Structured eigenvalue condition numbers for parameterized quasiseparable matrices

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Abstract

- Fast computations with $n \times n$ **rank structured matrices** have received recently a lot of attention and they rely on **parameterizing or representing** these matrices in terms of $O(n)$ parameters.
- If **parameters are the input of an algorithm for computing** Y, then the variation of Y under tiny relative **perturbations** of the parameters **determines the maximal/ideal accuracy of a computation**, and,
- in addition, this allows us to decide rigourously **whether or not one representation is better than another for computing** Y accurately.
- In this talk, we consider
  1. $Y = \lambda$, a simple eigenvalue.
  2. Mainly, **{1,1}-quasiseparable/semiseparable** matrices.
  3. Some results on **$\{n_L,n_U\}$-quasiseparable** matrices.
  4. **Two representations**: **Quasiseparable** (Eidelman-Gohberg) and **Givens-vector** (Vandebril-Van Barel-Mastronardi).
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Outline

1. Basics on eigenvalue condition numbers
2. Quasiseparable matrices
3. Condition numbers for $\{1,1\}$-quasiseparable matrices in the quasiseparable representation
4. Condition numbers for $\{1,1\}$-quasiseparable matrices in the Givens-vector representation
5. Numerical tests
6. Condition numbers for $\{1,1\}$-semiseparable matrices in the quasiseparable representation
7. Condition numbers for $\{n_L, n_U\}$-quasiseparable matrices in the quasiseparable representation
8. Conclusions and future work

Outline

Main characters: The matrix, the eigenvalue, the eigenvectors,...

Let $\lambda \in \mathbb{C}$ be a **simple eigenvalue** of $M \in \mathbb{R}^{n\times n}$, with left and right eigenvectors $\boldsymbol{y} \in \mathbb{C}^n$ and $\boldsymbol{x} \in \mathbb{C}^n$, that is,

$$M\,\boldsymbol{x} = \boldsymbol{x}\,\lambda \qquad \text{and} \qquad \boldsymbol{y}^*\,M = \lambda\,\boldsymbol{y}^*.$$

Theorem (The Wilkinson Condition Number (Wilkinson, 1965))

$$\kappa_\lambda := \lim_{\eta\to 0} \sup \left\{ \frac{|\delta\lambda|}{\eta} : (\lambda + \delta\lambda) \textit{ is eigenvalue of } (M + \delta M),\ \|\delta M\|_2 \le \eta \right\}$$

$$= \frac{\|\boldsymbol{y}\|_2 \|\boldsymbol{x}\|_2}{|\boldsymbol{y}^*\boldsymbol{x}|} = \frac{1}{\cos\angle(\boldsymbol{y},\boldsymbol{x})}$$

Wilkinson condition number is an **absolute-absolute normwise** condition number, so, it is **not convenient** in most applications.

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The Wilkinson Eigenvalue Condition Number

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Relative Wilkinson and Componentwise Condition Numbers ($\lambda \neq 0$)

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Relative Wilkinson vs. Componentwise Condition Numbers

- It is obvious that

$$\mathrm{cond}(\lambda) = \frac{|\boldsymbol{y}^*|\,|M|\,|\boldsymbol{x}|}{|\lambda|\,|\boldsymbol{y}^*\boldsymbol{x}|} \quad \leq \quad \sqrt{n}\,\frac{\|\boldsymbol{y}\|_2\|\boldsymbol{x}\|_2}{|\boldsymbol{y}^*\boldsymbol{x}|}\,\frac{\|M\|_2}{|\lambda|} = \sqrt{n}\,\kappa_\lambda^{rel}$$

- and, in some important situations, $\ll$,
- as, for instance, **if $A > 0$ entrywise and λ is the Perron-root**, then $y > 0$, $x > 0$, and

$$\mathrm{cond}(\lambda) = 1 \qquad \text{while } \kappa_\lambda^{rel} \text{ can be arbitrarily large.}$$

(Elsner, Koltracht, Neumann, Xiao, SIMAX, 1993).

- **Wilkinson and relative Wilkinson condition numbers are not invariant under diagonal similarity** (balancing), but **componentwise condition numbers are**

$$\mathrm{cond}(\lambda, M) = \mathrm{cond}(\lambda, KMK^{-1}),$$

with $K = \mathrm{diag}(k_1, \ldots, k_n) \in \mathbb{R}^{n\times n}$.

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- I am not claiming that this result is original, although I have not seen it written in this form.
- It was essentially presented in Ferreira, Parlett, D., *"Sensitivity of eigenvalues of an unsymmetric tridiagonal matrix"*, Numer. Math., 2012.
- The use of differential calculus in the Theory of Conditioning has been known from early days of Numerical Linear Algebra (Rice, SIAM J. Numer. Anal., 1966).
- If $\Omega = (m_{ij})$ are the entries of the matrix, then $\mathrm{cond}(\lambda; \Omega) = \mathrm{cond}(\lambda)$, i.e., the standard componentwise eigenvalue condition number.
- Sometimes, it will be necessary to specify both the matrix and the parameters. In these cases,

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- It was essentially presented in Ferreira, Parlett, D., *"Sensitivity of eigenvalues of an unsymmetric tridiagonal matrix"*, Numer. Math., 2012.
- The use of differential calculus in the Theory of Conditioning has been known from early days of Numerical Linear Algebra (Rice, SIAM J. Numer. Anal., 1966).
- If $\Omega = (m_{ij})$ are the entries of the matrix, then $\mathrm{cond}(\lambda; \Omega) = \mathrm{cond}(\lambda)$, i.e., the standard componentwise eigenvalue condition number.
- Sometimes, it will be necessary to specify both the matrix and the parameters. In these cases,

$$\mathrm{cond}(\lambda, M; \Omega) \equiv \mathrm{cond}(\lambda; \Omega)$$

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$$\mathrm{cond}(\lambda, M; \Omega) \equiv \mathrm{cond}(\lambda; \Omega)$$
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Outline

Quasiseparable matrices (I): Definition

Definition (Eidelman-Gohberg, Int. Eq. Op. Th., 1999)

A square matrix $C \in \mathbb{R}^{n\times n}$ is a $\{n_L, n_U\}$**-quasiseparable** matrix if

- every submatrix of C entirely located in the **strictly lower (resp. upper) triangular part** of C **have rank at most** n_L (resp. n_U), and
- at least one of these submatrices has rank equal to n_L (resp. n_U).

