NOTATION

$A_0, A_1 \in \mathbb{C}^{n\times n}$

- **Unperturbed Pencil:** $A_0 + \lambda A_1$ (**SINGULAR**).
- **Perturbation Pencil:** $E_0 + \lambda E_1 \equiv \epsilon(B_0 + \lambda B_1)$.
- **Perturbed Pencil:**

$$\begin{aligned}\widehat{A}_0 + \lambda \widehat{A}_1 &= (A_0 + \lambda A_1) + (E_0 + \lambda E_1)\\ &= (A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1).\end{aligned}$$

A natural choice for the small parameter ϵ is

$$\epsilon = \|[E_0 \;\; E_1]\|_F \qquad B_0 + \lambda B_1 \equiv \frac{E_0 + \lambda E_1}{\|[E_0 \;\; E_1]\|_F}$$

but others are possible.

REGULARIZING THE GENERAL PROBLEM (I)

Theorem (Smith Normal Form). Given a square singular pencil $A_0+\lambda A_1$, there exist matrix polynomials $P(\lambda)$ and $Q(\lambda)$ such that

- $\det P(\lambda) =$ number $\neq 0$,
- $\det Q(\lambda) =$ number $\neq 0$, and

$$P(\lambda)(A_0+\lambda A_1)Q(\lambda) = \begin{bmatrix} D(\lambda) & 0 \\ 0 & 0_{d\times d} \end{bmatrix},$$

where $D(\lambda) = \mathrm{diag}(i_1(\lambda), \ldots, i_r(\lambda))$.

The **finite eigenvalues** of $A_0 + \lambda A_1$ are the roots of the polynomial $\det D(\lambda)$.

REGULARIZING THE GENERAL PROBLEM (II)

- $P(\lambda)(\,(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)\,)Q(\lambda) \equiv$

$$\equiv \begin{bmatrix} D(\lambda) & 0 \\ 0 & 0_{d\times d} \end{bmatrix} + \epsilon \begin{bmatrix} G_{11}(\lambda) & G_{12}(\lambda) \\ G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix}$$

- $\det(\,(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)\,) =$

$$= C \,\det \begin{bmatrix} D(\lambda) + \epsilon G_{11}(\lambda) & \epsilon G_{12}(\lambda) \\ \epsilon G_{21}(\lambda) & \epsilon G_{22}(\lambda) \end{bmatrix}$$

- $\det(\,(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)\,) =$

$$= C\, \epsilon^d \,\det \begin{bmatrix} D(\lambda) + \epsilon G_{11}(\lambda) & G_{12}(\lambda) \\ \epsilon G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix}$$

REGULARIZING THE GENERAL PROBLEM (III)

- If $\epsilon \neq 0$ the eigenvalues of $(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)$ are the roots of the polynomial

$$\begin{aligned} p_\epsilon(\lambda) &= \det \begin{bmatrix} D(\lambda) + \epsilon G_{11}(\lambda) & G_{12}(\lambda) \\ \epsilon G_{21}(\lambda) & G_{22}(\lambda) \end{bmatrix} \\ &= \det D(\lambda) \det G_{22}(\lambda) + \epsilon\, q_\epsilon(\lambda), \end{aligned}$$

whose coefficients are polynomials in ϵ.

And

$$\lim_{\epsilon \to 0} p_\epsilon(\lambda) = \det D(\lambda) \det G_{22}(\lambda)$$

This polynomial is not identically zero if and only if $\det G_{22}(\lambda) \neq 0$.

THE GENERAL EXISTENCE THEOREM

Theorem: If $(B_0 + \lambda B_1)$ satisfies that $\det G_{22}(\lambda) \neq 0$ then

- There exists $K > 0$ such that

  $(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)$ is **regular** for $0 < |\epsilon| < K$.

- The n eigenvalues $\{\lambda_1(\epsilon), \ldots, \lambda_n(\epsilon)\}$ of $(A_0 + \lambda A_1) + \epsilon(B_0 + \lambda B_1)$ **are (fractional) power series** of ϵ.

