Inverses and condition numbers of Fiedler companion matrices

Froilán M. Dopico

joint work with Fernando De Terán and Javier Pérez-Álvaro

Department of Mathematics and
Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM,
Universidad Carlos III de Madrid (Spain)

LAA Editorial Board (current & former) 2012 Meeting
on the occasion of Hans Schneider's retirement as an Editor-in-Chief of LAA
Madison, October 10-14, 2012

Outline

Outline

1. **Introduction**
2. Definitions, examples, and properties of Fiedler matrices
3. The inverse of a Fiedler matrix
4. Condition numbers for inversion of Fiedler matrices
5. Singular values of Fiedler matrices and low rank matrices
6. Conclusion and future work

- Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be an **scalar polynomial**. Then, the **roots** of $p(z)$ are the **eigenvalues** of C_1 and C_2 (known as **Frobenius companion matrices** of $p(z)$)

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = C_1^T = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix}$$

because their characteristic polynomial is $p(z)$, i.e.,

$$\det(zI - C_1) = \det(zI - C_2) = p(z)$$

- **Frobenius companion matrices**
  - Very well-known since long long time ago (at least, since **1879**).
  - They are **very easy to construct** from the coefficients of $p(z)$.

- Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be an **scalar polynomial**. Then, the **roots** of $p(z)$ are the **eigenvalues** of C_1 and C_2 (known as **Frobenius companion matrices** of $p(z)$)

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = C_1^T = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix}$$

because their characteristic polynomial is $p(z)$, i.e.,

$$\det(zI - C_1) = \det(zI - C_2) = p(z)$$

- **Frobenius companion matrices**
  - Very well-known since long long time ago (at least, since **1879**).
  - They are **very easy to construct** from the coefficients of $p(z)$.

- Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be an **scalar polynomial**. Then, the **roots** of $p(z)$ are the **eigenvalues** of C_1 and C_2 (known as **Frobenius companion matrices** of $p(z)$)

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = C_1^T = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix}$$

because their characteristic polynomial is $p(z)$, i.e.,

$$\det(zI - C_1) = \det(zI - C_2) = p(z)$$

- **Frobenius companion matrices**
  - Very well-known since long long time ago (at least, since **1879**).
  - They are **very easy to construct** from the coefficients of $p(z)$.

Frobenius companion matrices (I)

- Let $p(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be an **scalar polynomial**. Then, the **roots** of $p(z)$ are the **eigenvalues** of C_1 and C_2 (known as **Frobenius companion matrices** of $p(z)$)

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = C_1^T = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix}$$

because their characteristic polynomial is $p(z)$, i.e.,

$$\det(zI - C_1) = \det(zI - C_2) = p(z)$$

- **Frobenius companion matrices**
  - Very well-known since long long time ago (at least, since **1879**).
  - They are **very easy to construct** from the coefficients of $p(z)$.

Frobenius companion matrices (II)

Frobenius companion matrices of scalar polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$,

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix},$$

are important in theory, numerical computations, and applications.

- They are the building blocks of the rational canonical form of a matrix.
- They are used to compute all the roots of a polynomial: **MATLAB** command `roots` uses Frobenius companion matrices and the QR algorithm to compute all the roots of $p(z)$.
- They are used in control: Single-input controllable systems can be transformed into companion form systems, which simplifies theoretical considerations as feedback analysis.
- They can be generalized to become linearizations of matrix polynomials: theoretical and numerical applications.

Frobenius companion matrices (II)

Frobenius companion matrices of scalar polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$,

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix},$$

are important in theory, numerical computations, and applications.

- They are the building blocks of the rational canonical form of a matrix.
- They are used to compute all the roots of a polynomial: **MATLAB** command `roots` uses Frobenius companion matrices and the QR algorithm to compute all the roots of $p(z)$.
- They are used in control: Single-input controllable systems can be transformed into companion form systems, which simplifies theoretical considerations as feedback analysis.
- They can be generalized to become linearizations of matrix polynomials: theoretical and numerical applications.

Frobenius companion matrices of scalar polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$,

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix},$$

are important in theory, numerical computations, and applications.

- They are the building blocks of the rational canonical form of a matrix.
- They are used to compute all the roots of a polynomial: **MATLAB** command `roots` uses Frobenius companion matrices and the QR algorithm to compute all the roots of $p(z)$.
- They are used in control: Single-input controllable systems can be transformed into companion form systems, which simplifies theoretical considerations as feedback analysis.
- They can be generalized to become linearizations of matrix polynomials: theoretical and numerical applications.

Frobenius companion matrices (II)

Frobenius companion matrices of scalar polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$,

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix},$$

are important in theory, numerical computations, and applications.

- They are the building blocks of the rational canonical form of a matrix.
- They are used to compute all the roots of a polynomial: **MATLAB** command `roots` uses Frobenius companion matrices and the QR algorithm to compute all the roots of $p(z)$.
- They are used in control: Single-input controllable systems can be transformed into companion form systems, which simplifies theoretical considerations as feedback analysis.
- They can be generalized to become linearizations of matrix polynomials: theoretical and numerical applications.

Frobenius companion matrices (II)

Frobenius companion matrices of scalar polynomial $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$,

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -a_{n-1} & 1 & & \\ \vdots & & \ddots & \\ -a_1 & & & 1 \\ -a_0 & & & \end{bmatrix},$$

are important in theory, numerical computations, and applications.

- They are the building blocks of the rational canonical form of a matrix.
- They are used to compute all the roots of a polynomial: **MATLAB** command `roots` uses Frobenius companion matrices and the QR algorithm to compute all the roots of $p(z)$.
- They are used in control: Single-input controllable systems can be transformed into companion form systems, which simplifies theoretical considerations as feedback analysis.
- They can be generalized to become linearizations of matrix polynomials: theoretical and numerical applications.

Frobenius companion matrices (III)

But as the degree n **increases**, Frobenius companion matrices **have many properties that are undesirable numerically**, which **limit their use in applications**. For instance,

- stable ones are nearly unstable,
- controllable ones are nearly uncontrollable,
- nonsingular ones are nearly singular as a consequence of having **large condition numbers for inversion**.

see, C. Kenney, A. J. Laub, *Controllability and stability radii for companion form systems*, Math. Control Signals Systems 1 (1988) 239-256.

Frobenius companion matrices (III)

But as the degree n **increases**, Frobenius companion matrices **have many properties that are undesirable numerically**, which **limit their use in applications**. For instance,

- stable ones are nearly unstable,
- controllable ones are nearly uncontrollable,
- nonsingular ones are nearly singular as a consequence of having **large condition numbers for inversion**.

see, C. Kenney, A. J. Laub, *Controllability and stability radii for companion form systems*, Math. Control Signals Systems 1 (1988) 239-256.

Frobenius companion matrices (III)

But as the degree n **increases**, Frobenius companion matrices **have many properties that are undesirable numerically**, which **limit their use in applications**. For instance,

- stable ones are nearly unstable,
- controllable ones are nearly uncontrollable,
- nonsingular ones are nearly singular as a consequence of having **large condition numbers for inversion**.

see, C. Kenney, A. J. Laub, *Controllability and stability radii for companion form systems*, Math. Control Signals Systems 1 (1988) 239-256.

Frobenius companion matrices (III)

But as the degree n **increases**, Frobenius companion matrices **have many properties that are undesirable numerically**, which **limit their use in applications**. For instance,

- stable ones are nearly unstable,
- controllable ones are nearly uncontrollable,
- nonsingular ones are nearly singular as a consequence of having **large condition numbers for inversion**.

see, C. Kenney, A. J. Laub, *Controllability and stability radii for companion form systems*, Math. Control Signals Systems 1 (1988) 239-256.

