Consistency and efficient solution of the Sylvester equation for $\star$-congruence: $AX + X^{\star}B = C$

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Abstract

Given $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{m\times m}$, we study the equations

$$A\,X + X^\star B = C\,, \qquad (X^\star = X^T \text{ or } X^*),$$

where $X \in \mathbb{C}^{n\times m}$ is the unknown to be determined. More precisely:

1. Necessary and sufficient conditions for consistency (Wimmer 1994, De Terán and D. 2011).
2. Necessary and sufficient conditions for uniqueness of solutions (Byers, Kressner, Schröder, Watkins, 2006, 2009).
3. Efficient and stable numerical algorithm for computing the unique solution (De Terán and D. 2011).
4. Very briefly, general solution and dimension of solution space of $A\,X + X^\star B = 0$ (De Terán, D., Guillery, Montealegre, Reyes, 2011)

We establish parallelism/differences with **well-known Sylvester equation**

$$A\,X - X\,B = C\,, \qquad A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{m\times n}.$$

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Outline

1. **Motivation**
2. **Consistency of the Sylvester equation for $\star$-congruence**
3. **Uniqueness of solutions**
4. **Efficient and stable algorithm to compute unique solutions**
5. **General "nonunique" solution of $AX + X^\star B = C$**
6. **Conclusions**

Outline

Motivation for studying $AX + X^\star B = C$ (I)

It is well known that given a block upper triangular matrix (computed by the QR-algorithm for eigenvalues, when the matrix is real or several eigenvalues form a cluster), then

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} A & C - (AX - XB) \\ 0 & B \end{bmatrix}.$$

Therefore, to find a solution of the **Sylvester equation** $AX - XB = C$ allows us to block-diagonalize block-triangular matrices via similarity

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This is indeed done in practice in numerical algorithms (LAPACK, MATLAB) to compute bases of invariant subspaces (eigenvectors) of matrices, via the classical Bartels-Stewart algorithm (Comm ACM, 1972) or level-3 BLAS variants of it Jonsson-Kågström (ACM TMS, 2002).

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Structured numerical algorithms for linear palindromic eigenproblems $(Z + \lambda Z^\star)$ compute an **anti-triangular Schur form** via unitary $\star$-congruence:

Theorem (Kressner, Schröder, Watkins (Numer. Alg., 2009) and Mackey^2, Mehl, Mehrmann (NLAA, 2009))

Let $Z \in \mathbb{C}^{n\times n}$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$M = U^\star Z\,U = \begin{bmatrix} * & \cdots & \cdots & * \\ \vdots & & \iddots & 0 \\ \vdots & \iddots & \iddots & \vdots \\ * & 0 & \cdots & 0 \end{bmatrix}$$

The matrix M can be computed, for instance, through structure-preserving

- QR-type methods for matrices in anti-Hessenberg form (Kressner, Schröder, Watkins (Numer. Alg., 2009)),
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Motivation for studying $AX + X^\star B = C$ (III)

Given a block upper ANTI-triangular matrix (computed via structured algorithms for linear palindromic eigenproblems, when the matrix is real or several eigenvalues form a cluster), then

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and to compute deflating subspaces of palindromic pencils.

GOAL: To understand **Sylvester equations for $\star$-congruence** and develop efficient and stable numerical algorithms for its solution in order to completely solve the linear palindromic eigenproblem numerically and to determine the conditioning of its deflating subspaces under structured perturbations.

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Outline

Theorem (Wimmer (LAA, 1994), De Terán and D. (ELA, 2011))

Let $\mathbb{F}$ be a field of characteristic different from two and let $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{n\times m}$, $C \in \mathbb{F}^{m\times m}$ be given. There is some $X \in \mathbb{F}^{n\times m}$ such that

$$AX + X^\star B = C$$

if and only if

$$\begin{bmatrix} C & A \\ B & 0 \end{bmatrix} \quad and \quad \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$$ *are $\star$-congruent.*

Remarks:

- The implication $\Longrightarrow$ very easy: done in previous slide.
- The implication $\Longleftarrow$ more challenging.
- Wimmer proved in 1994 the result, for $\mathbb{F} = \mathbb{C}$ and $\star = *$, without any reference to palindromic eigenproblems.
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Theorem (Roth (Proc. AMS, 1952))

Let $\mathbb{F}$ be any field and let $A \in \mathbb{F}^{m\times m}$, $B \in \mathbb{F}^{n\times n}$, $C \in \mathbb{F}^{m\times n}$ be given. There is some $X \in \mathbb{F}^{m\times n}$ such that

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if and only if

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \quad and \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$ *are similar.*

Outline

Remarks:

- If the matrices $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{n\times m}$ **are rectangular** ($m \neq n$), then the equation **does not have a unique solution for every right-hand side** C,
- that is, **the operator**

$$\begin{array}{ccc} \mathbb{F}^{n\times m} & \longrightarrow & \mathbb{F}^{m\times m} \\ X & \longmapsto & AX + X^\star B \end{array}$$

is never invertible.