- If $\det D(\lambda) = (\lambda - \mu_1) \ldots (\lambda - \mu_k)$, i.e., $\{\mu_1, \ldots, \mu_k\}$ **are the finite eigenvalues of** $A_0 + \lambda A_1$, then

  $$\lim_{\epsilon \to 0} \lambda_i(\epsilon) = \mu_i \quad \text{for } i = 1 : k,$$

  where

  $$P(\lambda)(A_0 + \lambda A_1)Q(\lambda) = \begin{bmatrix} D(\lambda) & 0 \\ 0 & 0_{d \times d} \end{bmatrix}.$$

FIRST ORDER TERM FOR SIMPLE EIGENVALUES (I)

- **Matrix case:** $Ax = \lambda_0 x$, $y^H A = \lambda_0 y^H$. There is a unique eigenvalue of $A + E$ such that

$$\widehat{\lambda}_0 = \lambda_0 + \frac{y^H E x}{y^H x} + O(\|E\|^2)$$

- **Regular Pencils:** $(A_0 + \lambda_0 A_1)x = 0$, $y^H(A_0 + \lambda_0 A_1) = 0$. There is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ such that

$$\widehat{\lambda}_0 = \lambda_0 - \frac{y^H (E_0 + \lambda_0 E_1) x}{y^H A_1 x} + O(\|[E_0\ E_1]\|^2)$$

- **Singular Square Pencils:**

$$\widehat{\lambda}_0 = \lambda_0 - \textbf{?} + O(\|[E_0\ E_1]\|^2)$$

FIRST ORDER TERM FOR SIMPLE EIGENVALUES (II)

Theorem: Let $A_0 + \lambda A_1$ be a square singular pencil and λ_0 a simple finite eigenvalue. Let

- X and Y be matrices whose columns are, respectively, bases of the right and the left null spaces of $A_0 + \lambda_0 A_1$.

Then, under certain generic assumptions, there is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ such that

$$\widehat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \ E_1]\|^2),$$

where ζ is the unique finite eigenvalue of the regular pencil

$$Y^H (E_0 + \lambda_0 E_1) X + \zeta Y^H A_1 X$$

FIRST ORDER TERM FOR SIMPLE EIGENVALUES (III)

Theorem: λ_0 eigenvalue of $A_0 + \lambda A_1$, X and Y bases of right and left null spaces of $A_0 + \lambda_0 A_1$.

There is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$

$$\widehat{\lambda} = \lambda_0 + \zeta + O(\|[E_0\ E_1]\|^2),$$

where ζ is the unique finite eigenvalue of the regular pencil

$$Y^H(E_0 + \lambda_0 E_1)X + \zeta Y^H A_1 X$$

If $A_0 + \lambda A_1$ is **regular** then X and Y only have one column vector, and

$$\widehat{\lambda}_0 = \lambda_0 - \frac{y^H(E_0 + \lambda_0 E_1)x}{y^H A_1 x} + O(\|[E_0\ E_1]\|^2)$$

EXAMPLE-EIGENVALUES

$$(A_0 + \lambda A_1) + (E_0 + \lambda E_1) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + 10^{-3} \begin{bmatrix} 1-\lambda & 2-\lambda/2 \\ 3+7\lambda & 4+9\lambda \end{bmatrix}$$

Exact Eigenvalues $= \{\boxed{0.000500375....}, -0.44438....\}$

Perturbation results for $\lambda_0 = 0$:

- $X = Y = I_2$
- $Y^H(E_0+\lambda_0 E_1)X + \zeta Y^H A_1 X = \begin{bmatrix} 10^{-3} & 2\cdot 10^{-3} \\ 3\cdot 10^{-3} & 4\cdot 10^{-3} \end{bmatrix} + \zeta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- $\zeta = \frac{1}{2}\cdot 10^{-3}$

$$\boxed{\hat{\lambda} = \frac{1}{2}\cdot 10^{-3} + 0(10^{-6}) = 0.0005 + 0(10^{-6})}$$

EIGENVECTORS ARE NOT DEFINED IN SINGULAR PENCILS

Example: Given the diagonal pencil

$$A_0 + \lambda A_1 = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \text{ with } 0 \text{ as unique eigenvalue,}$$

it is tempting to say that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the eigenvector of $\lambda = 0$, because $P = Q = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ diagonalize $P(A_0 + \lambda A_1)Q$.

However

$$\begin{bmatrix} 1/\alpha & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}$$

for every numbers α, β such that $\alpha \neq 0$.

The correct concept is **reducing subspace** (Van Dooren, 1982).

…but yes in the generically REGULAR perturbed pencils

$(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ is generically regular and has well defined eigenvectors.

It makes no sense to study the perturbations of non existent objects. But, a natural question is:

How are the left and right eigenvectors of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ corresponding to

$$\widehat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \ E_1]\|^2)$$

related to properties of $A_0 + \lambda A_1$?