Frobenius companion matrices (III)

But as the degree n **increases**, Frobenius companion matrices **have many properties that are undesirable numerically**, which **limit their use in applications**. For instance,

- stable ones are nearly unstable,
- controllable ones are nearly uncontrollable,
- nonsingular ones are nearly singular as a consequence of having **large condition numbers for inversion**.

see, C. Kenney, A. J. Laub, *Controllability and stability radii for companion form systems*, Math. Control Signals Systems 1 (1988) 239-256.

Outline

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix},$$

which satisfy $\quad M_i M_j = M_j M_i \quad$ for $|i-j| \neq 1$.

Lemma

Frobenius companion matrix can be factorized as

$$C_1 = M_{n-1} M_{n-2} \cdots M_1 M_0$$

What happens if the order of the factors is permuted?

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix},$$

which satisfy $\quad M_i M_j = M_j M_i \quad$ for $|i-j| \neq 1$.

Lemma

Frobenius companion matrix can be factorized as

$$C_1 = M_{n-1} M_{n-2} \cdots M_1 M_0$$

What happens if the order of the factors is permuted?

Definition of Fiedler companion matrices (Fiedler, LAA, 2003) (I)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix},$$

which satisfy $\quad M_i M_j = M_j M_i \quad$ for $|i-j| \neq 1$.

Lemma

Frobenius companion matrix can be factorized as

$$C_1 = M_{n-1} M_{n-2} \cdots M_1 M_0$$

What happens if the order of the factors is permuted?

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Definition (Fiedler Companion Matrix)

Given a permutation $\sigma = (i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}$$

Examples: Frobenius companion matrices

$$C_1 = M_{n-1} \cdots M_1 M_0, \qquad C_2 = M_0 M_1 \cdots M_{n-1}$$

Definition of Fiedler companion matrices (Fiedler, LAA, 2003) (II)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Definition (Fiedler Companion Matrix)

Given a permutation $\sigma = (i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}}$$

Examples: Frobenius companion matrices

$$C_1 = M_{n-1} \cdots M_1 M_0, \qquad C_2 = M_0 M_1 \cdots M_{n-1}$$

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Definition (Fiedler Companion Matrix)

Given a permutation σ = $(i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}}$$

Examples: Frobenius companion matrices

$$C_1 = M_{n-1} \cdots M_1 M_0, \qquad C_2 = M_0 M_1 \cdots M_{n-1}$$

Properties of Fiedler companion matrices (Fiedler, LAA, 2003)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Given a permutation $\sigma = (i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \ldots M_{i_{n-1}}}$$

Theorem

If M_{σ_1} and M_{σ_2} are two Fiedler matrices of $p(z)$, **then M_{σ_1} is similar to M_{σ_2}**.
(Note: But not similar via permutation matrices!!)

Properties of Fiedler companion matrices (Fiedler, LAA, 2003)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Given a permutation $\sigma = (i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \ldots M_{i_{n-1}}}$$

Theorem

If M_{σ_1} and M_{σ_2} are two Fiedler matrices of $p(z)$, **then M_{σ_1} is similar to M_{σ_2}**.

(Note: But not similar via permutation matrices!!)

Properties of Fiedler companion matrices (Fiedler, LAA, 2003)

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ and let us define the $n \times n$ matrices

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \qquad i = 1, 2, \ldots, n-1$$

$$M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix}$$

Given a permutation $\sigma = (i_0, i_1, \ldots, i_{n-1})$ of $(0, 1, \ldots, n-1)$, the Fiedler companion matrix of $p(z)$ associated with σ is

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \ldots M_{i_{n-1}}}$$

Theorem

If M_{σ_1} and M_{σ_2} are two Fiedler matrices of $p(z)$, **then M_{σ_1} is similar to M_{σ_2}**. *(Note: But not similar via permutation matrices!!)*

Fiedler matrices: Examples and structural properties (I)

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Second Frobenius companion matrix:

$$C_2 = M_0 M_1 M_2 M_3 M_4 M_5 = \begin{bmatrix} -a_5 & 1 & & & & \\ -a_4 & & 1 & & & \\ -a_3 & & & 1 & & \\ -a_2 & & & & 1 & \\ -a_1 & & & & & 1 \\ -a_0 & & & & & \end{bmatrix}$$

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Second Frobenius companion matrix:

$$C_2 = M_0 M_1 M_2 M_3 M_4 M_5 = \begin{bmatrix} -a_5 & 1 & & & & \\ -a_4 & & 1 & & & \\ -a_3 & & & 1 & & \\ -a_2 & & & & 1 & \\ -a_1 & & & & & 1 \\ -a_0 & & & & & \end{bmatrix}$$

Fiedler matrices: Examples and structural properties (I)

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Second Frobenius companion matrix:

$$C_2 = M_0 M_1 M_2 M_3 M_4 M_5 = \begin{bmatrix} -a_5 & 1 & & & & \\ -a_4 & & 1 & & & \\ -a_3 & & & 1 & & \\ -a_2 & & & & 1 & \\ -a_1 & & & & & 1 \\ -a_0 & & & & & \end{bmatrix}$$

Fiedler matrices: Examples and structural properties (I)

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Another Fiedler matrix:

$$F = M_0 M_1 M_5 M_4 M_3 M_2 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & 1 & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -a_1 & & 1 \\ & & & -a_0 & & \end{bmatrix}$$

Fiedler matrices: Examples and structural properties (I)

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Another Fiedler matrix:

$$F = M_0 M_1 M_5 M_4 M_3 M_2 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & 1 & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -a_1 & & 1 \\ & & & -a_0 & & \end{bmatrix}$$

Fiedler matrices: Examples and structural properties (II)

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Another Fiedler matrix:

$$F = M_0 M_1 M_5 M_4 M_3 M_2 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & 1 & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -a_1 & & 1 \\ & & & -a_0 & & \end{bmatrix}$$

Structural property 1 of Fiedler matrices

Every Fiedler matrix has exactly the **same entries** as the first Frobenius companion matrix but they are in **different positions**.

Fiedler matrices: Examples and structural properties (II)

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Another Fiedler matrix:

$$F = M_0 M_1 M_5 M_4 M_3 M_2 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & 1 & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & -a_1 & & 1 \\ & & & -a_0 & & \end{bmatrix}$$

Structural property 1 of Fiedler matrices

Every Fiedler matrix has exactly the **same entries** as the first Frobenius companion matrix but they are in **different positions**.

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

Very special Fiedler matrices: **Pentadiagonal matrices** (there are 4 for each degree n)

- $$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

- $$P_2 = (M_2 M_4)(M_1 M_3 M_5) M_0 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & -a_0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

Very special Fiedler matrices: Pentadiagonal matrices (there are 4 for each degree n)

- $$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

- $$P_2 = (M_2 M_4)(M_1 M_3 M_5) M_0 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & -a_0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fiedler matrices: Examples and structural properties (III)

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Pentadiagonal Fiedler matrix:

$$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

Structural property 2 of Fiedler matrices

Frobenius companion matrices are the Fiedler matrices with **largest bandwidth**.
Pentadiagonal Fiedler matrices are the ones with **smallest bandwidth**.

Fiedler matrices: Examples and structural properties (IV)

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Pentadiagonal Fiedler matrix:

$$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

Structural property 3 of Fiedler matrices

The “ones” are never in the main diagonal.

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Pentadiagonal Fiedler matrix:

$$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

Structural property 3 of Fiedler matrices

If a "one" is at the (i, j) entry, then either the ith row or the jth column has the remaining entries equal to 0.

