Diagonally dominant matrices: Surprising recent results on a classical class of matrices

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Outline

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Definition

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The matrix $A \in \mathbb{R}^{n\times n}$ is ROW DIAGONALLY DOMINANT (rdd) if

$$\sum_{j\neq i} |a_{ij}| \leq |a_{ii}|, \quad i = 1, 2, \ldots, n.$$

$A \in \mathbb{R}^{n\times n}$ is COLUMN DIAGONALLY DOMINANT (cdd) if A^T is row diagonally dominant.

Example

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix} \quad (rdd), \qquad B = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ -2 & 1 & 5 \end{bmatrix} \quad (cdd).$$

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Famous Example (I): Second difference matrix

$$K_n = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

- This matrix arises by discretizing one-dimensional boundary value problems (second derivatives).
- Numerical methods for solving PDEs are a source of many linear systems of equations whose coefficients form diagonally dominant matrices.
- In d-dimensional problems, it is necessary to order carefully the nodes of the grids used in the discretization to get diag. domin. matrices.

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To compute the cubic spline (with parabolic boundary conditions) of a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \quad \text{with} \quad x_1 < x_2 < \cdots < x_n \,,$$

one needs to solve a system of equations whose coefficient matrix is

$$\begin{bmatrix} 1 & 1 & & & & \\ h_2 & 2(h_1+h_2) & h_1 & & & \\ & h_3 & 2(h_2+h_3) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-2}+h_{n-1}) & h_{n-2} \\ & & & & 1 & 1 \end{bmatrix},$$

where $h_k = x_{k+1} - x_k > 0$.

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- DD matrices arise also in social sciences, biology, economy ...

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Famous Example (II): Collocation matrices in cubic splines

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Selected results for diagonally dominant matrices (I)

Theorem (Lévy-Desplanques Theorem, 1881-1886)

Let the matrix $A \in \mathbb{R}^{n\times n}$ be **strictly** *row diagonally dominant, that is,*

$$\sum_{j\neq i} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \ldots, n.$$

Then A is nonsingular.

Theorem

Let the matrix $A \in \mathbb{R}^{n\times n}$ be **strictly** *row diagonally dominant. Then the number of eigenvalues of A with positive (resp. negative) real part is equal to the number of positive (resp. negative) diagonal entries of A*

Example

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix} \quad (rdd), \quad \text{eigenvalues} = \{-4.2702, 5.6351 \pm 1.8363\, i\}$$

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Let $A \in \mathbb{R}^{n \times n}$ be row or column diagonally dominant. Then **all the Schur complements of A have the same kind of diagonal dominance as A.**

In plain words, **all matrices arising by applying** *(row)* **Gaussian elimination to A (without pivoting) have the same kind of diagonal dominance as A.**

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$$A = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & -4 & 2 & -10 \end{bmatrix} \quad (rdd) \quad \sim \quad \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & -2.5 & 2.75 & -10.75 \end{bmatrix}$$

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Parenthesis: Errors in Gaussian elimination (GE) (I)

Theorem (Wilkinson, 1961)

Let $B \in \mathbb{R}^{n \times n}$ be **ANY nonsingular matrix***, let $b \in \mathbb{R}^n$, and let*

$$\widehat{x}$$

be the approximate solution of

$$Bx = b$$

computed by GE in a computer in **double precision***. Then*

$$(B + \Delta B)\widehat{x} = b, \quad \frac{\|\Delta B\|_\infty}{\|B\|_\infty} \leq 6 \cdot n^3 \cdot 10^{-16} \cdot \boldsymbol{\rho}_n,$$

where

$$\boldsymbol{\rho}_n = \frac{\max_{ijk} |a_{ij}^{(k)}|}{\max_{ij} |a_{ij}|},$$

is the **growth factor of Gaussian elimination***. Here $A^{(1)} := A, A^{(2)}, \ldots, A^{(n)}$ are the matrices appearing in the Gaussian elimination process.*

Parenthesis: Errors in Gaussian elimination (GE) (II)

Example (Growth factor)

$$A = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & -4 & 2 & -10 \end{bmatrix} \sim A^{(2)} = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & -2.5 & 2.75 & -10.75 \end{bmatrix} \sim$$

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Wilkinson (1961)- Wendroff (1966) proved

Class of matrix	Method	Bound on ρ_n
General	GE without pivoting	unbounded
General	GE with partial pivoting	2^{n-1} (huge, but usually small)

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diag. dominant	GE **without** pivoting	2

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Theorem

If $A \in \mathbb{R}^{n \times n}$ is row or column diagonally dominant, then the Gaussian elimination algorithm **without pivoting** *for solving $Ax = b$ is* **backward stable**. *More precisely, the computed solution $\widehat{x}$ satisfies*

$$(A + \Delta A)\widehat{x} = b, \qquad \frac{\|\Delta A\|_\infty}{\|A\|_\infty} \leq 12 \cdot n^3 \cdot 10^{-16}$$

Remark

Very important for preserving simultaneously structures and backward stab.