PERTURBED EIGENVECTORS OF SIMPLE EIGENVALUES

Theorem: λ_0 eigenvalue of $A_0 + \lambda A_1$, X and Y bases of right and left null spaces of $A_0 + \lambda_0 A_1$.

There is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$

$$\widehat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \; E_1]\|^2),$$

where ζ is the unique finite eigenvalue of the regular pencil

$$Y^H(E_0 + \lambda_0 E_1)X + \zeta Y^H A_1 X \qquad (1)$$

In addition, if z (w) are right (left) eigenvectors of (1) corresponding to ζ, then

$$\widehat{x} = Xz + O(\|[E_0 \; E_1]\|) \qquad \widehat{y} = Yw + O(\|[E_0 \; E_1]\|)$$

are right and left eigenvectors of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ corresponding to $\widehat{\lambda}$.

EXAMPLE-EIGENVECTORS (I)

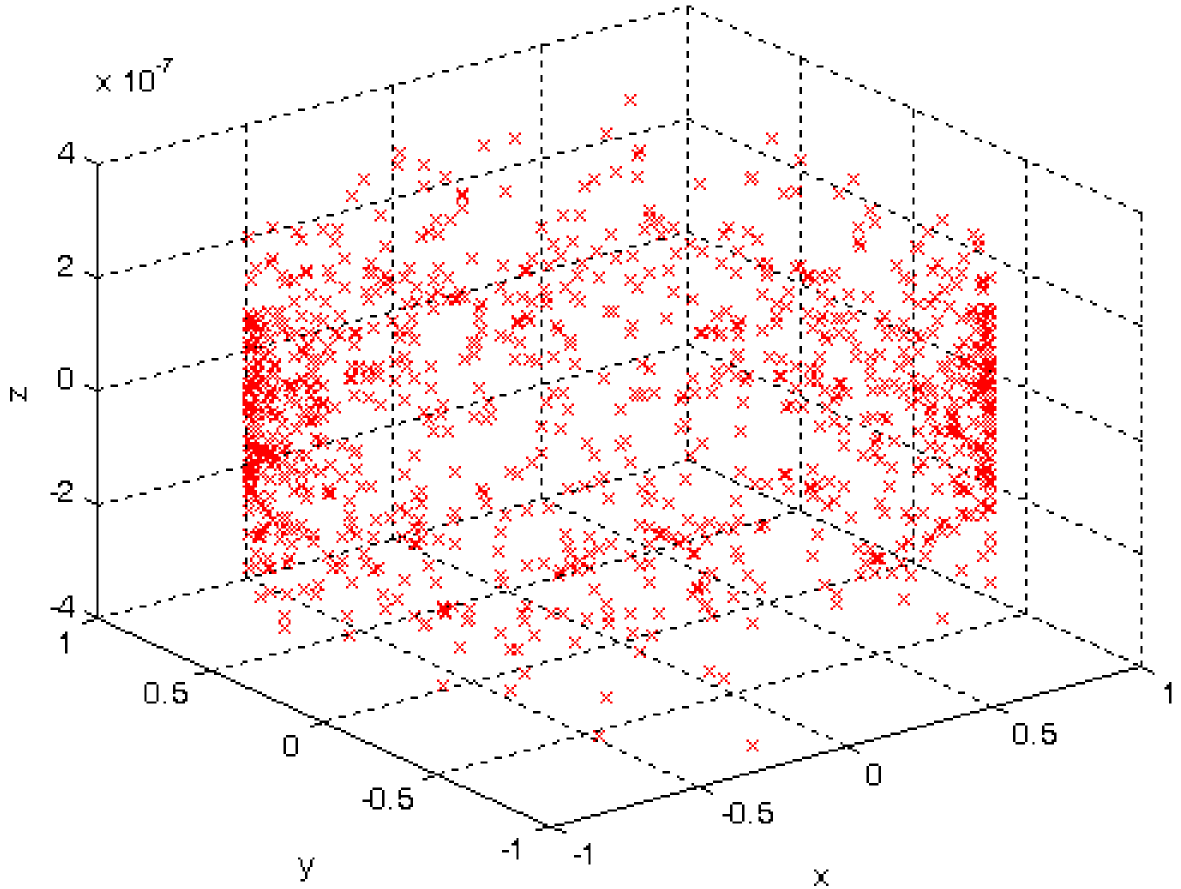
$$(A_0+\lambda A_1)+(E_0+\lambda E_1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda-2 \end{bmatrix} + \text{random } \|\cdot\| = 10^{-6}$$

$$\mathcal{N}(A_0+0\cdot A_1) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \text{ (where } \mathcal{N} \equiv \text{null space)}$$

We compute the right eigenvector for the eigenvalue closest to zero for 1000 perturbations, and plot a point for each of these eigenvectors.