Fiedler matrices: Examples and structural properties (V)

First Frobenius companion matrix:

$$C_1 = M_5 M_4 M_3 M_2 M_1 M_0 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & -a_2 & -a_1 & -a_0 \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \end{bmatrix}$$

Pentadiagonal Fiedler matrix:

$$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

Structural property 3 of Fiedler matrices

If a “one” is at the (i, j) entry, then either the ith row or the jth column has at least one of the coefficients of the polynomial: $-a_0, -a_1, \ldots, -a_{n-1}$

Consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0,1,\ldots,n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

Since $M_i M_j = M_j M_i$ for $|i-j| \neq 1$, the relative position in M_σ of the factors M_i and M_{i+1} plays a crucial role in many properties of Fiedler matrices. This motivates the following definitions.

For $i = 0, 1, 2, \ldots, n-2$, we say that M_σ has a

- consecution at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- inversion at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

Examples:

- $M_5 M_1 M_0 M_2 M_3 M_4$ has a consecution at 1.
- $M_5 M_1 M_0 M_2 M_3 M_4$ has an inversion at 4.

Consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

Since $M_i M_j = M_j M_i$ for $|i - j| \neq 1$, the relative position in M_σ of the factors M_i and M_{i+1} plays a crucial role in many properties of Fiedler matrices. This motivates the following definitions.

For $i = 0, 1, 2, \ldots, n-2$, we say that M_σ has a

- consecution at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- inversion at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

Examples:

- $M_5 M_1 M_0 M_2 M_3 M_4$ has a consecution at 1.
- $M_5 M_1 M_0 M_2 M_3 M_4$ has an inversion at 4.

Consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

Since $M_i M_j = M_j M_i$ for $|i-j| \neq 1$, the relative position in M_σ of the factors M_i and M_{i+1} plays a crucial role in many properties of Fiedler matrices. This motivates the following definitions.

For $i = 0, 1, 2, \ldots, n-2$, we say that M_σ has a

- consecution at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- inversion at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

Examples:

- $M_5 M_1 M_0 M_2 M_3 M_4$ has a consecution at 1.
- $M_5 M_1 M_0 M_2 M_3 M_4$ has an inversion at 4.

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

Since $M_i M_j = M_j M_i$ for $|i - j| \neq 1$, the relative position in M_σ of the factors M_i and M_{i+1} plays a crucial role in many properties of Fiedler matrices. This motivates the following definitions.

For $i = 0, 1, 2, \ldots, n-2$, we say that M_σ has a

- consecution at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_i \cdots M_{i+1} \cdots$$

- inversion at i if the matrix M_σ is of the form

$$M_\sigma = \cdots M_{i+1} \cdots M_i \cdots$$

Examples:

- $M_5 M_1 M_0 M_2 M_3 M_4$ has a consecution at 1.
- $M_5 M_1 M_0 M_2 M_3 M_4$ has an inversion at 4.

Consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

Since $M_i M_j = M_j M_i$ for $|i - j| \neq 1$, the relative position in M_σ of the factors M_i and M_{i+1} plays a crucial role in many properties of Fiedler matrices. This motivates the following definitions.

For $i = 0, 1, 2, \ldots, n-2$, we say that M_σ has a

- consecution at i if the matrix M_σ is of the form

$$\boxed{M_\sigma = \cdots M_i \cdots M_{i+1} \cdots}$$

- inversion at i if the matrix M_σ is of the form

$$\boxed{M_\sigma = \cdots M_{i+1} \cdots M_i \cdots}$$

Examples:

- $M_5 M_1 M_0 M_2 M_3 M_4$ has a consecution at 1.
- $M_5 M_1 M_0 M_2 M_3 M_4$ has an inversion at 4.

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \iff \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \dots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \dots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \to \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \iff \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \to \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \text{PCIS}(\sigma_1) = \text{PCIS}(\sigma_2).$$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \to \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \iff \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \dots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \dots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \to \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \iff \text{PCIS}(\sigma_1) = \text{PCIS}(\sigma_2).$$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \rightarrow \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Example:

$$M_\sigma = M_0 M_3 M_2 M_1 M_4 M_5 \to \text{PCIS}(\sigma) = (1, 0, 0, 1, 1)$$

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \iff \text{PCIS}(\sigma_1) = \text{PCIS}(\sigma_2).$$

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS(σ) $= (1, 0, 0, 1, 1)$

Consecutions and inversions (II)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the positional consecution-inversion sequence of M_σ, denoted by PCIS(σ), as the $(n-1)$-tuple $(v_0, v_1, \ldots, v_{n-2})$ such that:

- $v_i = 1$ if M_σ has a consecution at i.
- $v_i = 0$ if M_σ has an inversion at i.

Lemma

Let M_{σ_1} and M_{σ_2} be two Fiedler matrices of $p(z)$. Then

$$M_{\sigma_1} = M_{\sigma_2} \Longleftrightarrow \mathrm{PCIS}(\sigma_1) = \mathrm{PCIS}(\sigma_2).$$

Example:

$M_0 M_3 M_2 M_1 M_4 M_5 = M_0 M_3 M_4 M_2 M_5 M_1$ both have PCIS$(\sigma) = (1, 0, 0, 1, 1)$

Algorithm 1: Constructing M_σ without multiplications

DESCRIPTION: Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ and PCIS(σ) yields M_σ.

if σ has a consecution at 0 then

$$W = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix}$$

else

$$W = \begin{bmatrix} -a_1 & -a_0 \\ 1 & 0 \end{bmatrix}$$

end if

for $i = 1 : n-2$

if σ has a consecution at i then

$$W = \begin{bmatrix} -a_{i+1} & 1 & 0 \\ W(:,1) & 0 & W(:,2:i+1) \end{bmatrix}$$

else

$$W = \begin{bmatrix} -a_{i+1} & W(1,:) \\ 1 & 0 \\ 0 & W(2:i+1,:) \end{bmatrix}$$

endif

end for

$M_\sigma = W$

Corollary

Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *with* $n \geq 2$.

(a) *If* $a_0 \neq -1$*, then there are* $\mathbf{2^{n-1}}$ *different Fiedler matrices of* $p(z)$.

(b) *If* $a_0 = -1$*, then there are* $\mathbf{2^{n-2}}$ *different Fiedler matrices of* $p(z)$.

- Quadratic polynomials: Fiedler matrices are the two Frobenius companion forms.
- Cubic polynomials: There are two additional Fiedler matrices.

$$\begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & -a_0 \\ 1 & 0 & 0 \end{bmatrix}$$

Number of different Fiedler matrices

Corollary

Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *with* $n \geq 2$.

(a) *If* $a_0 \neq -1$, *then there are* $\mathbf{2^{n-1}}$ *different Fiedler matrices of* $p(z)$.

(b) *If* $a_0 = -1$, *then there are* $\mathbf{2^{n-2}}$ *different Fiedler matrices of* $p(z)$.

- Quadratic polynomials: Fiedler matrices are the two Frobenius companion forms.
- Cubic polynomials: There are two additional Fiedler matrices.

$$\begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & -a_0 \\ 1 & 0 & 0 \end{bmatrix}$$

Number of different Fiedler matrices

Corollary

Let $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ *with* $n \geq 2$.

(a) *If* $a_0 \neq -1$, *then there are* $\mathbf{2^{n-1}}$ *different Fiedler matrices of* $p(z)$.

(b) *If* $a_0 = -1$, *then there are* $\mathbf{2^{n-2}}$ *different Fiedler matrices of* $p(z)$.