Example

$$\begin{bmatrix} 2 & -1 & & \\ -4 & 5 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \sim \text{(only one row operation)} \sim \begin{bmatrix} 2 & -1 & & \\ & 3 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

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$$\frac{\|x - \widehat{x}\|_\infty}{\|x\|_\infty} \leq \kappa(A) \cdot 12 \cdot n^3 \cdot 10^{-16}, \quad \text{where } \kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

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$$A = \begin{bmatrix} 10^{16} & -10^8/5 & 1/10 \\ 10^{16}/3 & 10^8 & -1/10 \\ 10^{16}/3 & -10^8/5 & 1 \end{bmatrix} \quad \text{and} \quad b = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{\|x - \widehat{x}\|_\infty}{\|x\|_\infty} = 0.14 \quad \text{and} \quad \kappa(A) \approx 1.6 \cdot 10^{16} \quad \left(\frac{\|A\widehat{x} - b\|_\infty}{\|A\|_\infty \|\widehat{x}\|_\infty} = 1.3 \cdot 10^{-16} \right)$$

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Last basic concept on GE: LU (LDU) factorization (I)

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ -0.25 & -0.23 & 1 & 0 \\ -0.75 & -0.38 & 0.63 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & 0 & -11.76 \end{bmatrix} \equiv LU$$

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$$\begin{bmatrix} -4 & & & \\ & 6.5 & & \\ & & 5.77 & \\ & & & -11.76 \end{bmatrix} \begin{bmatrix} 1 & -0.5 & -0.25 & 0.25 \\ 0 & 1 & 0.35 & -0.35 \\ 0 & 0 & 1 & 0.04 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv LDU$$

Outline

Motivation (I)

- Q. Ye, in *Computing singular values of diagonally dominant matrices to high relative accuracy*, *Math. Comp. (2008)*,
- developed a very ingenuous algorithm for **computing accurately? in $2n^3$ flops** the LDU factorization (Gaussian Elimination) with complete pivoting of row diagonally dominant **(rDD)** matrices
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- best error bounds that Q. Ye proved after a direct error analysis that requires considerable efforts are

$$\frac{\|L - \widehat{L}\|_\infty}{\|L\|_\infty} \leq 6\,n\,8^{(n-1)}\epsilon, \quad \frac{\|U - \widehat{U}\|_\infty}{\|U\|_\infty} \leq 6 \cdot 8^{(n-1)}\epsilon, \quad \frac{|d_{ii} - \widehat{d}_{ii}|}{|d_{ii}|} \leq 5 \cdot 8^{(n-1)}\epsilon,$$

  where $n \times n$ is the size of the matrix and ϵ the unit roundoff.

- $\epsilon = 2^{-53} \approx 10^{-16}$ in double precision, **so the bounds are > 1 for $n > 20$...**

- However, there are no condition numbers in the bounds and numerical experiments indicated accuracy.

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- **Using a structured perturbation theory of LDU factorization of Diagonally Dominant matrices** and intricate error analysis, we proved *(D. and Koev, Numer. Math, 2011)*

$$\frac{\|L-\widehat{L}\|_M}{\|L\|_M} \leq 14\, n^3 \epsilon, \quad \frac{\|U-\widehat{U}\|_M}{\|U\|_M} \leq 14\, n^3 \epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 14\, n^3 \epsilon \;\; \forall i$$

for the errors of Q. Ye's algorithm (here $\|A\|_M = \max_{ij} |a_{ij}|$).

Motivation (IV)

- **Fundamental consequences:** Q. Ye's algorithm + other existing implicit algorithms for factorized matrices allow us **to compute for Diagonally Dominant matrices with guaranteed high relative accuracy**

  1. solutions of linear systems and least square problems for most right-hand-sides *(D. and Molera, IMA Journal of Numerical Analysis, 2011), (Castro, Ceballos, D., Molera, in preparation)*,
  2. SVDs and eigenvalues-vectors of positive definite matrices *(Demmel et al, SIMAX 1992, LAA 1999)*,
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Goal of the talk

- No algorithms, no error analysis!!!
- **To present a family of new perturbation bounds under structured perturbations** for several magnitudes corresponding to **Diagonally Dominant matrices**: inverses, solutions of linear systems, LDU factorization, singular values, eigenvalues.
- Common key point in (almost all) these perturbation bounds: they are always tiny for tiny structured perturbations, even for extremely ill conditioned matrices (independent of traditional condition numbers).

