Recent results on Fiedler linearizations for matrix polynomials

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joint work with

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ICIAM 2011. Minisymposium on Nonlinear Eigenvalue Problems.
Vancouver, Canada, July 18-22, 2011

We consider a matrix polynomial of degree k

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}\,. \quad A_k \neq 0.$$

A linearization for $P(\lambda)$ is an $nk \times nk$ **linear matrix poly (pencil)** $L(\lambda)$ s. t.

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{n(k-1)} & \\ & P(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

$L(\lambda)$ is "strong linearization" if, in addition, $\mathrm{rev}\, L(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$, where

$$\mathrm{rev}\, P(\lambda) := \lambda^k A_0 + \ldots + \lambda A_{k-1} + A_k$$

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Advantages and disadvantages of the use of linearizations

- Strong linearizations preserve the finite and infinite elementary divisors of $P(\lambda)$, but NOT the eigenvectors and minimal indices/minimal bases.
- Good numerical methods for computing eigenvalues/vectors and minimal indices/bases of pencils are available (QZ, GUPTRI (Staircase form)).
- Standard linearizations do not preserve structures that $P(\lambda)$ may have.
- Conditioning of eigenvalues in linearizations may be much larger than in $P(\lambda)$. Backward errors?
- These difficulties have motivated an intense research on linearizations in the last years by different groups of several countries (Amiraslani, Antoniou, Bueno, Corless, De Terán, D, Grammont, Higham, Lancaster, R-C. Li, Mackey^2, Mehl, Merhmann, Tisseur, Vologiannidis, ...)
- In this talk, we review advances for one of the most relevant classes of linearizations developed in the last years.
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Fiedler pencils of $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ satisfy:

- They are **strong linearizations**, $\lambda X + Y$, **for any** $P(\lambda)$, regular or singular ($\det P(\lambda) \equiv 0$), over an arbitrary field and
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Fiedler pencils (I): main ideas

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- Extended to regular matrix polynomials (and generalized to preserve symmetry for odd degree) by Antoniou and Vologiannidis (ELA, 2004, 2006), where it is proved that they are strong linearizations.
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- Palindromic strong linearizations for odd degree polynomials based on Fiedler pencils were introduced by De Terán, D, Mackey (submitted, 2010, MIMS eprint 2010.33).
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Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Second companion form:

$$C_2(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & I_n & & & & \\ -A_4 & & I_n & & & \\ -A_3 & & & I_n & & \\ -A_2 & & & & I_n & \\ -A_1 & & & & & I_n \\ -A_0 & & & & & \end{bmatrix}$$

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Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

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Structural property 1 of Fiedler Pencils

The one-degree coefficient of every Fiedler Pencil is always the same. The zero-degree coefficient of every Fiedler Pencil has exactly the same blocks as the first companion form but they are in different positions.

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Another Fiedler pencil:

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

Structural property 1 of Fiedler Pencils

The one-degree coefficient of every Fiedler Pencil is always the same. The zero-degree coefficient of every Fiedler Pencil has exactly the same blocks as the first companion form but they are in different positions.

Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & -A_0 \\ & & & I_n & 0 & 0 \end{bmatrix}$$

Fiedler pencils (III): Examples

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

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$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

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Fiedler pencils (III): Examples and structural properties

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & & & \\ I_n & 0 & 0 & 0 & & \\ 0 & -A_3 & 0 & -A_2 & I_n & \\ & I_n & 0 & 0 & 0 & 0 \\ & & 0 & -A_1 & 0 & -A_0 \\ & & & I_n & 0 & 0 \end{bmatrix}$$

Structural property 2 of Fiedler pencils

Companion forms are the Fiedler pencils with largest banwidth. Pentadiagonal Fiedler pencils are the ones with smallest bandwith.

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 3 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- The identity blocks are never in the main block diagonal.

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 4 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- If an I_n block is at block-entry (i, j), then either the ith block-row or the jth block column has the remaining blocks equal to 0_n.

First companion form:

$$C_1(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & -A_1 & -A_0 \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \end{bmatrix}$$

Special Fiedler pencils: Pentadiagonal pencils. There are 4 for each degree k.

$$F(\lambda) = \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & -A_0 \\ 0 & 0 & 0 & I_n & 0 & 0 \end{bmatrix}$$

Structural property 4 of Fiedler pencils

The zero degree coefficient of every Fiedler pencil satisfies:

- If an I_n block is at block-entry (i, j), then either the ith block-row or the jth block column has at least one matrix $-A_0, -A_1, \ldots, -A_{k-1}$.