- Quadratic polynomials: Fiedler matrices are the two Frobenius companion forms.
- Cubic polynomials: There are two additional Fiedler matrices.

$$\begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & 1 \\ -a_0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}, \begin{bmatrix} -a_2 & 1 & 0 \\ -a_1 & 0 & -a_0 \\ 1 & 0 & 0 \end{bmatrix}$$

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Some recent developments on Fiedler companion matrices

- We consider in this talk only scalar polynomials,
- but it should be stressed that Fiedler companion matrices have received recently considerable attention as **strong linearizations of matrix polynomials** (regular, singular, rectangular),
- and that based on them **structured linearizations of structured matrix polynomials** (symmetric, palindromic, skew-symmetric,...) have been constructed.
- Antoniou and Vologiannidis (ELA 2004, Math. Control Signals Syst 2011).
- De Terán, D., Mackey (SIMAX 2010 and 2011, JCAM 2011, LAA 2012)
- Bueno, Furtado (ELA 2012, LAMA to appear, LAA submitted)...
- Tisseur and coworkers are considering one particular Fiedler companion matrix for replacing the Frobenius companion form in MATLAB command `polyeig` for computing eigenvalues of matrix polynomials.
- Fiedler matrices of scalar polynomials have appeared also in the context of **quasiseparable and CMV matrices**: Bella, Olshevsky, Strang, Zhlobich (LAA 2010 and 2011).

Outline

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

- For $i = 1, 2, \ldots, n-1$, the matrices M_i are nonsingular for any value of the coefficients, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$. In this case,

$$M_i^{-1} = \begin{bmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & a_i & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}$$

- Let $\sigma = (i_0, i_1, \ldots, i_{n-1})$ be a permutation of $(0, 1, \ldots, n-1)$ and let $M_\sigma = M_{i_0} \cdots M_{i_{n-1}}$ be the Fiedler matrix of $p(z)$ associated to σ. Then

$$M_\sigma^{-1} = M_{i_{n-1}}^{-1} M_{i_{n-2}}^{-1} \cdots M_{i_1}^{-1} M_{i_0}^{-1}.$$

The inverse of a Fiedler matrix

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

- For $i = 1, 2, \ldots, n-1$, the matrices M_i are nonsingular for any value of the coefficients, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$. In this case,

$$M_i := \begin{bmatrix} I_{n-i-1} & & & \\ & -a_i & 1 & \\ & 1 & 0 & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -a_0 \end{bmatrix},$$

$$M_i^{-1} = \begin{bmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & a_i & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}$$

- Let $\sigma = (i_0, i_1, \ldots, i_{n-1})$ be a permutation of $(0, 1, \ldots, n-1)$ and let $M_\sigma = M_{i_0} \cdots M_{i_{n-1}}$ be the Fiedler matrix of $p(z)$ associated to σ. Then

$$M_\sigma^{-1} = M_{i_{n-1}}^{-1} M_{i_{n-2}}^{-1} \cdots M_{i_1}^{-1} M_{i_0}^{-1}.$$

The inverse of a Fiedler matrix

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

- For $i = 1, 2, \ldots, n-1$, the matrices M_i are nonsingular for any value of the coefficients, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$. In this case,

$$M_i^{-1} = \begin{bmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & a_i & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}$$

- Let $\sigma = (i_0, i_1, \ldots, i_{n-1})$ be a permutation of $(0, 1, \ldots, n-1)$ and let $M_\sigma = M_{i_0} \cdots M_{i_{n-1}}$ be the Fiedler matrix of $p(z)$ associated to σ. Then

$$M_\sigma^{-1} = M_{i_{n-1}}^{-1} M_{i_{n-2}}^{-1} \cdots M_{i_1}^{-1} M_{i_0}^{-1}.$$

The inverse of a Fiedler matrix

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

- For $i = 1, 2, \ldots, n-1$, the matrices M_i are nonsingular for any value of the coefficients, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$. In this case,

$$M_i^{-1} = \begin{bmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & a_i & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}$$

- Let $\sigma = (i_0, i_1, \ldots, i_{n-1})$ be a permutation of $(0, 1, \ldots, n-1)$ and let $M_\sigma = M_{i_0} \cdots M_{i_{n-1}}$ be the Fiedler matrix of $p(z)$ associated to σ. Then

$$M_\sigma^{-1} = M_{i_{n-1}}^{-1} M_{i_{n-2}}^{-1} \cdots M_{i_1}^{-1} M_{i_0}^{-1}.$$

The inverse of a Fiedler matrix

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$

- For $i = 1, 2, \ldots, n-1$, the matrices M_i are nonsingular for any value of the coefficients, while the matrix M_0 is nonsingular if and only if $a_0 \neq 0$. In this case,

$$M_i^{-1} = \begin{bmatrix} I_{n-i-1} & & & \\ & 0 & 1 & \\ & 1 & a_i & \\ & & & I_{i-1} \end{bmatrix}, \quad M_0^{-1} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1/a_0 \end{bmatrix}$$

- Let $\sigma = (i_0, i_1, \ldots, i_{n-1})$ be a permutation of $(0, 1, \ldots, n-1)$ and let $M_\sigma = M_{i_0} \cdots M_{i_{n-1}}$ be the Fiedler matrix of $p(z)$ associated to σ. Then

$$\boxed{M_\sigma^{-1} = M_{i_{n-1}}^{-1} M_{i_{n-2}}^{-1} \cdots M_{i_1}^{-1} M_{i_0}^{-1}.}$$

Algorithm 2: Constructing M_σ^{-1} without multiplications

DESCRIPTION: Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ and PCIS(σ) yields M_σ^{-1}.

if σ has a consecution at 0 then

$$W = \begin{bmatrix} 0 & -1/a_0 \\ 1 & -a_1/a_0 \end{bmatrix}$$

else

$$W = \begin{bmatrix} 0 & 1 \\ -1/a_0 & -a_1/a_0 \end{bmatrix}$$

end if

for $i = 1 : n-2$

if σ has a consecution at i then

$$W = \begin{bmatrix} 0 & W(1,:) \\ 1 & a_{i+1} W(1,:) \\ 0 & W(2:i+1,:) \end{bmatrix}$$

else

$$W = \begin{bmatrix} 0 & 1 & 0 \\ W(:,1) & a_{i+1} W(:,1) & W(:,2:i+1) \end{bmatrix}$$

end if

end for

$M_\sigma^{-1} = W$

Examples and properties of inverses of Fiedler Matrices (I)

$$p(z) = z^6 + a_5 z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

Inverse of first Frobenius companion matrix:

$$C_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/a_0 & -a_5/a_0 & -a_4/a_0 & -a_3/a_0 & -a_2/a_0 & -a_1/a_0 \end{bmatrix}$$

Inverse of pentadiagonal Fiedler matrix

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & a_4 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/a_0 \\ 0 & 0 & 0 & 1 & a_2 & -a_1/a_0 \end{bmatrix}$$

Examples and properties of inverses of Fiedler Matrices (I)

Inverse of first Frobenius companion matrix:

$$C_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/a_0 & -a_5/a_0 & -a_4/a_0 & -a_3/a_0 & -a_2/a_0 & -a_1/a_0 \end{bmatrix}$$

Inverse of pentadiagonal Fiedler matrix

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & a_4 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/a_0 \\ 0 & 0 & 0 & 1 & a_2 & -a_1/a_0 \end{bmatrix}$$

Property 1

The inverse of a Fiedler matrix has as entries

- $-1/a_0$ and some coefficients of $p(z)$ divided by a_0 (not always the same!! and to be determined later)

Examples and properties of inverses of Fiedler Matrices (II)

Inverse of first Frobenius companion matrix:

$$C_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/a_0 & -a_5/a_0 & -a_4/a_0 & -a_3/a_0 & -a_2/a_0 & -a_1/a_0 \end{bmatrix}$$

Inverse of pentadiagonal Fiedler matrix

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \boxed{a_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & \boxed{a_4} & 0 & \boxed{a_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/a_0 \\ 0 & 0 & 0 & 1 & \boxed{a_2} & -a_1/a_0 \end{bmatrix}$$

Property 2

The inverse of a Fiedler matrix has as entries

- the remaining coefficients of $p(z)$

Examples and properties of inverses of Fiedler Matrices (III)

Inverse of first Frobenius companion matrix:

$$C_1^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1/a_0 & -a_5/a_0 & -a_4/a_0 & -a_3/a_0 & -a_2/a_0 & -a_1/a_0 \end{bmatrix}$$

Inverse of pentadiagonal Fiedler matrix

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & a_4 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/a_0 \\ 0 & 0 & 0 & 1 & a_2 & -a_1/a_0 \end{bmatrix}$$

Property 3

The inverse of a Fiedler matrix has as entries

- $n-1$ "ones".