This is equivalent to

$$\max_i \operatorname{rank} C(i+1:n, 1:i) = n_L \quad \text{and} \quad \max_i \operatorname{rank} C(1:i, i+1:n) = n_U$$

Therefore the following submatrices have rank at most n_L or rank at most n_U:

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Quasiseparable matrices (II)

$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

Quasiseparable matrices (II): rank $\leq n_L$

$$C = \left[\begin{array}{c|cccc} \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{array}\right]$$

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Outline

Quasiseparable representation of $\{1,1\}$-quasiseparable matrices

Theorem (Eidelman and Gohberg, Int. Eq. Op. Th., 1999)

The set of $\{1,1\}$**-quasiseparable matrices of size** $n \times n$ **can be parameterized in terms of** $7n-8$ **independent scalar parameters**

$$\Omega_{QS} = (\{p_i\}_{i=2}^{n}, \{a_i\}_{i=2}^{n-1}, \{q_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{g_i\}_{i=1}^{n-1}, \{b_i\}_{i=2}^{n-1}, \{h_i\}_{i=2}^{n})$$

as follows: C is $\{1,1\}$-quasiseparable if and only if (5×5 example)

$$C = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

This is the most general representation of $\{1,1\}$-quasiseparable matrices and includes as special cases other representations explained in the books by Vandebril, Van Barel, and Mastronardi (2008).

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Properties of quasiseparable representation

- **It is not unique:** If the matrix C is fixed, then **there are infinite sets of parameters Ω_{QS} that give C**. Example

$$C = \begin{bmatrix} d_1 & g_1 h_2 & g_1 b_2 h_3 & g_1 b_2 b_3 h_4 & g_1 b_2 b_3 b_4 h_5 \\ (\alpha p_2)\frac{q_1}{\alpha} & d_2 & g_2 h_3 & g_2 b_3 h_4 & g_2 b_3 b_4 h_5 \\ p_3(\alpha a_2)\frac{q_1}{\alpha} & p_3 q_2 & d_3 & g_3 h_4 & g_3 b_4 h_5 \\ p_4 a_3(\alpha a_2)\frac{q_1}{\alpha} & p_4 a_3 q_2 & p_4 q_3 & d_4 & g_4 h_5 \\ p_5 a_4 a_3(\alpha a_2)\frac{q_1}{\alpha} & p_5 a_4 a_3 q_2 & p_5 a_4 q_3 & p_5 q_4 & d_5 \end{bmatrix}$$

- Therefore, **the natural perturbations to be considered are relative componentwise perturbations of Ω_{QS}** and not relative normwise perturbations of Ω_{QS}.
- Although the function

$$\Omega_{QS} \longrightarrow \text{"Set of } \{1,1\}\text{-quasiseparable matrices"} \subset \mathbb{R}^{n\times n}$$

is not injective, it is surjective and differentiable and we can deduce eigenvalue condition numbers.

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Componentwise eig. cond. number in the quasiseparable repr. (I)

Theorem

Let $C \in \mathbb{R}^{n\times n}$ be a $\{1,1\}$-quasiseparable matrix and

$$C = C_L + C_D + C_U$$

with C_L strictly lower triangular, C_D diagonal, and C_U strictly upper triangular. Then

$$\begin{aligned}\operatorname{cond}(\lambda;\Omega_{QS}) = & \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{|\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| \\ & + |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}| \\ & + \sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i-1) & 0\end{bmatrix}\boldsymbol{x}\right| \\ & + \sum_{j=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & C(1:j-1,j+1:n) \\ 0 & 0\end{bmatrix}\boldsymbol{x}\right|\Bigg\}\end{aligned}$$

Eig. cond. number in the quasiseparable repr. (II): d_i

$$\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$

$$+ |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}|$$

$$+ \sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i-1) & 0 \end{bmatrix}\boldsymbol{x}\right|$$

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$$C = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

Eig. cond. number in the quasiseparable repr. (II): p_i

$$\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^* C_L||\boldsymbol{x}|$$
$$+ |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^* C_U||\boldsymbol{x}|$$
$$+ \sum_{i=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix} \boldsymbol{x} \right|$$
$$+ \sum_{j=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & C(1:j-1, j+1:n) \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \right| \Bigg\}$$

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Eig. cond. number in the quasiseparable repr. (II): q_i

$$\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$

$$+ |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}|$$

$$+ \sum_{i=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix} \boldsymbol{x} \right|$$

$$+ \sum_{j=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & C(1:j-1, j+1:n) \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \right| \Bigg\}$$

$$C = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

Eig. cond. number in the quasiseparable repr. (II): g_i

$$\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$

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Eig. cond. number in the quasiseparable repr. (II): h_i

$$
\begin{aligned}
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& + |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}| \\
& + \sum_{i=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix} \boldsymbol{x} \right| \\
& + \sum_{j=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & C(1:j-1, j+1:n) \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \right| \Bigg\}
\end{aligned}
$$

$$
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\end{bmatrix}
$$

Eig. cond. number in the quasiseparable repr. (II): a_i

$$\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{|\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$
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& + \sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i-1) & 0\end{bmatrix}\boldsymbol{x}\right| \\
& + \sum_{j=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & C(1:j-1,j+1:n) \\ 0 & 0\end{bmatrix}\boldsymbol{x}\right| \Bigg\}
\end{aligned}
$$

$$
C = \begin{bmatrix}
d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\
p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\
p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\
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\end{bmatrix}
$$