EXAMPLE-EIGENVECTORS (I)-cont.

$$\mathcal{N}(A_0 + 0 \cdot A_1) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$



EXAMPLE-EIGENVECTORS (II)

$$(A_0 + \lambda A_1) + (E_0 + \lambda E_1) \; = \; \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} + 10^{-3} \begin{bmatrix} 1-\lambda & 2-\lambda/2 \\ 3+7\lambda & 4+9\lambda \end{bmatrix}$$

Exact Eigenvalues $= \{0.000500375....,\, -0.44438....\}$

Exact Eigenvector $= \boxed{\begin{bmatrix} 1 \\ -0.7500312... \end{bmatrix}}$

Perturbation results for $\lambda_0 = 0$:

- $Y^H(E_0 + \lambda_0 E_1)X + \zeta\, Y^H A_1 X = \begin{bmatrix} 10^{-3} & 2 \cdot 10^{-3} \\ 3 \cdot 10^{-3} & 4 \cdot 10^{-3} \end{bmatrix} + \zeta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
- Right Eigenvector corresponding to $\zeta = \frac{1}{2} \cdot 10^{-3}$ is $\begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix}$

$$\boxed{\widehat{x} = \begin{bmatrix} 1 \\ -0.75 \end{bmatrix} + O(10^{-3})}$$

Theorem (Kronecker Canonical Form) Let $A_0, A_1 \in \mathbb{C}^{m\times n}$. Then there exist two nonsingular matrices R and S such that

$$\boxed{\begin{aligned} R\,(A_0+\lambda A_1)\,S \;=\; & L_{\epsilon_1}(\lambda)\oplus\ldots\oplus L_{\epsilon_p}(\lambda)\oplus \\ & (J+\lambda I)\oplus(I+\lambda N)\oplus \\ & L^T_{\eta_1}(\lambda)\oplus\ldots\oplus L^T_{\eta_q}(\lambda), \end{aligned}}$$

J is square and is in Jordan canonical form,

N is square and is in Jordan canonical form with all its eigenvalues equal to zero,

$$L_{\epsilon_i}(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \end{bmatrix} \in \mathbb{C}^{\epsilon_i\times(\epsilon_i+1)}$$

$L_{\epsilon_i}(\lambda)$ ($L^T_{\eta_i}(\lambda)$) are called column (row) singular blocks.

REDUCING SUBSPACES AND GENERIC CONDITIONS (II)

Definition (Van Dooren 1982): Let $\mathcal{X} \in \mathbb{C}^n$ be a subspace. $\mathcal{X}$ is a **reducing subspace** of the $m \times n$ pencil $A_0 + \lambda A_1$ if

$$\dim(A_0\mathcal{X} + A_1\mathcal{X}) = \dim(\mathcal{X}) - \#(L_{\epsilon_i} \text{ blocks in KCF of } A_0 + \lambda A_1)$$

- For arbitrary subspaces $\mathcal{Z} \in \mathbb{C}^n$,

$$\dim(A_0\mathcal{Z} + A_1\mathcal{Z}) \geq \dim(\mathcal{Z}) - \#(L_{\epsilon_i} \text{ blocks in KCF of } A_0 + \lambda A_1)$$

- Given KCF

$$\begin{aligned} R\,(A_0 + \lambda A_1)\,S \;\; = \;\; & L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_p}(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N) \oplus \\ & L^T_{\eta_1}(\lambda) \oplus \ldots \oplus L^T_{\eta_q}(\lambda) \end{aligned}$$

Every reducing subspace $\mathcal{X}$ is spanned by the columns of S corresponding to all $L_{\epsilon_i}(\lambda)$ blocks, together with columns of S corresponding to some blocks of $(J + \lambda I) \oplus (I + \lambda N)$. These last columns are not necessarily present.

REDUCING SUBSPACES AND GENERIC CONDITIONS (III)

- Reducing subspaces play in singular pencils the role of invariant subspaces in matrices, and deflating subspaces in regular pencils.