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$

- $\mathfrak{i}_0$ **initial inversions** if it has inversions at

$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

Initial consecutions/inversions

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at
$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$
- $\mathfrak{i}_0$ **initial inversions** if it has inversions at
$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at
$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$
- $\mathfrak{i}_0$ **initial inversions** if it has inversions at
$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

Initial consecutions/inversions

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$

- $\mathfrak{i}_0$ **initial inversions** if it has inversions at

$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

Initial consecutions/inversions

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$

- $\mathfrak{i}_0$ **initial inversions** if it has inversions at

$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at
$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$
- $\mathfrak{i}_0$ **initial inversions** if it has inversions at
$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at
$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$
- $\mathfrak{i}_0$ **initial inversions** if it has inversions at
$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

This additional definition is necessary to determine those entries of M_σ^{-1} which are coefficients of $p(z)$ divided by a_0.

Let M_σ be the Fiedler matrix associated with the permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We say that M_σ has

- $\mathfrak{c}_0$ **initial consecutions** if it has consecutions at

$$0, 1, \ldots, \mathfrak{c}_0 - 1, \qquad \text{but not at } \mathfrak{c}_0.$$

- $\mathfrak{i}_0$ **initial inversions** if it has inversions at

$$0, 1, \ldots, \mathfrak{i}_0 - 1, \qquad \text{but no at } \mathfrak{i}_0.$$

Example:

- $M_\sigma = M_4 M_3 M_0 M_1 M_2 M_5 \longrightarrow \mathfrak{c}_0 = 2.$
- $M_\sigma = M_4 M_2 M_1 M_0 M_5 M_3 \longrightarrow \mathfrak{i}_0 = 2.$

The entries of M_σ^{-1}

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with σ:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

Let t_σ be the number of initial consecutions or inversions of M_σ, i.e,

$$t_\sigma = \begin{cases} \mathfrak{c}_0 & \text{if} \quad \mathfrak{c}_0 \neq 0 \\ \mathfrak{i}_0 & \text{if} \quad \mathfrak{c}_0 = 0 \end{cases}.$$

Lemma (Entries of M_σ^{-1})

(a) M_σ^{-1} *has* $t_\sigma + 1$ *entries equal to* $-\frac{1}{a_0}, -\frac{a_1}{a_0}, \ldots, -\frac{a_{t_\sigma}}{a_0}$.

(b) M_σ^{-1} *has* $n - 1 - t_\sigma$ *entries equal to* $a_{t_\sigma+1}, a_{t_\sigma+2}, \ldots, a_{n-1}$.

(c) M_σ^{-1} *has* $n - 1$ *entries equal to* 1.

(d) *The rest of the entries of* M_σ^{-1} *are equal to* 0.

The entries of M_σ^{-1}

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with σ:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

Let t_σ be the number of initial consecutions or inversions of M_σ, i.e,

$$t_\sigma = \begin{cases} \mathfrak{c}_0 & \text{if} \quad \mathfrak{c}_0 \neq 0 \\ \mathfrak{i}_0 & \text{if} \quad \mathfrak{c}_0 = 0 \end{cases} .$$

Lemma (Entries of M_σ^{-1})

(a) M_σ^{-1} *has* $t_\sigma + 1$ *entries equal to* $-\frac{1}{a_0}, -\frac{a_1}{a_0}, \ldots, -\frac{a_{t_\sigma}}{a_0}$.

(b) M_σ^{-1} *has* $n - 1 - t_\sigma$ *entries equal to* $a_{t_\sigma+1}, a_{t_\sigma+2}, \ldots, a_{n-1}$.

(c) M_σ^{-1} *has* $n - 1$ *entries equal to* 1.

(d) *The rest of the entries of* M_σ^{-1} *are equal to* 0.

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with σ:

$$\boxed{M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.}$$

Let t_σ be the number of initial consecutions or inversions of M_σ, i.e,

$$t_\sigma = \begin{cases} \mathfrak{c}_0 & \text{if} \quad \mathfrak{c}_0 \neq 0 \\ \mathfrak{i}_0 & \text{if} \quad \mathfrak{c}_0 = 0 \end{cases}.$$

Lemma (Entries of M_σ^{-1})

(a) M_σ^{-1} *has* $t_\sigma + 1$ *entries equal to* $-\frac{1}{a_0}, -\frac{a_1}{a_0}, \ldots, -\frac{a_{t_\sigma}}{a_0}$.

(b) M_σ^{-1} *has* $n - 1 - t_\sigma$ *entries equal to* $a_{t_\sigma+1}, a_{t_\sigma+2}, \ldots, a_{n-1}$.

(c) M_σ^{-1} *has* $n - 1$ *entries equal to* 1.

(d) *The rest of the entries of* M_σ^{-1} *are equal to* 0.

The entries of M_σ^{-1}. Example.

Pentadiagonal Fiedler matrix (with $t_\sigma = \mathfrak{c}_0 = 1$)

$$P_1 = (M_0 M_2 M_4)(M_1 M_3 M_5) = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

Inverse of Pentadiagonal Fiedler matrix

$$P_1^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & a_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & a_4 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/a_0 \\ 0 & 0 & 0 & 1 & a_2 & -a_1/a_0 \end{bmatrix}$$

Frobenius norm of the inverse of a Fiedler matrix

Corollary

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ, and let t_σ **be the number of initial consecutions or inversions of M_σ**. *Then:*

$$||M_\sigma^{-1}||_F^2 = (n-1) + \frac{1 + |a_1|^2 + \cdots + |a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2.$$

Example:

- Pentadiagonal: $P_1 = (M_0 M_2 M_4 \cdots)(M_1 M_3 M_5 \cdots) \longrightarrow \mathfrak{c}_0 = 1$

$$||P_1^{-1}||_F = \sqrt{n - 1 + \frac{1 + |a_1|^2}{|a_0|^2} + |a_2|^2 + \cdots + |a_{n-1}|^2}$$

- Frobenius companion matrix: $C_1 = M_{n-1} M_{n-2} \cdots M_1 M_0 \longrightarrow \mathfrak{i}_0 = n - 1$

$$||C_1^{-1}||_F = \sqrt{n - 1 + \frac{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_{n-1}|^2}{|a_0|^2}}$$

Frobenius norm of the inverse of a Fiedler matrix

Corollary

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *with* $a_0 \neq 0$, *let* M_σ *be the Fiedler matrix of* $p(z)$ *associated with a permutation* σ, *and let* t_σ **be the number of initial consecutions or inversions of** M_σ. *Then:*

$$||M_\sigma^{-1}||_F^2 = (n-1) + \frac{1 + |a_1|^2 + \cdots + |a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2.$$

Example:

- Pentadiagonal: $P_1 = (M_0 M_2 M_4 \cdots)(M_1 M_3 M_5 \cdots) \longrightarrow \mathfrak{c}_0 = 1$

$$||P_1^{-1}||_F = \sqrt{n - 1 + \frac{1 + |a_1|^2}{|a_0|^2} + |a_2|^2 + \cdots + |a_{n-1}|^2}$$

- Frobenius companion matrix: $C_1 = M_{n-1} M_{n-2} \cdots M_1 M_0 \longrightarrow \mathfrak{i}_0 = n - 1$

$$||C_1^{-1}||_F = \sqrt{n - 1 + \frac{1 + |a_1|^2 + |a_2|^2 + \cdots + |a_{n-1}|^2}{|a_0|^2}}$$

Outline

Expression of condition numbers of Fiedler matrices

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ, and let t_σ be the number of initial consecutions or inversions of M_σ.