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Outline

- We will assume that $A \in \mathbb{R}^{n\times n}$ **satisfies** $a_{ii} \geq 0$ for all i, unless otherwise stated.

  (**No restriction** for inverses, linear systems, least square problems, SVD, but **yes** for eigenvalues).

Example

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix} \Longrightarrow B = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

Parameterizing row diagonally dominant matrices (Q. Ye) (I)

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- Define the diagonal dominances of A and store them in a column vector $v = (v_1, v_2, \ldots, v_n)^T$ where

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.

- $$A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$$

- The pair (A_D, v) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through pairs of this type. A matrix A parameterized is this way will be denoted as

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Diagonal dominances of collocation matrices in cubic splines

To compute the cubic spline (with parabolic b. c.) of a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \quad \text{with} \quad x_1 < x_2 < \cdots < x_n \,,$$

one needs to solve a system of equations whose coefficient matrix is

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where $h_k = x_{k+1} - x_k > 0$.

- $v_1 = 0, v_2 = h_1 + h_2, \ldots, v_{n-1} = h_{n-2} + h_{n-1}, v_n = 0$
- Diagonal dominances can be computed accurately (without computing the entries) directly from the parameters defining the problem!!
- This happens in many other applications.

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Fundamental contribution: updating diagonal dominances

$$v_i^{(k)} := a_{ii}^{(k)} - \sum_{j\neq i} |a_{ij}^{(k)}|$$

Lemma (Q. Ye, 2008)

For $k+1 \leq i \leq n$,

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Example of updating diagonal dominances

$$A^{(1)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ -3 & 4 & -2 & 10 \end{bmatrix}, \quad v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \sim$$

$$A^{(2)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & 2.5 & -2.75 & 10.75 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5.5 \end{bmatrix} \sim$$

$$A^{(3)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & -3.62 & 11.62 \end{bmatrix}, \quad v^{(3)} = \begin{bmatrix} 0 \\ 2 \\ 5.54 \\ 8 \end{bmatrix} \sim$$

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Example of updating diagonal dominances

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What happens if the vector v in $\mathcal{D}(A_D, v)$ is not known?

- If only the entries of the starting matrix A are known, then one can compute with the usual *recursive summation* method

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}| \qquad \text{for all } i,$$

  but it may produce large relative cancellation errors if $a_{ii} \approx \sum_{j \neq i} |a_{ij}|$ and this would spoil the accuracy of the whole computation.

- In case of severe cancellation, one can compute the v_i with *doubly compensated summation* (Priest, 1992) that computes the sum of n numbers with relative error $2 \cdot 10^{-16}$ with cost of $10(n-1)$ flops.

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Example: Good perturbation properties of this parametrization (I)

Example: Two small relative componentwise perturbations of a row diag. dominant matrix A. **Both preserve the diag. dominant structure.**

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

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Singular values of A, B and C

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
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Outline

Bounds for the inverse under structured perturbations

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be row diagonally dominant matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Then

- A *is nonsingular if and only if* $\widetilde{A}$ *is nonsingular.*
- $$\frac{\|\widetilde{A}^{-1} - A^{-1}\|_M}{\|A^{-1}\|_M} \leq \frac{(3n-2)\delta}{1-2n\delta}$$

To be compared with

$$\frac{\|\widetilde{A}^{-1} - A^{-1}\|_\infty}{\|A^{-1}\|_\infty} \leq \left(\|A\|_\infty \|A^{-1}\|_\infty\right) \frac{\|\widetilde{A} - A\|_\infty}{\|A\|_\infty}$$

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Example: Good perturbation properties for the inverse

Two small ($\approx 10^{-3}$) relative componentwise perturbations of a rDD matrix A:

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Inverses of A, B and C:

$$A^{-1} = \begin{bmatrix} 500.56 & 500.00 & -500.33 \\ 500.00 & 500.00 & -500.00 \\ -500.44 & -500.00 & 500.67 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1000.6 & 1000.0 & -1000.3 \\ 1000.0 & 1000.0 & -1000.0 \\ -1000.4 & -1000.0 & 1000.7 \end{bmatrix}$$

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Two small ($\approx 10^{-3}$) relative componentwise perturbations of a rDD matrix A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}$$