Main ideas on Reducing Subspaces:

- If $\mathcal{X}$ is a reducing subspace and $\mathcal{Y} = A_0\mathcal{X} + A_1\mathcal{X}$. Let $\mathcal{X} = \mathrm{col}(X_1)$, $\mathcal{Y} = \mathrm{col}(Y_1)$, $X = [X_1\ X_2]$, and $Y = [Y_1\ Y_2]$ nonsingular, then

$$(A_0 + \lambda A_1)[X_1\ X_2] = [Y_1\ Y_2]\begin{bmatrix} C_0 + \lambda C_1 & D_0 + \lambda D_1 \\ 0 & F_0 + \lambda F_1 \end{bmatrix},$$

  and $\text{E-val}(A_0 + \lambda A_1) = \text{E-val}(C_0 + \lambda C_1) \cup \text{E-val}(F_0 + \lambda F_1)$.

- If $\text{E-val}(C_0 + \lambda C_1) \cap \text{E-val}(F_0 + \lambda F_1) = \emptyset$ then

$$\mathrm{KCF}(A_0 + \lambda A_1) = \mathrm{KCF}(C_0 + \lambda C_1) \oplus \mathrm{KCF}(F_0 + \lambda F_1),$$

  and the reducing subspace $\mathcal{X}$ is unique.

REDUCING SUBSPACES AND GENERIC CONDITIONS (IV)

- These properties are not obvious in singular pencils, for instance

$$\left[\begin{array}{c|cc} \lambda & 1 & 0 \\ \hline 0 & \lambda & 1 \\ 0 & 0 & 0 \end{array}\right], \qquad \text{but 0 is not eigenvalue}$$

- Among all reducing subspaces of $A_0 + \lambda A_1$, we will be interested in the **minimal** one, i.e., the one that is contained in any other reducing subspace.

- Given the KCF

$$\begin{aligned} R\,(A_0 + \lambda A_1)\,S \quad = \quad & L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_p}(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N) \oplus \\ & L_{\eta_1}^T(\lambda) \oplus \ldots \oplus L_{\eta_q}^T(\lambda), \end{aligned}$$

the minimal reducing subspace is spanned by the columns of S corresponding to all blocks L_{ϵ_i}.

FIRST ORDER TERM FOR SIMPLE EIGENVALUES (II)

Theorem: Let $A_0 + \lambda A_1$ be a square singular pencil and λ_0 a simple finite eigenvalue. Let

- X and Y be matrices whose columns are, respectively, bases of the right and the left null spaces of $A_0 + \lambda_0 A_1$.

Then, **UNDER CERTAIN GENERIC ASSUMPTIONS**, there is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ such that

$$\widehat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \; E_1]\|^2),$$

where ζ is the unique finite eigenvalue of the regular pencil

$$Y^H(E_0 + \lambda_0 E_1)X + \zeta \, Y^H A_1 X$$

Theorem: Let $A_0 + \lambda A_1$ be a square singular pencil and λ_0 a simple finite eigenvalue. Let

- $\mathcal{R}$ ($\mathcal{R}_H$) be the minimal reducing subspace of $A_0 + \lambda A_1$ ($A_0^H + \lambda A_1^H$).
- $\mathcal{N}(A_0 + \lambda_0 A_1)$ and $\mathcal{N}(A_0^H + \bar{\lambda}_0 A_1^H)$ be right and left null spaces associated to λ_0.
- X_1 be a matrix whose columns are a basis of $\mathcal{R} \cap \mathcal{N}(A_0 + \lambda_0 A_1)$.
- Y_1 be a matrix whose columns are a basis of $\mathcal{R}_H \cap \mathcal{N}(A_0^H + \bar{\lambda}_0 A_1^H)$.

If $Y_1^H (E_0 + \lambda_0 E_1) X_1$ **is nonsingular**, then there is a unique eigenvalue of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ such that

$$\hat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \; E_1]\|^2).$$

Generic condition on the perturbation, for the first order perturbation expansion around λ_0 to be valid:

$$\boxed{\text{Nonsingularity of the restriction of the linear mapping } E_0+\lambda_0 E_1 \text{ to the subspaces } \mathcal{R}\cap\mathcal{N}(A_0+\lambda_0 A_1) \text{ (domain), and } \mathcal{R}_H\cap\mathcal{N}(A_0^H+\bar{\lambda}_0 A_1^H) \text{ (codomain).}}$$

This condition

- is intrinsic, i.e, it does not depend on particular bases, or special normalizations.
- is specific for λ_0.