Define

$$N(p)^2 := (n-1) + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2,$$

which is the square of the Frobenius norm of any Fiedler matrix of $p(z)$ (same for all Fiedler matrices!!!).

Theorem

In the Frobenius norm, the condition number for inversion of M_σ, i.e., $\kappa_F(M_\sigma) = \|M_\sigma\|_F \, \|M_\sigma^{-1}\|_F$, is equal to:

$$\kappa_F^2(M_\sigma) = N(p)^2 \left((n-1) + \frac{1 + |a_1|^2 + \cdots + |a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2 \right)$$

Remark: If $p(z)$ is fixed, then all Fiedler matrices with the same t_σ have the same condition number!!

Expression of condition numbers of Fiedler matrices

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ, and let t_σ be the number of initial consecutions or inversions of M_σ.

Define

$$N(p)^2 := (n-1) + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2,$$

which is the square of the Frobenius norm of any Fiedler matrix of $p(z)$ (same for all Fiedler matrices!!!).

Theorem

In the Frobenius norm, the condition number for inversion of M_σ, i.e., $\kappa_F(M_\sigma) = \|M_\sigma\|_F \, \|M_\sigma^{-1}\|_F$, is equal to:

$$\kappa_F^2(M_\sigma) = N(p)^2 \left((n-1) + \frac{1 + |a_1|^2 + \cdots + |a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2 \right)$$

Remark: If $p(z)$ is fixed, then all Fiedler matrices with the same t_σ have the same condition number!!

Expression of condition numbers of Fiedler matrices

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ, and let t_σ be the number of initial consecutions or inversions of M_σ.

Define

$$N(p)^2 := (n-1) + |a_0|^2 + |a_1|^2 + \cdots + |a_{n-1}|^2,$$

which is the square of the Frobenius norm of any Fiedler matrix of $p(z)$ (same for all Fiedler matrices!!!).

Theorem

In the Frobenius norm, the condition number for inversion of M_σ, i.e., $\kappa_F(M_\sigma) = \|M_\sigma\|_F \, \|M_\sigma^{-1}\|_F$, is equal to:

$$\kappa_F^2(M_\sigma) = N(p)^2 \left((n-1) + \frac{1 + |a_1|^2 + \cdots + |a_{t_\sigma}|^2}{|a_0|^2} + |a_{t_\sigma+1}|^2 + \cdots + |a_{n-1}|^2 \right)$$

Remark: If $p(z)$ is fixed, then all Fiedler matrices with the same t_σ have the same condition number!!

Simple bounds for the condition numbers of Fiedler matrices

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, and let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ.

(a) If $|a_0| \leq 1$, then

$$\frac{\sqrt{n-1+|a_1|^2+\cdots+|a_{n-1}|^2}}{|a_0|} \leq \kappa_F(M_\sigma) \leq \frac{n+|a_1|^2+\cdots+|a_{n-1}|^2}{|a_0|}.$$

(b) If $|a_0| > 1$, then

$$\sqrt{n-1+|a_0|^2+|a_1|^2+\cdots+|a_{n-1}|^2} \leq \kappa_F(M_\sigma) \leq n-1+|a_0|^2+|a_1|^2+\cdots+|a_{n-1}|^2.$$

In plain words: $\kappa_F(M_\sigma)$ is large if and only if

- $|a_0|$ is small or
- $|a_i|$ is large for some $i = 0, 1, \ldots, n-1$.

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, and let M_σ be the Fiedler matrix of $p(z)$ associated with a permutation σ.

(a) If $|a_0| \leq 1$, then

$$\frac{\sqrt{n-1+|a_1|^2+\cdots+|a_{n-1}|^2}}{|a_0|} \leq \kappa_F(M_\sigma) \leq \frac{n+|a_1|^2+\cdots+|a_{n-1}|^2}{|a_0|}.$$

(b) If $|a_0| > 1$, then

$$\sqrt{n-1+|a_0|^2+|a_1|^2+\cdots+|a_{n-1}|^2} \leq \kappa_F(M_\sigma) \leq n-1+|a_0|^2+|a_1|^2+\cdots+|a_{n-1}|^2.$$

In plain words: $\kappa_F(M_\sigma)$ is large if and only if

- $|a_0|$ is small or
- $|a_i|$ is large for some $i = 0, 1, \ldots, n-1$.

In plain words: $\kappa_F(M_\sigma)$ is large if and only if

- $|a_0|$ is small or
- $|a_i|$ is large for some $i = 0, 1, \ldots, n-1$.

Theorem

Let A be nonsingular. Then

$$\frac{1}{\kappa_2(A)} = \frac{1}{\|A\|_2 \, \|A^{-1}\|_2} = \min \left\{ \frac{\|\delta A\|_2}{\|A\|_2} \; : \; A + \delta A \text{ is singular} \right\} .$$

Example:

$$M_\sigma = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & -a_1 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & \mathbf{1} & 0 \\ -a_1 & 0 & 0 & \mathbf{1} \\ -a_0 & 0 & 0 & 0 \end{bmatrix}$$

If some $|a_i| \gg 1$, then a **tiny relative normwise perturbation** can turn an entry 1 in M_σ into a 0 and **make the matrix singular.**

In plain words: $\kappa_F(M_\sigma)$ is large if and only if

- $|a_0|$ is small or
- $|a_i|$ is large for some $i = 0, 1, \ldots, n-1$.

Theorem

Let A be nonsingular. Then

$$\frac{1}{\kappa_2(A)} = \frac{1}{\|A\|_2 \, \|A^{-1}\|_2} = \min \left\{ \frac{\|\delta A\|_2}{\|A\|_2} \ : \ A + \delta A \text{ is singular} \right\} .$$

Example:

$$M_\sigma = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & -a_1 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & \mathbf{1} & 0 \\ -a_1 & 0 & 0 & \mathbf{1} \\ -a_0 & 0 & 0 & 0 \end{bmatrix}$$

If some $|a_i| \gg 1$, then a **tiny relative normwise perturbation** can turn an entry 1 in M_σ into a 0 and **make the matrix singular**.

Explanation on bounds for condition numbers of Fiedler matrices

In plain words: $\kappa_F(M_\sigma)$ is large if and only if

- $|a_0|$ is small or
- $|a_i|$ is large for some $i = 0, 1, \ldots, n-1$.

Theorem

Let A be nonsingular. Then

$$\frac{1}{\kappa_2(A)} = \frac{1}{\|A\|_2 \, \|A^{-1}\|_2} = \min \left\{ \frac{\|\delta A\|_2}{\|A\|_2} \; : \; A + \delta A \textit{ is singular} \right\} .$$

Example:

$$M_\sigma = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & -a_1 & \mathbf{1} \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & -a_0 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -a_3 & \mathbf{1} & 0 & 0 \\ -a_2 & 0 & \mathbf{1} & 0 \\ -a_1 & 0 & 0 & \mathbf{1} \\ -a_0 & 0 & 0 & 0 \end{bmatrix}$$

If some $|a_i| \gg 1$, then a **tiny relative normwise perturbation** can turn an entry 1 in M_σ into a 0 and **make the matrix singular**.