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Inverses of A, B and C:

$$A^{-1} = \begin{bmatrix} 500.56 & 500.00 & -500.33 \\ 500.00 & 500.00 & -500.00 \\ -500.44 & -500.00 & 500.67 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1000.6 & 1000.0 & -1000.3 \\ 1000.0 & 1000.0 & -1000.0 \\ -1000.4 & -1000.0 & 1000.7 \end{bmatrix}$$

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Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be row diagonally dominant nonsingular matrices such that

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Consider the systems

$$A\, x = b \quad \text{and} \quad \widetilde{A}\, \widetilde{x} = \widetilde{b}$$

with $\|b - \widetilde{b}\|_\infty \leq \mu \|b\|_\infty$. If $2n\delta < 1$, then

$$\frac{\|\widetilde{x} - x\|_\infty}{\|x\|_\infty} \leq \left(\frac{n(3n-2)\delta}{1-2n\delta} + \mu + \frac{n(3n-2)\mu\,\delta}{1-2n\delta} \right) \frac{\|A^{-1}\|_\infty \|b\|_\infty}{\|x\|_\infty}$$

To be compared with

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For most vectors b, $\|A^{-1}\|_\infty \, \|b\|_\infty / \|x\|_\infty$ is a moderate number and for A ill-conditioned,

$$\|A^{-1}\|_\infty \, \|b\|_\infty / \|x\|_\infty \ll \kappa_\infty(A)$$

Example: Good perturbation properties for linear systems

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Solutions of $Ax = b$, $Bx = b$ and $Cx = b$ for $b = [1, 1, -1]^T$:

	$Ax_A = b$	$Bx_B = b$	$Cx_C = b$
x_1	$1.5009 \cdot 10^3$	$3.0009 \cdot 10^3$	$1.4987 \cdot 10^3$
x_2	$1.5000 \cdot 10^3$	$3.0000 \cdot 10^3$	$1.4978 \cdot 10^3$
x_3	$-1.5011 \cdot 10^3$	$-3.0011 \cdot 10^3$	$-1.4989 \cdot 10^3$

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Theorem (D. and Koev, Numer. Math., 2011)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be row diagonally dominant matrices, and $A = LDU$ and $\widetilde{A} = \widetilde{L}\,\widetilde{D}\,\widetilde{U}$ be their factorizations. If

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then

- For $i = 1 : n$

$$\widetilde{d}_{ii} = d_{ii} \frac{(1+\eta_1)\cdots(1+\eta_i)}{(1+\alpha_1)\cdots(1+\alpha_{i-1})} \qquad |\eta_k| \leq \delta, \; |\alpha_k| \leq \delta.$$

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$$|\widetilde{u}_{ij} - u_{ij}| \leq 3\, i \delta$$

Recall: $\max_{ij} |u_{ij}| = \max_{ii} |u_{ii}| = 1$.

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Theorem (continuation)

- For $i > j$,

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$$= (\, j\delta + O(\delta^2)\,) \left(|\ell_{ij}| + 2 \left| \frac{a_{ii}^{(j)}}{a_{jj}^{(j)}} \right| \right),$$

  where $A^{(j)}$ is the matrix obtained after $(j-1)$ steps of Gaussian elimination.

- If the matrix A is ordered for complete (diagonal) pivoting, then $|\ell_{ij}| \leq 1$, $|a_{ii}^{(j)}| \leq |a_{jj}^{(j)}|$ and

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Complete pivoting is essential for good behavior of L: Example

Matrix ordered according to a pivoting strategy designed to make the **factor L column diagonally dominant** and as much as possible.

$$A = \begin{bmatrix} 1000 & 100 & 500 \\ 0 & 0.1 & 0.05 \\ 100 & 10 & 120 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 400 \\ 0.05 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0.1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 & & \\ & 0.1 & \\ & & 70 \end{bmatrix} \begin{bmatrix} 1 & 0.1 & 0.5 \\ & 1 & 0.5 \\ & & 1 \end{bmatrix}$$

Example: $\delta \approx 10^{-2}$ perturbation in $\mathcal{D}(A_D, v)$.