Outline

1. Definition of Fiedler pencils. Consecutions and inversions.
2. Recovery of eigenvectors from Fiedler pencils
3. Recovery of minimal indices and bases from Fiedler pencils
4. Fiedler pencils of rectangular matrix polynomials
5. Preservation of structures and generalized Fiedler pencils
6. Latest developments and conclusions

Outline

Let $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$. We define $nk \times nk$ matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix}, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix}, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix}.$$

Given any **permutation** $\sigma = (j_0, j_1, \ldots, j_{k-1})$ of $(0, 1, \ldots, k-1)$, the **Fiedler pencil associated with** σ is

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

Examples: Companion forms-Pentadiagonal Fiedler pencils

$$C_1(\lambda) = \lambda M_k - M_{k-1} \cdots M_1 M_0$$
$$C_2(\lambda) = \lambda M_k - M_0 M_1 \cdots M_{k-1}$$
$$T(\lambda) = \lambda M_k - (M_1 M_3 M_5 \cdots)(M_2 M_4 M_6 \cdots) M_0$$

Definition (Fiedler, 2003–Antoniou & Vologiannidis, 2004)

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Observe that $M_i M_j = M_j M_i$ for $|i-j| \neq 1$. This implies:

Lemma

Let $P(\lambda)$ be an arbitrary matrix polynomial of degree k. Then there exist 2^{k-1} distinct Fiedler pencils associated with $P(\lambda)$.

Consequences:

- **Quadratic polys: Fiedler pencils are the two companion forms.**
- For degree $k=3$, there are two more Fiedler pencils:
- The potential applications of Fiedler pencils are in degrees $k \geq 3$.

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Let us consider the Fiedler pencil associated to $\sigma = (j_0, j_1, \ldots, j_{k-1})$, permutation of $(0, 1, \ldots, k-1)$, i.e.,

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

For $i = 0, 1, \ldots k-2$, we say that $F_\sigma(\lambda)$ has a

- **consecution at** i, if the product $M_\sigma := M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

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We say that $F_\sigma(\lambda)$ has $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, 2, \ldots, \mathfrak{c}_0 - 1,$$

but not at $\mathfrak{c}_0$. Analogous for $\mathfrak{i}_0$ **initial inversions**.

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Consecutions and inversions

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Example of consecutions and inversions

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

$F_\sigma(\lambda)$ has

- Consecutions at $0, 1,$
- Inversions at $2, 3, 4,$
- $\mathfrak{c}_0 = 2$, and
- $\mathfrak{i}_0 = 0$.

Example of consecutions and inversions

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$F_\sigma(\lambda) = \lambda M_6 - {\color{red} M_5 M_4 M_3 M_0 M_1 M_2}$$

$$= \lambda \begin{bmatrix} A_6 & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & I_n & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} -A_5 & -A_4 & -A_3 & -A_2 & I_n & \\ I_n & & & & & \\ & I_n & & & & \\ & & I_n & & & \\ & & & -A_1 & & I_n \\ & & & -A_0 & & \end{bmatrix}$$

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- Consecutions at 0, 1,
- Inversions at 2, 3, 4,
- $\mathfrak{c}_0 = 2$, and
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Outline

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **regular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{c}_0$* **initial consecutions** *and $\mathfrak{i}_0$* **initial inversions**, *and suppose that λ_0* **is a *finite* eigenvalue** *of $P(\lambda)$.*

- *If*

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_k \end{bmatrix} \in \mathbb{F}^{nk \times 1}, \qquad x_i \in \mathbb{F}^{n \times 1},$$

is a right λ_0-eigenvector of $F_\sigma(\lambda)$, **then $x_{k-\mathfrak{c}_0}$ is a right λ_0-eigenvector of $P(\lambda)$.**

- *If*

$$w^T = \left[\ w_1^T \mid w_2^T \mid \ldots \mid w_k^T\ \right] \in \mathbb{F}^{1 \times nk}, \qquad w_i^T \in \mathbb{F}^{1 \times n},$$

is a left λ_0-eigenvector of $F_\sigma(\lambda)$, **then $w_{k-\mathfrak{i}_0}^T$ is a left λ_0-eigenvector of $P(\lambda)$.**

For first companion form $\mathfrak{c}_0 = 0, \mathfrak{i}_0 = k-1$, and for second $\mathfrak{c}_0 = k-1, \mathfrak{i}_0 = 0$.

For the **infinite e-value**, one has to extract the first blocks.