Eig. cond. number in the quasiseparable repr. (II): a_i

$$\begin{aligned}\mathrm{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|} \Bigg\{ & |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| \\ & + |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}| \\ & + \sum_{i=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i-1) & 0 \end{bmatrix} \boldsymbol{x} \right| \\ & + \sum_{j=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & C(1:j-1,j+1:n) \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \right| \Bigg\}\end{aligned}$$

$$C = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

Eig. cond. number in the quasiseparable repr. (II): b_i

$$\mathrm{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{|\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$

$$+ |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}|$$

$$+ \sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix}\boldsymbol{x}\right|$$

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Eig. cond. number in the quasiseparable repr. (II): b_i

$$
\begin{aligned}
\operatorname{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|} \Bigg\{ & |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| \\
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Eig. cond. number in the quasiseparable repr. (II): b_i

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\begin{aligned}
\operatorname{cond}(\lambda;\Omega_{QS}) = & \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| \\
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Properties of $\mathrm{cond}(\lambda;\Omega_{QS})$. (I)

$$\mathrm{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^* C_L||\boldsymbol{x}| + |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^* C_U||\boldsymbol{x}| + \sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i-1) & 0\end{bmatrix}\boldsymbol{x}\right| + \sum_{j=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & C(1:j-1,j+1:n) \\ 0 & 0\end{bmatrix}\boldsymbol{x}\right| \Bigg\}$$

Property 1

$\mathrm{cond}(\lambda;\Omega_{QS})$ **does not depend on the parameters**. It only depends on the matrix entries, the eigenvalue, and the left-right eigenvectors.

Therefore, for any two sets Ω_{QS} and Ω'_{QS} of quasiseparable parameters of the same matrix C

$$\mathrm{cond}(\lambda;\Omega_{QS}) = \mathrm{cond}(\lambda;\Omega'_{QS})$$

Properties of $\mathrm{cond}(\lambda; \Omega_{QS})$. (II)

$$\mathrm{cond}(\lambda; \Omega_{QS}) = \frac{1}{|\lambda| |\boldsymbol{y}^* \boldsymbol{x}|} \Bigg\{ |\boldsymbol{y}^*| |C_D| |\boldsymbol{x}| + |\boldsymbol{y}^*| |C_L \boldsymbol{x}| + |\boldsymbol{y}^* C_L| |\boldsymbol{x}| + |\boldsymbol{y}^*| |C_U \boldsymbol{x}| + |\boldsymbol{y}^* C_U| |\boldsymbol{x}| + \sum_{i=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix} \boldsymbol{x} \right| + \sum_{j=2}^{n-1} \left| \boldsymbol{y}^* \begin{bmatrix} 0 & C(1:j-1, j+1:n) \\ 0 & 0 \end{bmatrix} \boldsymbol{x} \right| \Bigg\}$$

Property 2 (Comparison with unstructured condition number)

$$\mathrm{cond}(\lambda; \Omega_{QS}) \quad \leq \quad n \, \mathrm{cond}(\lambda) = n \, \frac{|\boldsymbol{y}^*| \, |C| \, |\boldsymbol{x}|}{|\lambda| \, |\boldsymbol{y}^* \boldsymbol{x}|},$$

and "potentially" $\mathrm{cond}(\lambda; \Omega_{QS}) \ll \mathrm{cond}(\lambda)$. The factor n comes from

$$\widetilde{c}_{n1} = \widetilde{p}_n \widetilde{a}_{n-1} \cdots \widetilde{a}_2 \widetilde{q}_1 = p_n (1 + \eta_{p_n}) \, a_{n-1} (1 + \eta_{a_{n-1}}) \cdots a_2 (1 + \eta_{a_2}) q_1 (1 + \eta_{q_1}) = c_{n1} \, (1 + \eta_{p_n})(1 + \eta_{a_{n-1}}) \cdots (1 + \eta_{a_2})(1 + \eta_{q_1})$$

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$$= c_{n1}\,(1+\eta_{p_n})(1+\eta_{a_{n-1}})\cdots(1+\eta_{a_2})(1+\eta_{q_1})$$

Properties of $\mathrm{cond}(\lambda;\Omega_{QS})$. (III)

$$\mathrm{cond}(\lambda;\Omega_{QS}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{ |\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}|$$
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Property 3

Assume that λ, x, $\boldsymbol{y}$, and Ω_{QS} are known. Then

$$\mathrm{cond}(\lambda;\Omega_{QS})$$

can be computed in $42n - 57$ **flops**. The main "trick" for this is to compute the terms in the summations via recurrence relations.

Properties of $\mathrm{cond}(\lambda;\Omega_{QS})$. (IV)

$$\mathrm{cond}_{eff}(\lambda;\Omega_{QS}) := \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Big\{|\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*||C_L\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| + |\boldsymbol{y}^*||C_U\boldsymbol{x}| + |\boldsymbol{y}^*C_U||\boldsymbol{x}|\Big\}$$

Property 4

Then

$$\mathrm{cond}_{eff}(\lambda;\Omega_{QS}) \le \mathrm{cond}(\lambda;\Omega_{QS}) \le (n-1)\,\mathrm{cond}_{eff}(\lambda;\Omega_{QS}).$$

It should be observed that C_L and C_U are "never out of $|\cdot|$". To be compared with "unstructured" componentwise condition number:

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Properties of $\mathrm{cond}(\lambda;\Omega_{QS})$. (V)

Lemma

Let $K = \mathrm{diag}(k_1,\ldots,k_n)$ *and* $C \in \mathbb{R}^{n\times n}$ *be* $\{n_L, n_U\}$*-quasiseparable.*