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Definition: a set $\{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is *$\star$-reciprocal free* if $\lambda_i \neq 1/\lambda_j^{\star}$ for any $1 \leq i, j \leq n$. We admit 0 and/or ∞ as elements of $\{\lambda_1, \ldots, \lambda_n\}$.

Theorem (Byers, Kressner (SIMAX, 2006), Kressner, Schröder, Watkins, (Num. Alg., 2009))

Let $A, B \in \mathbb{C}^{n\times n}$ be given. Then:

- *$AX + X^T B = C$ has a unique solution X for every right-hand side $C \in \mathbb{C}^{n\times n}$ if and only if the following conditions hold:*
  1) *The pencil $A - \lambda B^T$ is regular, and*
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Theorem

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- $A\,X \;-\; X\,B = C$ *has a unique solution X for every right-hand side $C \in \mathbb{C}^{m\times n}$ if and only if A and B have no eigenvalues in common.*

Remark: Comparison of both results brings to our attention **a key difference** that appears always between solution methods for $AX + X^\star B = C$ and $AX - XB = C$:

- In $AX + X^\star B = C$, one starts dealing with the eigenproblem of $A - \lambda B^\star$, that is, one deals from the very beginning **simultaneously with A and B**.
- By contrast in $AX - XB = C$, one starts dealing **independently** with the eigenproblems **of A and B**.

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Outline

The fundamental transformation

- In this section in $A\,X + X^\star B = C$ all matrices are in $\mathbb{C}^{n\times n}$ and the solution is unique for every C.
- $A\,X + X^\star B = C$ is equivalent to a linear system for the n^2 entries of X if $\star = T$ and to a linear system for the $2\,n^2$ entries of $(\mathcal{R}e\,X\,,\,\mathcal{I}m\,X)$ if $\star = *$. From now on, we say simply "linear system" for X.
- Then, it is possible to use Gaussian elimination on the equivalent system (constructed via $\mathrm{vec}(X)$, $\mathrm{vec}(C)$, $\otimes$), but it costs $O(n^6)$ flops, which is prohibitive except for small n.
- **IDEA: transform $A\,X + X^\star B = C$ into an equation of the same type but with much simpler coefficients instead of A and B and that can be easily solved to get a total cost of $O(n^3)$ flops.**
- To this purpose, use **QZ algorithm** to compute in $O(n^3)$ flops the **generalized Schur decomposition** of

$$A - \lambda B^\star = U(R - \lambda S)V\,, \quad \text{where} \quad \begin{cases} R,\ S & \text{are upper triangular} \\ U,\ V & \text{are unitary matrices} \end{cases}$$

If A, B real matrices: use real arithmetic to get *quasi-triangular* R. We do not consider this for brevity.

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If A, B real matrices: use real arithmetic to get *quasi-triangular* R. We do not consider this for brevity.

The fundamental transformation

- In this section in $A\,X + X^\star B = C$ all matrices are in $\mathbb{C}^{n\times n}$ and the solution is unique for every C.
- $A\,X + X^\star B = C$ is equivalent to a linear system for the n^2 entries of X if $\star = T$ and to a linear system for the $2\,n^2$ entries of $(\mathcal{R}e\,X\,,\,\mathcal{I}m\,X)$ if $\star = *$. From now on, we say simply "linear system" for X.
- Then, it is possible to use Gaussian elimination on the equivalent system (constructed via $\mathrm{vec}(X)$, $\mathrm{vec}(C)$, $\otimes$), but it costs $O(n^6)$ flops, which is prohibitive except for small n.
- **IDEA: transform $A\,X + X^\star B = C$ into an equation of the same type but with much simpler coefficients instead of A and B and that can be easily solved to get a total cost of $O(n^3)$ flops.**
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If A, B real matrices: use real arithmetic to get *quasi-triangular* R. We do not consider this for brevity.