Theorem for eigenvector recovery: extracting blocks

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **regular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{c}_0$* **initial consecutions** *and $\mathfrak{i}_0$* **initial inversions**, *and suppose that λ_0* **is a *finite* eigenvalue** *of $P(\lambda)$.*

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$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_k \end{bmatrix} \in \mathbb{F}^{nk \times 1}, \qquad x_i \in \mathbb{F}^{n \times 1},$$

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- *If*

$$w^T = \left[\begin{array}{c|c|c|c} w_1^T & w_2^T & \dots & w_k^T \end{array} \right] \in \mathbb{F}^{1 \times nk}, \qquad w_i^T \in \mathbb{F}^{1 \times n},$$

is a left λ_0-eigenvector of $F_\sigma(\lambda)$, **then $w_{k-\mathfrak{i}_0}^T$ is a left λ_0-eigenvector of $P(\lambda)$**.

For first companion form $\mathfrak{c}_0 = 0, \mathfrak{i}_0 = k-1$, and for second $\mathfrak{c}_0 = k-1, \mathfrak{i}_0 = 0$.

For the **infinite e-value**, one has to extract the first blocks.

Example of eigenvector recovery

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$ has size $n \times n$

$$F_\sigma(\lambda) = \lambda M_6 - M_5 M_4 M_3 M_0 M_1 M_2$$

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$F_\sigma(\lambda)$ has $\mathfrak{c}_0 = 2$

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline x_4 \\ \hline x_5 \\ \hline x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$$ be such that $F_\sigma(\lambda_0)z = 0 \Longrightarrow P(\lambda_0)x_4 = 0$

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- This type of results will be useful to compare the **conditioning and backward error** of a number λ_0 as an eigenvalue of the matrix polynomial $P(\lambda)$ and as an eigenvalue of the Fiedler linearization $F_\sigma(\lambda)$.
- The complete description of these expressions requires to introduce more notation and submatrices, and we have no time to present it here.
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Outline

Theorem (De Terán, D, Mackey (SIMAX, 2010))

Let $P(\lambda)$ be an $n \times n$ **singular** *matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ with permutation σ having $\mathfrak{i}(\sigma)$* **total number of inversions** *and $\mathfrak{c}(\sigma)$* **total number of consecutions**.

(a) *If $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are the right minimal indices of $P(\lambda)$, then*

$$\varepsilon_1 + \mathfrak{i}(\sigma) \leq \varepsilon_2 + \mathfrak{i}(\sigma) \leq \cdots \leq \varepsilon_p + \mathfrak{i}(\sigma)\,,$$

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- The extension of Fiedler pencils from square to rectangular matrix polynomials is delicate.
- A definition via products of M_j-matrices has an important drawback: the sizes of the factors M_j depend on their positions in the product, even for the same Fiedler pencil.
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Let $P(\lambda)$ be an $m \times n$ matrix poly. **Then any Fiedler pencil $F_\sigma(\lambda)$ of $P(\lambda)$ is a strong linearization for $P(\lambda)$.**

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Minimal indices and bases of $P(\lambda)$ can be recovered from those of $F_\sigma(\lambda)$ with the same rules as for square matrix polys.

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Fiedler pencils of an $m \times n$ matrix poly have different sizes.

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The extension to rectangular polys is not always possible for other types of linearizations

Example: Pencil in $\mathbb{L}_1(P)$ (Mackey[2], Mehl, Mehrmann, (SIMAX, 2006))

$$P(\lambda) = A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0, \quad n \times n$$

$$L(\lambda) = \lambda \begin{bmatrix} A_3 & 0 & 2A_1 \\ -2A_3 & -A_2 - A_1 & A_0 - 4A_1 \\ 0 & A_3 & -I \end{bmatrix} + \begin{bmatrix} A_2 & -A_1 & A_0 \\ A_1 - A_2 & 2A_1 - A_0 & -2A_0 \\ -A_3 & I & 0 \end{bmatrix}$$

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There are structural obstacles for an easy extension to rectangular polys.

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Which pencils in $\mathbb{L}_1(P)$ are linearizations for rectangular polys?