Ordering Fiedler matrices according to condition numbers

Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 3$ and $a_0 \neq 0$, and a number t such that $1 \leq t \leq n-1$, we define the set

$$\mathcal{S}_t(p) := \{M_\sigma : t_\sigma = t\}. \qquad (t_\sigma \text{ number of initial cons/invs})$$

- All Fiedler matrices in $\mathcal{S}_t(p)$ have the same condition number $\kappa_F(M_\sigma)$.
- The cardinality of $\mathcal{S}_t(p)$ is

$$|\mathcal{S}_t(p)| = \begin{cases} 2^{n-1-t}, & \text{if} \quad t < n-1 \\ 2, & \text{if} \quad t = n-1 \end{cases}$$

Corollary

Define $\kappa(t) := \kappa_F(M_\sigma)$, for $M_\sigma \in \mathcal{S}_t(p)$. Then,

(a) *If $|a_0| < 1$, then $\kappa(1) \leq \kappa(2) \leq \cdots \leq \kappa(n-1)$.*

(b) *If $|a_0| = 1$, then $\kappa(1) = \kappa(2) = \cdots = \kappa(n-1)$.*

(c) *If $|a_0| > 1$, then $\kappa(1) \geq \kappa(2) \geq \cdots \geq \kappa(n-1)$.*

If $|a_0| < 1$, then the **two Frobenius companion matrices** ($t_\sigma = n-1$) **have the largest condition numbers**.

Ordering Fiedler matrices according to condition numbers

Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 3$ and $a_0 \neq 0$, and a number t such that $1 \leq t \leq n-1$, we define the set

$$\mathcal{S}_t(p) := \{M_\sigma : t_\sigma = t\}. \qquad (t_\sigma \text{ number of initial cons/invs})$$

- All Fiedler matrices in $\mathcal{S}_t(p)$ have the same condition number $\kappa_F(M_\sigma)$.
- The cardinality of $\mathcal{S}_t(p)$ is

$$|\mathcal{S}_t(p)| = \begin{cases} 2^{n-1-t}, & \text{if} \quad t < n-1 \\ 2, & \text{if} \quad t = n-1 \end{cases}$$

Corollary

Define $\kappa(t) := \kappa_F(M_\sigma)$, for $M_\sigma \in \mathcal{S}_t(p)$. Then,

(a) *If $|a_0| < 1$, then $\kappa(1) \leq \kappa(2) \leq \cdots \leq \kappa(n-1)$.*

(b) *If $|a_0| = 1$, then $\kappa(1) = \kappa(2) = \cdots = \kappa(n-1)$.*

(c) *If $|a_0| > 1$, then $\kappa(1) \geq \kappa(2) \geq \cdots \geq \kappa(n-1)$.*

If $|a_0| < 1$, then the **two Frobenius companion matrices** ($t_\sigma = n-1$) **have the largest condition numbers**.

Ordering Fiedler matrices according to condition numbers

Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 3$ and $a_0 \neq 0$, and a number t such that $1 \leq t \leq n-1$, we define the set

$$\mathcal{S}_t(p) := \{M_\sigma : t_\sigma = t\}\,. \qquad (t_\sigma \text{ number of initial cons/invs})$$

- All Fiedler matrices in $\mathcal{S}_t(p)$ have the same condition number $\kappa_F(M_\sigma)$.
- The cardinality of $\mathcal{S}_t(p)$ is

$$|\mathcal{S}_t(p)| = \begin{cases} 2^{n-1-t}, & \text{if} \quad t < n-1 \\ 2, & \text{if} \quad t = n-1 \end{cases}$$

Corollary

Define $\kappa(t) := \kappa_F(M_\sigma)$, for $M_\sigma \in \mathcal{S}_t(p)$. Then,

(a) *If $|a_0| < 1$, then $\kappa(1) \leq \kappa(2) \leq \cdots \leq \kappa(n-1)$.*

(b) *If $|a_0| = 1$, then $\kappa(1) = \kappa(2) = \cdots = \kappa(n-1)$.*

(c) *If $|a_0| > 1$, then $\kappa(1) \geq \kappa(2) \geq \cdots \geq \kappa(n-1)$.*

If $|a_0| < 1$, then the **two Frobenius companion matrices** ($t_\sigma = n-1$) **have the largest condition numbers**.

Ordering Fiedler matrices according to condition numbers

Given $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$, with $n \geq 3$ and $a_0 \neq 0$, and a number t such that $1 \leq t \leq n-1$, we define the set

$$\mathcal{S}_t(p) := \{M_\sigma : t_\sigma = t\}\,. \qquad (t_\sigma \text{ number of initial cons/invs})$$

- All Fiedler matrices in $\mathcal{S}_t(p)$ have the same condition number $\kappa_F(M_\sigma)$.
- The cardinality of $\mathcal{S}_t(p)$ is

$$|\mathcal{S}_t(p)| = \begin{cases} 2^{n-1-t}, & \text{if} \quad t < n-1 \\ 2, & \text{if} \quad t = n-1 \end{cases}$$

Corollary

Define $\kappa(t) := \kappa_F(M_\sigma)$, *for* $M_\sigma \in \mathcal{S}_t(p)$. *Then,*

(a) *If* $|a_0| < 1$, *then* $\kappa(1) \leq \kappa(2) \leq \cdots \leq \kappa(n-1)$.

(b) *If* $|a_0| = 1$, *then* $\kappa(1) = \kappa(2) = \cdots = \kappa(n-1)$.

(c) *If* $|a_0| > 1$, *then* $\kappa(1) \geq \kappa(2) \geq \cdots \geq \kappa(n-1)$.

If $|a_0| < 1$, then the **two Frobenius companion matrices ($t_\sigma = n-1$) have the largest condition numbers**.

Ratio of the condition numbers of two Fiedler matrices (I)

- For any pair of permutations σ and μ of $(0, 1, \ldots, n-1)$ such that $t_\sigma \neq t_\mu$,
- it is possible to find monic polynomials $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ of degree n, such that the ratio
- $$\frac{\kappa_F(M_\mu(p))}{\kappa_F(M_\sigma(p))}$$
- **is arbitrarily large (or small) and, so, different Fiedler matrices behave very differently.**
- Loosely speaking these polynomials must satisfy
  1. at least one of $|a_2|, |a_3|, \ldots, |a_{n-1}|$ is very large, and
  2. $|a_0|$ is very small or very large.

  Note that a_1 does not appear!! and that these conditions are necessary but not sufficient.

- For any pair of permutations σ and μ of $(0, 1, \ldots, n-1)$ such that $t_\sigma \neq t_\mu$,
- it is possible to find monic polynomials $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ of degree n, such that the ratio
- $$\frac{\kappa_F(M_\mu(p))}{\kappa_F(M_\sigma(p))}$$
- **is arbitrarily large (or small) and, so, different Fiedler matrices behave very differently.**
- Loosely speaking these polynomials must satisfy
  1. at least one of $|a_2|, |a_3|, \ldots, |a_{n-1}|$ is very large, and
  2. $|a_0|$ is very small or very large.

  Note that a_1 does not appear!! and that these conditions are necessary but not sufficient.

Ratio of the condition numbers of two Fiedler matrices (I)

- For any pair of permutations σ and μ of $(0, 1, \ldots, n-1)$ such that $t_\sigma \neq t_\mu$,
- it is possible to find monic polynomials $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ of degree n, such that the ratio
- $$\frac{\kappa_F(M_\mu(p))}{\kappa_F(M_\sigma(p))}$$
- **is arbitrarily large (or small) and, so, different Fiedler matrices behave very differently.**
- Loosely speaking these polynomials must satisfy
  1. at least one of $|a_2|, |a_3|, \ldots, |a_{n-1}|$ is very large, and
  2. $|a_0|$ is very small or very large.

  Note that a_1 does not appear!! and that these conditions are necessary but not sufficient.