Algorithm to solve $A\,X + X^\star B = C$ in $O(n^3)$ flops

INPUT: $A, B, C \in \mathbb{C}^{n\times n}$
OUTPUT: $X \in \mathbb{C}^{n\times n}$

Step 1. Compute via QZ algorithm R, S, U and V such that

$$A = URV\,, \quad B^\star = USV\,, \text{ where } \begin{cases} R,\ S & \text{are upper triangular} \\ U,\ V & \text{are unitary matrices} \end{cases}$$

Step 2. Compute $E = U^*\,C\,(U^\star)^*$

Step 3. Solve for $W \in \mathbb{C}^{n\times n}$ the transformed equation

$$R\,W + W^\star S^\star = E$$

Step 4. Compute $X = V^*\,W\,U^\star$

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Step 1. Compute via QZ algorithm R, S, U and V such that

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Step 2. Compute $E = U^* \, C\, (U^\star)^*$

Step 3. How to solve for $W \in \mathbb{C}^{n\times n}$ the transformed equation

$$RW + W^\star S^\star = E \ ?$$

Step 4. Compute $X = V^* \, W \, U^\star$

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (4,4)-entry, then we get

$$r_{44} \; w_{44} + w_{44}^\star \; s_{44}^\star = e_{44} \, ,$$

a scalar equation that allows us to determine w_{44}.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w^\star_{11} & w^\star_{21} & w^\star_{31} & w^\star_{41} \\ w^\star_{12} & w^\star_{22} & w^\star_{32} & w^\star_{42} \\ w^\star_{13} & w^\star_{23} & w^\star_{33} & w^\star_{43} \\ w^\star_{14} & w^\star_{24} & w^\star_{34} & w^\star_{44} \end{bmatrix} \begin{bmatrix} s^\star_{11} & 0 & 0 & 0 \\ s^\star_{12} & s^\star_{22} & 0 & 0 \\ s^\star_{13} & s^\star_{23} & s^\star_{33} & 0 \\ s^\star_{14} & s^\star_{24} & s^\star_{34} & s^\star_{44} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (4,4)-entry, then we get

$$r_{44} \; w_{44} + w^\star_{44} \; s^\star_{44} = e_{44} \;,$$

a scalar equation that allows us to determine w_{44}.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w^\star_{11} & w^\star_{21} & w^\star_{31} & w^\star_{41} \\ w^\star_{12} & w^\star_{22} & w^\star_{32} & w^\star_{42} \\ w^\star_{13} & w^\star_{23} & w^\star_{33} & w^\star_{43} \\ w^\star_{14} & w^\star_{24} & w^\star_{34} & w^\star_{44} \end{bmatrix} \begin{bmatrix} s^\star_{11} & 0 & 0 & 0 \\ s^\star_{12} & s^\star_{22} & 0 & 0 \\ s^\star_{13} & s^\star_{23} & s^\star_{33} & 0 \\ s^\star_{14} & s^\star_{24} & s^\star_{34} & s^\star_{44} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (3,4) and (4,3) entries, then we get

$$\begin{aligned} s_{33}\, w_{34} + w^\star_{43}\, r^\star_{44} &= e^\star_{43} - s_{34}\, w_{44} \\ r_{33}\, w_{34} + w^\star_{43}\, s^\star_{44} &= e_{34} - r_{34}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{34} and w_{43} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (3,4) and (4,3) entries, then we get

$$\begin{aligned} s_{33}\, w_{34} + w_{43}^\star\, r_{44}^\star &= e_{43}^\star - s_{34}\, w_{44} \\ r_{33}\, w_{34} + w_{43}^\star\, s_{44}^\star &= e_{34} - r_{34}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{34} and w_{43} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (2,4) and (4,2) entries, then we get

$$\begin{aligned} s_{22}\, w_{24} + w_{42}^\star\, r_{44}^\star &= e_{42}^\star - s_{23}\, w_{34} - s_{24}\, w_{44} \\ r_{22}\, w_{24} + w_{42}^\star\, s_{44}^\star &= e_{24} - r_{23}\, w_{34} - r_{24}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{24} and w_{42} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w^\star_{11} & w^\star_{21} & w^\star_{31} & w^\star_{41} \\ w^\star_{12} & w^\star_{22} & w^\star_{32} & w^\star_{42} \\ w^\star_{13} & w^\star_{23} & w^\star_{33} & w^\star_{43} \\ w^\star_{14} & w^\star_{24} & w^\star_{34} & w^\star_{44} \end{bmatrix} \begin{bmatrix} s^\star_{11} & 0 & 0 & 0 \\ s^\star_{12} & s^\star_{22} & 0 & 0 \\ s^\star_{13} & s^\star_{23} & s^\star_{33} & 0 \\ s^\star_{14} & s^\star_{24} & s^\star_{34} & s^\star_{44} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (2,4) and (4,2) entries, then we get