- *Then* KSK^{-1} *is* $\{n_L, n_U\}$*-quasiseparable.*
- *If* $(\{p_i\}_{i=2}^{n}, \{a_i\}_{i=2}^{n-1}, \{q_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{g_i\}_{i=1}^{n-1}, \{b_i\}_{i=2}^{n-1}, \{h_i\}_{i=2}^{n})$ *are quasiseparable parameters of* C*, then*

$$(\{k_i p_i\}_{i=2}^{n}, \{a_i\}_{i=2}^{n-1}, \{q_i/k_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{k_i g_i\}_{i=1}^{n-1}, \{b_i\}_{i=2}^{n-1}, \{h_i/k_i\}_{i=2}^{n})$$

are quasiseparable parameters of KCK^{-1}

Property 5

Let $C \in \mathbb{R}^{n\times n}$ be $\{1,1\}$-quasiseparable, $K \in \mathbb{R}^{n\times n}$ be diagonal and nonsingular, Ω_{QS} be any set of quasiseparable parameters of C, and Ω'_{QS} be any set of quasiseparable parameters of KCK^{-1}. Then

$$\mathrm{cond}(\lambda, C;\Omega_{QS}) = \mathrm{cond}(\lambda, KCK^{-1};\Omega'_{QS}).$$

Properties of $\mathrm{cond}(\lambda; \Omega_{QS})$. (V)

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Outline

Givens-vector representation of $\{1,1\}$-quasiseparable matrices (I)

Theorem (Vandebril-Van Barel-Mastronardi, Num. Lin. Alg. Appl., 2005)

The set of **$\{1,1\}$-quasiseparable matrices of size $n \times n$ can be parameterized in terms of**

- $\{c_i, s_i\}_{i=2}^{n-1}$ *pairs of cosines-sines,*
- $\{v_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{e_i\}_{i=1}^{n-1}$ *independent scalar parameters,*
- $\{r_i, t_i\}_{i=2}^{n-1}$ *pairs of cosines-sines,*

as follows: C is $\{1,1\}$-quasiseparable if and only if (5×5 example)

$$C = \begin{bmatrix} d_1 & e_1 r_2 & e_1 t_2 r_3 & e_1 t_2 t_3 r_4 & e_1 t_2 t_3 t_4 \\ c_2 v_1 & d_2 & e_2 r_3 & e_2 t_3 r_4 & e_2 t_3 t_4 \\ c_3 s_2 v_1 & c_3 v_2 & d_3 & e_3 r_4 & e_3 t_4 \\ c_4 s_3 s_2 v_1 & c_4 s_3 v_2 & c_4 v_3 & d_4 & e_4 \\ s_4 s_3 s_2 v_1 & s_4 s_3 v_2 & s_4 v_3 & v_4 & d_5 \end{bmatrix}$$

The "vectors" are $\{v_i\}_{i=1}^{n-1}$ and $\{e_i\}_{i=1}^{n-1}$.

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Givens-vector representation of $\{1,1\}$-quasiseparable matrices (II)

- Givens-vector representation was introduced to improve the numerical stability in solving eigenproblems with respect other representations.
- **Givens-vector representation is a particular case of quasiseparable representation seen before, i.e., a particular choice of Ω_{QS},**

$$\Omega_{QS}^{GV} := \big(\{c_i, s_i\}_{i=2}^{n-1}, \{v_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{e_i\}_{i=1}^{n-1}, \{t_i, r_i\}_{i=2}^{n-1}\big)$$

$$p_i, a_i \quad , \quad q_i \quad , \quad d_i \quad , \quad g_i \quad , \quad b_i, h_i,$$

with $p_n = h_n = 1$,

- therefore, one might think that it makes no sense to study again eigenvalue condition numbers since they are independent of the particular choice of Ω_{QS},
- **but the subtle point here is that independent componentwise perturbations of Ω_{QS}^{GV} destroy the pairs cosine-sine**, and
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- therefore, one might think that it makes no sense to study again eigenvalue condition numbers since they are independent of the particular choice of Ω_{QS},
- **but the subtle point here is that independent componentwise perturbations of Ω_{QS}^{GV} destroy the pairs cosine-sine**, and
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- Givens-vector representation is considered by some authors as the "most stable" representation of quasiseparable matrices, but I am not aware of a formal proof of this fact.
- I pretend to give a first step in this direction.
- **Givens-vector representation** has 4 variants, but if one of these variants is chosen, then it **is unique** by fixing $c_i \geq 0$ and $r_i \geq 0$.
- **Givens-vector representation is NOT a parametrization**, since the pairs of cosines-sines $\{c_i, s_i\}_{i=2}^{n-1}$ are not independent parameters. The same happens for $\{r_i, t_i\}_{i=2}^{n-1}$.
- **We need an additional parametrization of these pairs.** Avoiding the use of trigonometric functions, **we have essentially two options.**

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1^{st} (bad) option for additional parametrization of Givens-vector repr.

$$\{c_i, s_i\}_{i=2}^{n-1} = \left\{ \sqrt{1 - s_i^2}, s_i \right\}_{i=2}^{n-1} \quad \text{and} \quad \{r_i, t_i\}_{i=2}^{n-1} = \left\{ \sqrt{1 - t_i^2}, t_i \right\}_{i=2}^{n-1}$$

But this is well-known to be a bad idea: if $s_i \approx 1$, then tiny relative perturbations of s_i produce huge relative variation of $c_i = \sqrt{1 - s_i^2}$.

This is reflected in

$$\frac{s_i}{\lambda}\frac{\partial \lambda}{\partial s_i} = \frac{1}{\lambda\,(\boldsymbol{y}^*\,\boldsymbol{x})}\;\boldsymbol{y}^* \begin{bmatrix} 0 & 0 \\ -\left(\frac{s_i}{c_i}\right)^2 C(i, 1:i-1) & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix} \boldsymbol{x}$$

which may be huge if $\left(\frac{s_i}{c_i}\right)^2$ is huge!!!

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2^{nd} (good) option for additional parametrization of Givens-vector repr.