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Example: Two small relative componentwise perturbations of a row diagonally dominant matrix A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1.5 & 3 & 1.5 \\ 1.5 & 1.5 & 3.001 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 0 \\ 0 \\ 0.001 \end{bmatrix}$$

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Eigenvalues of A, B and C

	A	B	C
λ_1	4.5007	4.5010	4.5025
λ_2	4.5000	4.5003	4.5001
λ_3	$3.3328 \cdot 10^{-4}$	$6.6662 \cdot 10^{-4}$	$3.3328 \cdot 10^{-4}$

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Parameterizing rDD matrices with diagonal entries of any sign

- Let $A \in \mathbb{R}^{n\times n}$.
- Define $v = (v_1, v_2, \ldots, v_n)$ where

$$v_i := |a_{ii}| - \sum_{j\neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$
- Define $\boxed{S = \text{diag}(\text{sign}(a_{11}), \ldots, \text{sign}(a_{nn}))}$ $(\text{sign}(0) := 1)$.
- The triplet (A_D, v, S) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through triplets of this type. Any matrix A parameterized is this way will be denoted as

$$\boxed{A = \mathcal{D}(A_D, v, S)}$$

Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n \times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S) \in \mathbb{R}^{n \times n}$ *be diagonally dominant symmetric matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ *and* $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n \geq 0$ *be, respectively, the eigenvalues of* $A = \mathcal{D}(A_D, v, S)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S)$ *.*

Assume $n^3\delta < 1/5$ *and define* $\nu := \dfrac{4n^3\delta}{1 - n\delta}$.

Then

$$\begin{aligned} |\widetilde{\lambda}_i - \lambda_i| &\leq (2\nu + \nu^2)\, |\lambda_i| \\ &= (\, 8n^3\delta + O(\delta^2)\,)\, |\lambda_i|, \qquad i = 1, \ldots, n \end{aligned}$$

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Define

$$\nu_1 := \frac{n^2\delta}{1-n\delta}\left(3 + \frac{2n\delta}{1-n\delta}\right)\|L^{-1}\|_2 \quad \text{and} \quad \nu_2 = \frac{5n^3\delta}{1-2n\delta},$$

where L *is the* LDU *factor corresponding to complete (diagonal) pivoting. If* $n\delta < 1$ *and* $\nu := \max\{\nu_1, \nu_2\} < 1$*, then*

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Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *be a rDD matrix with nonnegative diagonal entries and* λ *a simple eigenvalue of* A*. Consider left and right eigenvectors*

$$y^* A = \lambda y^* \qquad \text{and} \qquad Ax = \lambda x$$

and define

$$\operatorname{relcond}(\lambda; A_D, v) := \lim_{\delta\to 0} \sup \left\{ \frac{|\widetilde{\lambda} - \lambda|}{\delta\,|\lambda|} \ : \widetilde{\lambda} \text{ eigenvalue of } \widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}), \right.$$
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If $s_{ij} = sign(a_{ij})$*, then*

$$\operatorname{relcond}(\lambda; A_D, v) = \frac{1}{|\lambda|\,|y^* x|} \sum_{i=1}^{n} |y_i| \left(v_i\,|x_i| + \sum_{j\neq i} |a_{ij}|\,|x_i + s_{ij} x_j| \right)$$

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Example: Good perturbation properties of eigenvalues of nonsymmetric row diagonally dominant matrices

Two types of small ($\approx 10^{-3}$) relative componentwise perturbations of a rDD matrix A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(B) = \begin{bmatrix} 0 \\ 0.001 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(C) = \begin{bmatrix} 0 \\ 0.002002 \\ 0 \end{bmatrix}$$

Eigenvalues and condition numbers:

	A	B	C	$\frac{\lvert y\rvert^T \lvert A\rvert \lvert x\rvert}{\lvert\lambda\rvert \lvert y^* x\rvert}$	relcond$(\lambda; A_D, v)$
λ_1	$8.5686 \cdot 10^{-4}$	$4.2850 \cdot 10^{-4}$	$8.5803 \cdot 10^{-4}$	$6.75 \cdot 10^3$	2.14
λ_2	3.5011	3.5006	3.5023	1.44	1.00
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Outline

General advices

- To work with matrices only through their entries may be not convenient, even in the case we force the preservation of structures.
- To parameterize important classes of matrices in order to preserve explicitly their structure may be important both in theory and in applications.
- A good set of parameters should have better perturbation properties than the entries.
- A good set of parameters should allow us to work numerically with the matrices, that is, to construct algorithms based on these parameters for the fundamental tasks of Numerical Linear Algebra.
- This is easy to say, but it might be hard to do.

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Conclusions and open problems

- We have presented structured perturbation results of rDD matrices for all basic problems in Numerical Linear Algebra, **except least square problems and eigenvectors**.

- Except in the case of eigenvalues of nonsymmetric matrices, **the perturbation bounds that we have obtained are rigorous and we have proved that are always tiny for tiny perturbations**.

- **Numerical methods to perform accurate and efficient dense Numerical Linear Algebra with parameterized rDD matrices are available**, **except in the case of eigenvalues of nonsymmetric matrices (open problem!!)**.

- We believe that some of the presented bounds can be improved, in particular the one for singular values.