$$\begin{aligned} s_{22}\, w_{24} + w^\star_{42}\, r^\star_{44} &= e^\star_{42} - s_{23}\, w_{34} - s_{24}\, w_{44} \\ r_{22}\, w_{24} + w^\star_{42}\, s^\star_{44} &= e_{24} - r_{23}\, w_{34} - r_{24}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{24} and w_{42} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (1,4) and (4,1) entries, then we get

$$\begin{aligned} s_{11}\, w_{14} + w_{41}^\star\, r_{44}^\star &= e_{41}^\star - s_{12}\, w_{24} - s_{13}\, w_{34} - s_{14}\, w_{44} \\ r_{11}\, w_{14} + w_{41}^\star\, s_{44}^\star &= e_{14} - r_{12}\, w_{24} - r_{13}\, w_{34} - r_{14}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{14} and w_{41} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (1,4) and (4,1) entries, then we get

$$\begin{aligned} s_{11}\, w_{14} + w_{41}^\star\, r_{44}^\star &= e_{41}^\star - s_{12}\, w_{24} - s_{13}\, w_{34} - s_{14}\, w_{44} \\ r_{11}\, w_{14} + w_{41}^\star\, s_{44}^\star &= e_{14} - r_{12}\, w_{24} - r_{13}\, w_{34} - r_{14}\, w_{44} \end{aligned},$$

a 2×2 system of scalar equations that allows us to determine w_{14} and w_{41} **simultaneously**.

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star & w_{41}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star & w_{42}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star & w_{43}^\star \\ w_{14}^\star & w_{24}^\star & w_{34}^\star & w_{44}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star & 0 \\ s_{14}^\star & s_{24}^\star & s_{34}^\star & s_{44}^\star \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (1:3,1:3) submatrices , then we get

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} + \begin{bmatrix} w_{11}^\star & w_{21}^\star & w_{31}^\star \\ w_{12}^\star & w_{22}^\star & w_{32}^\star \\ w_{13}^\star & w_{23}^\star & w_{33}^\star \end{bmatrix} \begin{bmatrix} s_{11}^\star & 0 & 0 \\ s_{12}^\star & s_{22}^\star & 0 \\ s_{13}^\star & s_{23}^\star & s_{33}^\star \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} - \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix} \begin{bmatrix} w_{41} & w_{42} & w_{43} \end{bmatrix} - \begin{bmatrix} w_{41}^\star \\ w_{42}^\star \\ w_{43}^\star \end{bmatrix} \begin{bmatrix} s_{14}^\star & s_{24}^\star & s_{34}^\star \end{bmatrix}$$

which is a 3×3 **matrix equation of the same type as the original one**.

Algorithm to solve the transformed equation $RW + W^\star S^\star = E$ (I)

We illustrate with 4×4 example for simplicity:

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{bmatrix}$$

$$+ \begin{bmatrix} w^\star_{11} & w^\star_{21} & w^\star_{31} & w^\star_{41} \\ w^\star_{12} & w^\star_{22} & w^\star_{32} & w^\star_{42} \\ w^\star_{13} & w^\star_{23} & w^\star_{33} & w^\star_{43} \\ w^\star_{14} & w^\star_{24} & w^\star_{34} & w^\star_{44} \end{bmatrix} \begin{bmatrix} s^\star_{11} & 0 & 0 & 0 \\ s^\star_{12} & s^\star_{22} & 0 & 0 \\ s^\star_{13} & s^\star_{23} & s^\star_{33} & 0 \\ s^\star_{14} & s^\star_{24} & s^\star_{34} & s^\star_{44} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{bmatrix}$$

If we equate the (1:3,1:3) submatrices , then we get

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \begin{bmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{bmatrix} + \begin{bmatrix} w^\star_{11} & w^\star_{21} & w^\star_{31} \\ w^\star_{12} & w^\star_{22} & w^\star_{32} \\ w^\star_{13} & w^\star_{23} & w^\star_{33} \end{bmatrix} \begin{bmatrix} s^\star_{11} & 0 & 0 \\ s^\star_{12} & s^\star_{22} & 0 \\ s^\star_{13} & s^\star_{23} & s^\star_{33} \end{bmatrix}$$

$$= \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} - \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix} \begin{bmatrix} w_{41} & w_{42} & w_{43} \end{bmatrix} - \begin{bmatrix} w^\star_{41} \\ w^\star_{42} \\ w^\star_{43} \end{bmatrix} \begin{bmatrix} s^\star_{14} & s^\star_{24} & s^\star_{34} \end{bmatrix}$$

which is a 3×3 **matrix equation of the same type as the original one**.