It is better to **use "tangents"**

$$\{c_i, s_i\}_{i=2}^{n-1} = \left\{ \frac{1}{\sqrt{1+l_i^2}}, \frac{l_i}{\sqrt{1+l_i^2}} \right\}_{i=2}^{n-1},$$

$$\{r_i, t_i\}_{i=2}^{n-1} = \left\{ \frac{1}{\sqrt{1+u_i^2}}, \frac{u_i}{\sqrt{1+u_i^2}} \right\}_{i=2}^{n-1},$$

with $l_i, u_i \in \mathbb{R}$, **since tiny relative perturbations of l_i produce tiny relative perturbations of $\{c_i, s_i\}$**, same for u_i and $\{r_i, t_i\}$.

Therefore, we use the following parameters:

Definition (Givens-vector parameters via tangents)

$$\Omega_{GV} := \left(\{l_i\}_{i=2}^{n-1}, \{v_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{e_i\}_{i=1}^{n-1}, \{u_i\}_{i=2}^{n-1} \right)$$

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Componentwise eig. cond. number in the Givens-vector repr.

Theorem

Let $C \in \mathbb{R}^{n\times n}$ *be a* $\{1,1\}$*-quasiseparable matrix and*

$$C = C_L + C_D + C_U$$

with C_L *strictly lower triangular,* C_D *diagonal, and* C_U *strictly upper triangular. Then*

$$\operatorname{cond}(\lambda;\Omega_{GV}) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\Bigg\{|\boldsymbol{y}^*||C_D||\boldsymbol{x}| + |\boldsymbol{y}^*C_L||\boldsymbol{x}| + |\boldsymbol{y}^*||C_U\boldsymbol{x}|$$
$$+\sum_{i=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0\\ -s_i^2\,C(i,1:i-1) & 0\\ c_i^2\,C(i+1:n,1:i-1) & 0\end{bmatrix}\boldsymbol{x}\right|$$
$$+\sum_{j=2}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & -t_j^2C(1:j-1,j) & r_j^2C(1:j-1,j+1:n)\\ 0 & 0 & 0\end{bmatrix}\boldsymbol{x}\right|\Bigg\}$$

Properties of $\mathrm{cond}(\lambda;\Omega_{GV})$. (I)

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\end{aligned}
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Property 1

$\mathrm{cond}(\lambda;\Omega_{GV})$ **does depend on the parameters** $\{c_i, s_i\}$ **and** $\{r_i, t_i\}$, but these parameters are uniquely determined by the entries.

Properties of $\mathrm{cond}(\lambda;\Omega_{GV})$. (II)

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Theorem (Property 2)

Let $C \in \mathbb{R}^{n\times n}$ be a $\{1,1\}$-quasiseparable matrix, Ω_{GV} be the tangent-Givens-vector parameters of C, and Ω_{QS} be any set of quasiseparable parameters of C, then

$$\mathrm{cond}(\lambda, C; \Omega_{GV}) \leq \mathrm{cond}(\lambda, C; \Omega_{QS})$$

This **proves rigorously** that Givens-vector is the **“most stable representation”** among all quasiseparable representations of $\{1,1\}$-quasiseparable matrices for eigen-computations.

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Property 3

Let $C \in \mathbb{R}^{n\times n}$ be a $\{1,1\}$-quasiseparable matrix, $K \in \mathbb{R}^{n\times n}$ be diagonal and nonsingular, Ω_{GV} and Ω'_{GV} be the tangent-Givens-vector parameters of C and KCK^{-1} respectively, then

$$\mathrm{cond}(\lambda, C; \Omega_{GV}) \neq \mathrm{cond}(\lambda, KCK^{-1}; \Omega'_{GV})$$

Important difference with $\mathrm{cond}(\lambda, C; \Omega_{QS})$ related to the fact that Ω_{GV} does not change trivially under diagonal similarities.

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Property 4

Assume that λ, $\boldsymbol{x}$, $\boldsymbol{y}$, and Ω_{GV} are known. Then

$$\mathrm{cond}(\lambda; \Omega_{GV})$$

can be computed in $56\,n -$ constant **flops**.

Outline

Numerical tests (I)

- Thousands of random numerical tests have been performed in MATLAB
- generating matrices via different types of random tangent-Givens-vector parameters, and then
- computing their eigenvalues and eigenvectors with `eig` command and finally computing

$$\mathrm{cond}(\lambda), \qquad \mathrm{cond}(\lambda; \Omega_{QS}), \qquad \mathrm{cond}(\lambda; \Omega_{GV}).$$

- The sizes of the matrices have been $n \times n$ with $n = 10, 20, 30, 100, 200, 300, 400$.
- Most of the times $\mathrm{cond}(\lambda) \approx \mathrm{cond}(\lambda; \Omega_{QS}) \approx \mathrm{cond}(\lambda; \Omega_{GV})$,
- but there are distributions of the tangent-Givens-vector parameters that produce $\{1,1\}$-quasiseparable matrices such that $\mathrm{cond}(\lambda) \gg \mathrm{cond}(\lambda; \Omega_{QS})$, we have found

$$\frac{\mathrm{cond}(\lambda)}{\mathrm{cond}(\lambda; \Omega_{QS})} = 5.02 \cdot 10^{12}$$

in a certain 300×300 matrix.

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- The sizes of the matrices have been $n \times n$ with $n = 10, 20, 30, 100, 200, 300, 400$.
- Most of the times $\mathrm{cond}(\lambda) \approx \mathrm{cond}(\lambda; \Omega_{QS}) \approx \mathrm{cond}(\lambda; \Omega_{GV})$,
- but there are distributions of the tangent-Givens-vector parameters that produce $\{1, 1\}$-quasiseparable matrices such that $\mathrm{cond}(\lambda) \gg \mathrm{cond}(\lambda; \Omega_{QS})$, we have found

$$\frac{\mathrm{cond}(\lambda)}{\mathrm{cond}(\lambda; \Omega_{QS})} = 5.02 \cdot 10^{12}$$

in a certain 300×300 matrix.