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Lemma

- There are no Fiedler pencils that are symmetric whenever $P(\lambda)$ is symmetric.
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An $n \times n$ matrix polynomial $P(\lambda)$ is

- **symmetric** if $P(\lambda) = P(\lambda)^T$.
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No existence of structure-preserving Fiedler pencils

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Idea: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,,\quad A_i \in \mathbb{F}^{n\times n}$, recall the matrices:

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Then multiply any Fiedler pencil

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by **some of the factors** $M_1^{-1}, M_2^{-1}, \ldots, M_{k-1}^{-1}$ in a certain order to obtain pencils **strictly equivalent to** $F_\sigma(\lambda)$ (so **strong linearizations for** $P(\lambda)$) of the type

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Generalized Fiedler pencils (I) (Fiedler (2003), Antoniou-Vologiannidis (2004))

Idea: Given $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0\,, \quad A_i \in \mathbb{F}^{n\times n}$, recall the matrices:

$$M_j := \begin{bmatrix} I_{n(k-j-1)} & & & \\ & -A_j & I_n & \\ & I_n & 0 & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad j = 1, \ldots, k-1\,,$$

$$M_0 := \begin{bmatrix} I_{n(k-1)} & \\ & -A_0 \end{bmatrix} \in \mathbb{F}^{nk\times nk}\,, \quad M_k := \begin{bmatrix} A_k & \\ & I_{n(k-1)} \end{bmatrix} \in \mathbb{F}^{nk\times nk},$$

and note that $M_1, M_2, \ldots, M_{k-1}$ **are always invertible**.

Then multiply any Fiedler pencil

$$F_\sigma(\lambda) = \lambda M_k - M_{j_0} M_{j_1} \cdots M_{j_{k-1}}$$

by **some of the factors** $M_1^{-1}, M_2^{-1}, \ldots, M_{k-1}^{-1}$ in a certain order to obtain pencils **strictly equivalent to** $F_\sigma(\lambda)$ (so **strong linearizations for** $P(\lambda)$) of the type

$$\lambda M_{\sigma_1} - M_{\sigma_0} := \lambda\,(M_{p_0}^{-1} \cdots M_{p_{s_1}}^{-1}) M_k (M_{q_0}^{-1} \cdots M_{q_{s_2}}^{-1}) - M_{r_0} M_{r_1} \cdots M_{r_{s_3}}$$

Definition

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is a **proper generalized Fiedler pencil (strong linearization)** for $P(\lambda)$ if

- $(p_0, \ldots, p_{s_1}, k, q_0, \ldots, q_{s_2}, r_0, \ldots, r_{s_3})$ is a permutation of $(0, 1, \ldots, k)$.
- $0 \in \{r_0, r_1, \ldots, r_{s_3}\}$.

Remark

If A_k and/or A_0 in $P(\lambda) = \lambda^k A_k + \cdots + \lambda A_1 + A_0$ are nonsingular, then it is possible to multiply a Fiedler pencil $F_\sigma(\lambda)$ by M_k^{-1} and/or M_0^{-1} and construct a **wider class of Generalized Fiedler pencils**, which

- contains symmetry-preserving linearizations of **even degree** polys.
- These pencils are not easy to construct: A_k^{-1} and/or A_0^{-1} are required.

For brevity, in this talk, we do not consider these pencils.

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Generalized Fiedler (GF) pencils that preserve symmetry for odd degree

Theorem (Antoniou & Vologiannidis, ELA, 2004)

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix poly of **odd degree**, then the proper generalized Fiedler linearization for $P(\lambda)$

$$S(\lambda) = \lambda\, M_k M_{k-2}^{-1} \cdots M_3^{-1} M_1^{-1} - M_{k-1} M_{k-3} \cdots M_2 M_0$$

is symmetric whenever $P(\lambda)$ is symmetric.

It follows easily from

$$M_j^{-1} = \begin{bmatrix} I_{n(k-j-1)} & & & \\ & 0 & I_n & \\ & I_n & A_j & \\ & & & I_{n(j-1)} \end{bmatrix} \in \mathbb{F}^{nk \times nk}, \quad j = 1, \dots, k-1\,.$$

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$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$S(\lambda) = \lambda\, M_5 M_3^{-1} M_1^{-1} - M_4 M_2 M_0$$

$$= \lambda \begin{bmatrix} A_5 & & & & \\ & 0 & I_n & & \\ & I_n & A_3 & & \\ & & & 0 & I_n \\ & & & I_n & A_1 \end{bmatrix} - \begin{bmatrix} -A_4 & I_n & & & \\ I_n & 0 & & & \\ & & -A_2 & I_n & \\ & & I_n & 0 & \\ & & & & -A_0 \end{bmatrix}$$

Theorem (De Terán, D, Mackey, MIMS e-print, 2010)

Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix poly of **odd degree**. Consider any **proper generalized Fiedler pencil** of the type

$$L(\lambda) = \lambda(\cdots M_k \cdots M_{k-i_1}^{-1} M_{k-i_0}^{-1}) - (M_{i_0} M_{i_1} \cdots M_0 \cdots),$$

and define

$$R = \begin{bmatrix} & & I_n \\ & \cdot^{\cdot^{\cdot}} & \\ I_n & & \end{bmatrix} \in \mathbb{F}^{nk \times nk} \text{ and } S = \begin{bmatrix} \pm I_n & & \\ & \ddots & \\ & & \pm I_n \end{bmatrix} \in \mathbb{F}^{nk \times nk},$$

where the signs are easily determined by the consecutions/inversions of the factors in $(M_{i_0} M_{i_1} \cdots M_0 \cdots)$. Then

$$L_{palin}(\lambda) = S\, R\, L(\lambda)$$

is a strong linearization of $P(\lambda)$ that is palindromic whenever $P(\lambda)$ is palindromic.

$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

There are many, let us illustrate one with lowest (anti-)bandwidth

$$L_{palin}(\lambda) = S\,R\,(\lambda\, M_1^{-1} M_3^{-1} M_5 - M_0 M_2 M_4)$$

$$= \lambda \begin{bmatrix} & & & I_n & A_1 \\ & & & 0 & -I_n \\ & I_n & A_3 & & \\ & 0 & -I_n & & \\ A_5 & & & & \end{bmatrix} + \begin{bmatrix} & & & & A_0 \\ & & I_n & 0 & \\ & & A_2 & -I_n & \\ I_n & 0 & & & \\ A_4 & -I_n & & & \end{bmatrix},$$

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Recovery of eigenvectors, minimal indices and bases from proper GF pencils (Bueno, De Terán, D, SIMAX, 2011)

- Exactly the same recovery rules via block-extraction for
  - minimal bases, and
  - eigenvectors of finite eigenvalues of $P(\lambda)$,

  considering consecutions and inversions only for the zero degree term of the pencil.
- Different, but simple, rules for eigenvectors of the infinite eigenvalue also via block-extraction. They involve consecutions and inversions only for the first degree term of the pencil.
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Example of eigenvector recovery in proper GF pencils

$$P(\lambda) = A_6\lambda^6 + A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

$$G(\lambda) = \lambda\, M_3^{-1} M_6 M_5^{-1} - {\color{red} M_4 M_0 M_2 M_1}$$

$$= \lambda \begin{bmatrix} & A_6 & & & & \\ I_n & A_5 & & & & \\ & & & I_n & & \\ & & I_n & A_3 & & \\ & & & & I_n & \\ & & & & & I_n \end{bmatrix} - \begin{bmatrix} I_n & & & & & \\ & -A_4 & I_n & & & \\ & I_n & & & & \\ & & & -A_2 & -A_1 & I_n \\ & & & I_n & & \\ & & & & -A_0 & \end{bmatrix}$$

$M_4 M_0 M_2 M_1$ has $\mathfrak{c}_0 = 1$ initial consecutions (consecution at 0, inversion at 1, nothing at $2, 3, 4, 5$

$$z = \begin{bmatrix} x_1 \\ \hline x_2 \\ \hline x_3 \\ \hline x_4 \\ \hline {\color{red} x_5} \\ \hline x_6 \end{bmatrix}, \quad (x_i \in \mathbb{F}^{n\times 1})$$ be such that $G(\lambda_0) z = 0 \Longrightarrow P(\lambda_0) {\color{red} x_5} = 0$

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Outline

- Explicit expressions of e-vectors of GF pencils in terms of e-vectors of $P(\lambda)$ (Bueno, De Terán, D, to be presented at ILAS 2011) and also
- for the new class of GF pencils with repeated factors (Vologiannidis and Antoniou, MCSS, 2011) (complicated expressions but easy recovery).

$$P(\lambda) = A_5\lambda^5 + A_4\lambda^4 + A_3\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0$$

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Latest developments

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- The **fundamental open problem to ascertain the practical relevance of Fiedler pencils** is to compare the **conditioning and backward errors** of eigenvalues in Fiedler pencils with respect conditioning and backward errors in companion forms and in the original polynomial $P(\lambda)$.
- This is a difficult problem. Two outgoing works
  - De Terán and Tisseur: cubic matrix polynomials.
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- Probably, the most relevant numerical applications in eigenvalue/vector computations of (generalized) Fiedler pencils will be in
  - Symmetric and palindromic matrix polynomials of odd-degree.
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Ongoing developments. Conclusions.

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