INPUT: $R, S, E \in \mathbb{C}^{n\times n}$, with R and S upper triangular
OUTPUT: $W \in \mathbb{C}^{n\times n}$

for $j = p : -1 : 1$
  solve $r_{jj} w_{jj} + w_{jj}^\star s_{jj}^\star = e_{jj}$ to get w_{jj}
  for $i = j-1 : -1 : 1$
    solve $\left\{ \begin{array}{l} s_{ii} w_{ij} + w_{ji}^\star r_{jj}^\star = e_{ji}^\star - \sum_{k=i+1}^{j} s_{ik} w_{kj} \\ r_{ii} w_{ij} + w_{ji}^\star s_{jj}^\star = e_{ij} - \sum_{k=i+1}^{j} r_{ik} w_{kj} \end{array} \right\}$ to get w_{ij}, w_{ji}
  end
  $E(1:j-1, 1:j-1) = E(1:j-1, 1:j-1) - R(1:j-1, j)W(j, 1:j-1)$
  $\qquad - (S(1:j-1, j)W(j, 1:j-1))^\star$
end

Cost

- $2n^3 + O(n^2)$ flops for real input matrices and a total cost $76n^3 + O(n^2)$ flops for the whole algorithm for $AX + X^\star B = C$.

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Remarks on algorithm to solve $A\,X + X^\star\,B = C$

- **Roundoff errors:** $\widehat{X}$, computed solution of $AX + X^\star B = C$, satisfies

$$\|A\widehat{X} + \widehat{X}^\star\,B - C\|_F \leq \alpha\,\mathbf{u}\,n^{5/2}\,\left(\|A\|_F + \|B\|_F\right)\,\|\widehat{X}\|_F,$$

  with $\mathbf{u}$ unit roundoff and α small integer constant.

- The algorithm for solving $RW + W^\star S^\star = E$ is dominated by level-2 BLAS operations. In modern computers, a blocked-version dominated by level-3 BLAS operations would be more efficient (future work), but...
- for $AX + X^\star B = C$, cost is dominated by the QZ-alg on $A - \lambda B^\star$.
- The algorithm should be compared with Bartels-Stewart algorithm for Sylvester equation $AX - XB = C$:
  1. Compute independently triang. Schur forms T_A and T_B of A and B.
  2. Solve $T_A\,Y - Y\,T_B = D$ for Y.
  3. Recover X from Y.
- Same flavor, but also differences: $A - \lambda B^\star$ and $T_A\,Y - Y\,T_B = D$ allows us to compute each entry of Y in terms on previous ones (from left and from bottom) **without using 2×2 linear-systems**.

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Outline

Theoretical method to solve $AX + X^\star B = 0$

- In case of consistency, but “nonuniqueness”, general solution of $AX + X^\star B = C$ is $X = X_p + X_h$, where
  1. X_p is a particular solution and
  2. X_h is the general solution of $AX + X^\star B = 0$.

  The latter found a few weeks ago by De Terán, D., Guillery, Montealegre, Reyes (REU program, U. of California at S. Barbara, M.I. Bueno).
- **KEY IDEA:** If $E - \lambda F^\star$ is the *Kronecker Canonical form (KCF)* of $A - \lambda B^\star$, then $AX + X^\star B = 0$ can be transformed into

$$EY + Y^\star F = 0.$$

- If $E = E_1 \oplus \cdots \oplus E_d$, $F^\star = F_1^\star \oplus \cdots \oplus F_d^\star$, and $Y = [Y_{ij}]$ is partitioned into blocks accordingly, then this equation decouples in

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- which produce **14** different types of matrix (systems) equations, whose explicit solutions have been found. Much more complicated general solution than standard Sylvester eq: $AX - XB = 0$.

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Conclusions

- Several questions related to the Sylvester equation for $\star$-congruence $AX + X^\star B = C$ are well-understood:
  1. Necessary and sufficient conds for consistency/uniqueness of sols.
  2. Efficient and stable nume. algorithm for computing unique solution.
  3. General solution of $AX + X^\star B = 0$.
- Connections with strd. Syl. eq $AX - XB = C$ but also relevant diffs:
  1. Use of QZ-algor for pencil $A - \lambda B^\star$ instead of QR-algor for matrices.
  2. Use of KCF for pencil $A - \lambda B^\star$ instead of JCF for matrices in general homogeneous solution: much more complicated solution.
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- Several problems still remain open. Among them:
  1. Combine the algor for $AX + X^\star B = C$ with algors for computing the anti-triangular Schur form for completely solving the linear palindromic eigenproblem via congruence.
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