Numerical tests (I)

- Thousands of random numerical tests have been performed in MATLAB
- generating matrices via different types of random tangent-Givens-vector parameters, and then
- computing their eigenvalues and eigenvectors with `eig` command and finally computing

$$\operatorname{cond}(\lambda), \qquad \operatorname{cond}(\lambda;\Omega_{QS}), \qquad \operatorname{cond}(\lambda;\Omega_{GV}).$$

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Numerical tests (II)

- However in all our tests

$$\frac{\operatorname{cond}(\lambda; \Omega_{QS})}{\operatorname{cond}(\lambda; \Omega_{GV})} < 15.$$

- This raises the question if one can prove

$$1 \leq \frac{\operatorname{cond}(\lambda; \Omega_{QS})}{\operatorname{cond}(\lambda; \Omega_{GV})} \leq \alpha(n)???,$$

with $\alpha(n)$ some moderately increasing function of n or a constant.

- We will try also to look for non-random proper tests via direct search optimization methods (Higham) in order to maximize

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Outline

Definition ($\{1,1\}$-semiseparable or Green's quasiseparable matrices)

A square matrix $C \in \mathbb{R}^{n\times n}$ is $(1,1)$**-semiseparable** if

- every submatrix of C entirely located in the **lower (resp. upper) triangular part** (**including the diagonal**) of C **have rank at most** 1 (resp. 1), and
- at least one of these submatrices has rank equal to 1 (resp. 1).

This is equivalent to

$$\max_i \operatorname{rank} C(i:n, 1:i) = 1 \quad \text{and} \quad \max_i \operatorname{rank} C(1:i, i:n) = 1$$

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Therefore the following submatrices have rank at most 1:

$\{1,1\}$-semiseparable matrices (II)

$$C = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

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$\{1,1\}$-semiseparable matrices (II)

$$C = \left[\begin{array}{c|cccc} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \hline \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{array}\right]$$

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Theorem

The set of $\{1,1\}$**-semiseparable matrices of size** $n \times n$ **can be parameterized in terms of** $6n-2$ **scalar parameters**

$$\Omega_S = (\,\{p_i\}_{i=1}^{n}, \{a_i\}_{i=1}^{n-1}, \{q_i\}_{i=1}^{n}, \{g_i\}_{i=1}^{n}, \{b_i\}_{i=1}^{n-1}, \{h_i\}_{i=1}^{n}\,)$$

with **the constraints** $p_i q_i = g_i h_i$ **for** $i = 1:n$ *as follows:* C *is* $\{1,1\}$*-semiseparable if and only if (*5×5 *example)*

$$C = \begin{bmatrix} p_1q_1 & g_1b_1h_2 & g_1b_1b_2h_3 & g_1b_1b_2b_3h_4 & g_1b_1b_2b_3b_4h_5 \\ p_2a_1q_1 & p_2q_2 & g_2b_2h_3 & g_2b_2b_3h_4 & g_2b_2b_3b_4h_5 \\ p_3a_2a_1q_1 & p_3a_2q_2 & p_3q_3 & g_3b_3h_4 & g_3b_3b_4h_5 \\ p_4a_3a_2a_1q_1 & p_4a_3a_2q_2 & p_4a_3q_3 & p_4q_4 & g_4b_4h_5 \\ p_5a_4a_3a_2a_1q_1 & p_5a_4a_3a_2q_2 & p_5a_4a_3q_3 & p_5a_4q_4 & p_5q_5 \end{bmatrix}$$

Four possible sets of independent parameters

The constraints $p_i q_i = g_i h_i$ for $i = 1 : n$ allow us to obtain from

$$\Omega_S = (\, \{p_i\}_{i=1}^n, \{a_i\}_{i=1}^{n-1}, \{q_i\}_{i=1}^n, \{g_i\}_{i=1}^n, \{b_i\}_{i=1}^{n-1}, \{h_i\}_{i=1}^n \,)$$

the following **four sets of independent parameters**

$$\begin{aligned}
\Omega_S(g) &:= (\, \{p_i\}_{i=1}^n, \{a_i\}_{i=1}^{n-1}, \{q_i\}_{i=1}^n, \{b_i\}_{i=1}^{n-1}, \{h_i\}_{i=1}^n \,) \\
\Omega_S(h) &:= (\, \{p_i\}_{i=1}^n, \{a_i\}_{i=1}^{n-1}, \{q_i\}_{i=1}^n, \{g_i\}_{i=1}^n, \{b_i\}_{i=1}^{n-1} \,) \\
\Omega_S(q) &:= (\, \{p_i\}_{i=1}^n, \{a_i\}_{i=1}^{n-1}, \{g_i\}_{i=1}^n, \{b_i\}_{i=1}^{n-1}, \{h_i\}_{i=1}^n \,) \\
\Omega_S(p) &:= (\, \{a_i\}_{i=1}^{n-1}, \{q_i\}_{i=1}^n, \{g_i\}_{i=1}^n, \{b_i\}_{i=1}^{n-1}, \{h_i\}_{i=1}^n \,)
\end{aligned}$$

for $\{1, 1\}$-semiseparable matrices **and the corresponding eigenvalue condition numbers**.

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for $\{1, 1\}$-semiseparable matrices **and the corresponding eigenvalue condition numbers**.