REDUCING SUBSPACES AND GENERIC CONDITIONS (VII)

If $Y_1^H(E_0+\lambda_0 E_1)X_1$ **is nonsingular**, then there is a unique eigenvalue of $(A_0+\lambda A_1)+(E_0+\lambda E_1)$ such that

$$\hat{\lambda} = \lambda_0 + \zeta + O(\|[E_0 \; E_1]\|^2).$$

In addition,

if the columns of $X = [x \; X_1]$ are a basis of $\mathcal{N}(A_0+\lambda_0 A_1)$, and

the columns of $Y = [y \; Y_1]$ are a basis of $\mathcal{N}(A_0^H+\bar{\lambda}_0 A_1^H)$ (x, y are vectors), then

$$\zeta = -\frac{\det(Y^H(E_0+\lambda_0 E_1)X)}{(y^H A_1 x)\cdot\det(Y_1^H(E_0+\lambda_0 E_1)X_1)}$$

MULTIPLE EIGENVALUES. MAIN IDEAS (I)

- Given the KCF

$$
\begin{aligned}
R\,(A_0 + \lambda A_1)\,S \quad = \quad & L_{\epsilon_1}(\lambda) \oplus \ldots \oplus L_{\epsilon_d}(\lambda) \oplus (J + \lambda I) \oplus (I + \lambda N) \oplus \\
& L_{\eta_1}^T(\lambda) \oplus \ldots \oplus L_{\eta_d}^T(\lambda).
\end{aligned}
$$

- Let us assume that the regular part $J + \lambda I$ corresponding to the finite eigenvalue λ_0 is

$$
J_{n_1}(\lambda_0) \oplus \cdots \oplus J_{n_g}(\lambda_0), \quad \text{where} \quad n_1 \leq n_2 \leq \ldots \leq n_g,
$$

and

$$
J_{n_i}(\lambda_0) = \begin{bmatrix} \lambda - \lambda_0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \lambda_0 \end{bmatrix} \quad \text{is} \quad n_i \times n_i
$$

MULTIPLE EIGENVALUES. MAIN IDEAS (II)

- Let us consider a group of r_i equal λ_0-Jordan blocks

$$\cdots < n_{i+1} = n_{i+2} = \cdots = n_{i+r_i} < \cdots$$

- Let us define for simplicity

$$\tilde{n}_i \equiv n_{i+1} = n_{i+2} = \cdots = n_{i+r_i}$$

Then, **UNDER CERTAIN GENERIC ASSUMPTIONS**, there are $r_i \, \tilde{n}_i$ eigenvalues of $(A_0 + \lambda A_1) + (E_0 + \lambda E_1)$ such that

$$\hat{\lambda} = \lambda_0 + \zeta_r^{1/\tilde{n}_i} + o(\|[E_0 \; E_1]\|^{1/\tilde{n}_i}),$$

where

- the $\tilde{n}_i$ different $\tilde{n}_i$-th roots are considered, and
- ζ_r, $r = 1 : r_i$, are the finite eigenvalues of a regular pencil.

MULTIPLE EIGENVALUES. MAIN IDEAS (III)

$$\widehat{\lambda} = \lambda_0 + {\zeta_r}^{1/\tilde{n}_i} + o(\|[E_0 \; E_1]\|^{1/\tilde{n}_i}),$$

with ζ_r, $r = 1, \ldots, r_i$, the finite eigenvalues of a regular pencil.

This pencil is of the type

$$\widetilde{Y}^H (E_0 + \lambda_0 E_1) \widetilde{X} + \zeta \begin{bmatrix} I_{r_i} & 0 \\ 0 & 0 \end{bmatrix}$$

- $\widetilde{X}$ is a particular basis, properly normalized, of the intersection of $\mathcal{N}(A_0 + \lambda_0 A_1)$ with the right reducing subspace corresponding to those blocks $J_{n_j}(\lambda_0)$ with $n_j \geq \tilde{n}_i$.
- Similar for $\widetilde{Y}$ with left null and reducing subspaces.
- Generic conditions are more difficult to state, but same flavor as in the simple case.
- Drawbacks on normalizations of bases are related to the multiple regular case, not to singularity.

CONCLUSIONS

- Generic sets of perturbations of square singular pencils for which an eigenvalue perturbation theory is possible have been presented.

- These sets are characterized in terms of the nonsingularity of the restriction of $E_0 + \lambda_0 E_1$ to intrinsic "spectral" subspaces of $A_0 + \lambda A_1$. They are related to reducing subspaces.

- For perturbations in these sets, first order perturbation expansions of simple and multiple eigenvalues have been developed.

- The perturbed eigenvectors are close to the corresponding null space of the singular unperturbed pencil, and the zero order term for these eigenvectors has been determined.