Not equal but equivalent condition numbers

Lemma

Let $C \in \mathbb{R}^{n \times n}$ be a $\{1,1\}$-semiseparable matrix and Ω_S be any set of semiseparable parameters of C in the quasisepararable representation. Then

1. $\operatorname{cond}(\lambda; \Omega_S(g)) = \operatorname{cond}(\lambda; \Omega_S(q))$.
2. $\operatorname{cond}(\lambda; \Omega_S(h)) = \operatorname{cond}(\lambda; \Omega_S(p))$.
3. $\frac{1}{3} \operatorname{cond}(\lambda; \Omega_S(g)) \leq \operatorname{cond}(\lambda; \Omega_S(h)) \leq 3 \operatorname{cond}(\lambda; \Omega_S(g))$

Lemma (Comparison with considering C as $\{1,1\}$-quasiseparable)

If $C \in \mathbb{R}^{n \times n}$ **is** *a* $\{1,1\}$**-semiseparable matrix**, *then $C \in \mathbb{R}^{n \times n}$* **is** *a* $\{1,1\}$**-quasiseparable** *matrix. Let Ω_{QS} be any set of quasiseparable parameters of C considered as a* $\{1,1\}$**-quasiseparable** *matrix. Then*

$$\max\{\operatorname{cond}(\lambda; \Omega_S(g)), \operatorname{cond}(\lambda; \Omega_S(h)), \operatorname{cond}(\lambda; \Omega_S(p)), \operatorname{cond}(\lambda; \Omega_S(q))\} \leq 4 \operatorname{cond}(\lambda; \Omega_{QS}).$$

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Theorem

Let $C \in \mathbb{R}^{n\times n}$ *be a* $\{1,1\}$*-semiseparable matrix. Then*

$$\mathrm{cond}(\lambda;\Omega_S(g)) = \frac{1}{|\lambda||\boldsymbol{y}^*\boldsymbol{x}|}\left\{ |\lambda||\boldsymbol{y}^*||\boldsymbol{x}| + \sum_{i=1}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & 0 \\ C(i+1:n,1:i) & 0\end{bmatrix}\boldsymbol{x}\right| + \sum_{j=1}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & C(1:j,j+1:n) \\ 0 & 0\end{bmatrix}\boldsymbol{x}\right| + \sum_{i=1}^{n}\left|\boldsymbol{y}^*\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & c_{ii} & C(i,i+1:n) \\ \hline 0 & C(i+1:n,i) & 0\end{array}\right]\boldsymbol{x}\right| + \sum_{i=1}^{n}\left|\boldsymbol{y}^*\left[\begin{array}{c|c|c} 0 & C(1:i-1,i) & 0 \\ \hline 0 & 0 & -C(i,i+1:n) \\ \hline 0 & 0 & 0\end{array}\right]\boldsymbol{x}\right| \right\}$$

Expression of $\mathrm{cond}(\lambda; \Omega_S(g))$ (II)

Theorem

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This expression is very different that $\mathrm{cond}(\lambda; \Omega_{QS})$. In particular, the yellow summand that merges lower-upper triangular parts of the matrix.

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$$+\sum_{j=1}^{n-1}\left|\boldsymbol{y}^*\begin{bmatrix} 0 & C(1:j,j+1:n) \\ 0 & 0\end{bmatrix}\boldsymbol{x}\right|$$

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This expression is very different that $\mathrm{cond}(\lambda;\Omega_{QS})$. In particular, the yellow summand that merges lower-upper triangular parts of the matrix.

1. $\mathrm{cond}(\lambda;\Omega_S(g))$ **does not depend on the parameters**. It only depends on the matrix entries, the eigenvalues, and the left-right eigenvectors.
2. If $\lambda, \boldsymbol{x}, \boldsymbol{y}$, and Ω_S are known, then $\mathrm{cond}(\lambda;\Omega_S(g))$ can be computed in $38n$ - constant **flops**.
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Outline

Quasiseparable representation of $\{n_L, n_U\}$-quasiseparable matrices

Theorem (Eidelman and Gohberg, Int. Eq. Op. Th., 1999)

The set of $\{n_L, n_U\}$**-quasiseparable matrices of size** $n \times n$ **can be parameterized in terms of parameters**

$$\Omega_{QS} = \left(\underset{1\times n_L}{\{p_i\}_{i=2}^{n}}, \underset{n_L\times n_L}{\{a_i\}_{i=2}^{n-1}}, \underset{n_L\times 1}{\{q_i\}_{i=1}^{n-1}}, \underset{1\times 1}{\{d_i\}_{i=1}^{n}}, \underset{1\times n_U}{\{g_i\}_{i=1}^{n-1}}, \underset{n_U\times n_U}{\{b_i\}_{i=2}^{n-1}}, \underset{n_U\times 1}{\{h_i\}_{i=2}^{n}} \right)$$

as follows: *C is $\{n_L, n_U\}$-quasiseparable if and only if (5×5 example)*

$$C(\Omega_{QS}) = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & g_1b_2b_3h_4 & g_1b_2b_3b_4h_5 \\ p_2q_1 & d_2 & g_2h_3 & g_2b_3h_4 & g_2b_3b_4h_5 \\ p_3a_2q_1 & p_3q_2 & d_3 & g_3h_4 & g_3b_4h_5 \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & d_4 & g_4h_5 \\ p_5a_4a_3a_2q_1 & p_5a_4a_3q_2 & p_5a_4q_3 & p_5q_4 & d_5 \end{bmatrix}$$

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$$1 \times n_L \,,\; n_L \times n_L \,,\; n_L \times 1 \,,\; 1 \times 1 \,,\; 1 \times n_U \,,\; n_U \times n_U \,,\; n_U \times 1$$

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Properties $\mathrm{cond}(\lambda; \Omega_{QS})$ of $\{n_L, n_U\}$-quasiseparable matrices (I)

- The explicit expression of $\mathrm{cond}(\lambda; \Omega_{QS})$ is omitted, since it is somewhat messy although computable in $O((n_L^2 + n_U^2)n)$ flops.
- **Property 1:** $\mathrm{cond}(\lambda; \Omega_{QS})$ **depends on the particular choice of parameters** Ω_{QS} **and not only on the matrix entries.**
- **Property 2:**

$$\mathrm{cond}(\lambda; \Omega_{QS}) \quad \not\leq \quad n \, \mathrm{cond}(\lambda) = n \, \frac{|\boldsymbol{y}^*| \, |C| \, |\boldsymbol{x}|}{|\lambda| \, |\boldsymbol{y}^* \boldsymbol{x}|},$$

  and "potentially" $\mathrm{cond}(\lambda; \Omega_{QS}) \gg \mathrm{cond}(\lambda)$ may happen, i.e., **unstructured smaller than structured!!**.
- **Property 3:** It can be proved

$$\mathrm{cond}(\lambda; \Omega_{QS}) \quad \leq \quad n \, \frac{|\boldsymbol{y}^*| \, C(|\Omega_{QS}|) \, |\boldsymbol{x}|}{|\lambda| \, |\boldsymbol{y}^* \boldsymbol{x}|},$$

  where $C(|\Omega_{QS}|)$ is the $\{n_L, n_U\}$-quasiseparable matrix corresponding to

$$|\Omega_{QS}| \equiv \left(\{|p_i|\}_{i=2}^{n}, \{|a_i|\}_{i=2}^{n-1}, \{|q_i|\}_{i=1}^{n-1}, \{|d_i|\}_{i=1}^{n}, \{|g_i|\}_{i=1}^{n-1}, \{|b_i|\}_{i=2}^{n-1}, \{|h_i|\}_{i=2}^{n} \right)$$

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  and "potentially" $\mathrm{cond}(\lambda;\Omega_{QS}) \gg \mathrm{cond}(\lambda)$ may happen, i.e., **unstructured smaller than structured!!**.
- **Property 3:** It can be proved

$$\mathrm{cond}(\lambda;\Omega_{QS}) \quad \leq \quad n\,\frac{|y^*|\,C(|\Omega_{QS}|)\,|x|}{|\lambda|\,|y^* x|},$$

  where $C(|\Omega_{QS}|)$ is the $\{n_L, n_U\}$-quasiseparable matrix corresponding to

$$|\Omega_{QS}| \equiv \left(\{|p_i|\}_{i=2}^{n}, \{|a_i|\}_{i=2}^{n-1}, \{|q_i|\}_{i=1}^{n-1}, \{|d_i|\}_{i=1}^{n}, \{|g_i|\}_{i=1}^{n-1}, \{|b_i|\}_{i=2}^{n-1}, \{|h_i|\}_{i=2}^{n}\right)$$

- The explicit expression of $\mathrm{cond}(\lambda;\Omega_{QS})$ is omitted, since it is somewhat messy although computable in $O((n_L^2 + n_U^2)n)$ flops.
- **Property 1:** $\mathrm{cond}(\lambda;\Omega_{QS})$ **depends on the particular choice of parameters Ω_{QS} and not only on the matrix entries.**
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Properties $\mathrm{cond}(\lambda; \Omega_{QS})$ of $\{n_L, n_U\}$-quasiseparable matrices (I)

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- **Property 3 is natural since tiny componentwise perturbations of parameters $|\delta \Omega_{QS}| \leq \eta \, |\Omega_{QS}|$ may change the matrix as follows**

$$|C(\Omega_{QS} + \delta\Omega_{QS}) - C(\Omega_{QS})| \leq \, [(1+\eta)^n - 1] \, C(|\Omega_{QS}|),$$

(only in the $\{1,1\}$-case we can replace $C(|\Omega_{QS}|)$ by $|C(\Omega_{QS})|$, and

- the **unstructured condition number** with respect these perturbations (Higham-Higham, 1998) is

$$\mathrm{cond}_a(\lambda) := \lim_{\eta \to 0} \sup \left\{ \frac{|\delta\lambda|}{\eta|\lambda|} : (\lambda + \delta\lambda) \text{ evalue of } (C + \delta C), \; |\delta C| \leq \eta \, C(|\Omega_{QS}|) \right\}$$

$$= \frac{|\boldsymbol{y}^*| \, C(|\Omega_{QS}|) \, |\boldsymbol{x}|}{|\lambda| \, |\boldsymbol{y}^* \boldsymbol{x}|}$$

- **Property 5:** Invariance under diagonal similarities KCK^{-1} still holds:

$$\mathrm{cond}(\lambda, C; \Omega_{QS}) = \mathrm{cond}(\lambda, KCK^{-1}; \Omega'_{QS})$$

with

$$\Omega'_{QS} = \left(\{k_i p_i\}_{i=2}^{n}, \{a_i\}_{i=2}^{n-1}, \{q_i/k_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{k_i g_i\}_{i=1}^{n-1}, \{b_i\}_{i=2}^{n-1}, \{h_i/k_i\}_{i=2}^{n} \right).$$

Properties $\mathrm{cond}(\lambda; \Omega_{QS})$ of $\{n_L, n_U\}$-quasiseparable matrices (II)

- **Property 3 is natural since tiny componentwise perturbations of parameters $|\delta\Omega_{QS}| \leq \eta\,|\Omega_{QS}|$ may change the matrix as follows**

$$|C(\Omega_{QS} + \delta\Omega_{QS}) - C(\Omega_{QS})| \leq \left[(1+\eta)^n - 1\right] C(|\Omega_{QS}|),$$

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Outline

Conclusions and future work

- We have presented a framework that allows us to obtain structured eigenvalue condition numbers for different representations of quasiseparable matrices.
- **For $\{1,1\}$-quasi. matrices the structure plays a key role and leads to eigenvalue condition numbers that can be much smaller, but in the $\{n_L, n_U\}$-case the selection of proper parameters is essential.**
- We still need to understand well the relationship between the eigenvalue condition numbers in the quasiseparable and Givens-vector representations for $\{1,1\}$-quasi. matrices,
- and to develop the Givens-vector condition numbers for $\{n_L, n_U\}$-quasiseparable matrices.
- **Next step:** condition numbers for the solution of linear systems.
- **Next step:** posteriori "residual" backward errors for different representations both for eigenvalues and linear systems.
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