- For any pair of permutations σ and μ of $(0, 1, \ldots, n-1)$ such that $t_\sigma \neq t_\mu$,
- it is possible to find monic polynomials $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ of degree n, such that the ratio
- $$\frac{\kappa_F(M_\mu(p))}{\kappa_F(M_\sigma(p))}$$
- **is arbitrarily large (or small) and, so, different Fiedler matrices behave very differently.**
- Loosely speaking these polynomials must satisfy
  1. at least one of $|a_2|, |a_3|, \ldots, |a_{n-1}|$ is very large, and
  2. $|a_0|$ is very small or very large.

  Note that a_1 does not appear!! and that these conditions are necessary but not sufficient.

- For any pair of permutations σ and μ of $(0, 1, \ldots, n-1)$ such that $t_\sigma \neq t_\mu$,
- it is possible to find monic polynomials $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ of degree n, such that the ratio
- $$\frac{\kappa_F(M_\mu(p))}{\kappa_F(M_\sigma(p))}$$
- **is arbitrarily large (or small) and, so, different Fiedler matrices behave very differently.**
- Loosely speaking these polynomials must satisfy
  1. at least one of $|a_2|, |a_3|, \ldots, |a_{n-1}|$ is very large, and
  2. $|a_0|$ is very small or very large.

  Note that a_1 does not appear!! and that these conditions are necessary but not sufficient.

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small? No

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small?

No

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small?
  No

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small?
  No

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small?
  No

Example: Large ratio of cond. numbers of two Fiedler matrices

Let $p(z) = z^3 + 10^7 z^2 + 10^{-5}$ with degree $n = 3$ and consider

$$C_1 = \begin{bmatrix} -a_2 & -a_1 & -a_0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} -a_2 & -a_1 & 1 \\ 1 & 0 & 0 \\ 0 & -a_0 & 0 \end{bmatrix}$$

with $a_2 = 10^7$ and $a_0 = 10^{-5}$.

Then

$$\kappa_F(C_1) = 10^{21}, \qquad \kappa_F(P_2) = \sqrt{2} \cdot 10^{14}, \qquad \frac{\kappa_F(C_1)}{\kappa_F(P_2)} = \frac{10^7}{\sqrt{2}}$$

- By increasing a_2 and/or by decreasing a_0, one can increase $\kappa_F(C_1)/\kappa_F(P_2)$ arbitrarily, but
- also $\kappa_F(C_1)$ and $\kappa_F(P_2)$.
- Is it possible to find polynomials with $\kappa_F(C_1)$ large and $\kappa_F(P_2)$ small? **No**

Theorem

Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $a_0 \neq 0$, let σ and μ be two permutations of $(0, 1, \ldots, n-1)$, let M_σ and M_μ be the Fiedler matrices of $p(z)$ associated with σ and μ, and let t_σ and t_μ be the number of initial consecutions or inversions of M_σ and M_μ. Assume $t_\sigma < t_\mu$.

(a) *If $|a_0| < 1$, then*

$$1 \leq \left(\frac{\kappa_F(M_\mu)}{\kappa_F(M_\sigma)} \right)^2 \leq \kappa_F(M_\sigma) \leq \kappa_F(M_\mu).$$

(b) *If $|a_0| > 1$, then*

$$1 \leq \frac{\kappa_F(M_\sigma)}{\kappa_F(M_\mu)} \leq \kappa_F(M_\mu) \leq \kappa_F(M_\sigma).$$

Condition numbers in other norms.

- We have considered only condition numbers in the Frobenius norm,
- because in other norms are more complicated
- and Fiedler matrices cannot be ordered (at least in a simple way) in terms of condition numbers in other norms.
- Other properties are also lost for other norms.

Condition numbers in other norms.

- We have considered only condition numbers in the Frobenius norm,
- because in other norms are more complicated
- and Fiedler matrices cannot be ordered (at least in a simple way) in terms of condition numbers in other norms.
- Other properties are also lost for other norms.

Condition numbers in other norms.

- We have considered only condition numbers in the Frobenius norm,
- because in other norms are more complicated
- and Fiedler matrices cannot be ordered (at least in a simple way) in terms of condition numbers in other norms.
- Other properties are also lost for other norms.

Condition numbers in other norms.

- We have considered only condition numbers in the Frobenius norm,
- because in other norms are more complicated
- and Fiedler matrices cannot be ordered (at least in a simple way) in terms of condition numbers in other norms.
- Other properties are also lost for other norms.

Outline

Singular values of Frobenius companion matrices (I)

The Frobenius companion matrix

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

has the remarkable property that its singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ can be determined explicitly in terms of the coefficients. (Kenney and Laub, 1988)

- $\sigma_{1,n}^2 = \dfrac{1 + \sum_{k=0}^{n-1} |a_k|^2 \pm \sqrt{\left(1 + \sum_{k=0}^{n-1} |a_k|^2\right)^2 - 4|a_0|^2}}{2}$
- $\sigma_i = 1$ for $i = 2, 3, \ldots, n-1$.

Therefore, the **spectral condition number** $\kappa_2(C_1)$ can also be determined explicitly.

Singular values of Frobenius companion matrices (I)

The Frobenius companion matrix

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

has the remarkable property that its singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ can be determined explicitly in terms of the coefficients. (Kenney and Laub, 1988)

- $\sigma_{1,n}^2 = \dfrac{1 + \sum_{k=0}^{n-1} |a_k|^2 \pm \sqrt{\left(1 + \sum_{k=0}^{n-1} |a_k|^2\right)^2 - 4|a_0|^2}}{2}$
- $\sigma_i = 1$ for $i = 2, 3, \ldots, n-1$.

Therefore, the **spectral condition number** $\kappa_2(C_1)$ can also be determined explicitly.

Singular values of Frobenius companion matrices (I)

The Frobenius companion matrix

$$C_1 = \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

has the remarkable property that its singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$ can be determined explicitly in terms of the coefficients. (Kenney and Laub, 1988)

- $\sigma_{1,n}^2 = \dfrac{1 + \sum_{k=0}^{n-1} |a_k|^2 \pm \sqrt{\left(1 + \sum_{k=0}^{n-1} |a_k|^2\right)^2 - 4|a_0|^2}}{2}$
- $\sigma_i = 1$ for $i = 2, 3, \ldots, n-1$.

Therefore, the **spectral condition number** $\kappa_2(C_1)$ can also be determined explicitly.

Singular values of Frobenius companion matrices (II)

- These formulas, that can be easily proved, are linked to the fact that C_1 is the sum of a unitary (permutation) matrix plus a matrix of rank one.

$$\begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 - 1 \end{bmatrix}$$

- No other Fiedler companion matrices satisfy this property, and this makes other Fiedler matrices more difficult.
- However, one can prove that every Fiedler matrix can be expressed as a sum of a permutation matrix plus a matrix whose rank varies from 1 to $\lfloor (n+1)/2 \rfloor$,
- but this is not enough to find explicit expressions of their singular values, although yes for simplifying their computations.

Singular values of Frobenius companion matrices (II)

- These formulas, that can be easily proved, are linked to the fact that C_1 is the sum of a unitary (permutation) matrix plus a matrix of rank one.

$$\begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 - 1 \end{bmatrix}$$

- No other Fiedler companion matrices satisfy this property, and this makes other Fiedler matrices more difficult.
- However, one can prove that every Fiedler matrix can be expressed as a sum of a permutation matrix plus a matrix whose rank varies from 1 to $\lfloor (n+1)/2 \rfloor$,
- but this is not enough to find explicit expressions of their singular values, although yes for simplifying their computations.

Singular values of Frobenius companion matrices (II)

- These formulas, that can be easily proved, are linked to the fact that C_1 is the sum of a unitary (permutation) matrix plus a matrix of rank one.

$$\begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 - 1 \end{bmatrix}$$

- No other Fiedler companion matrices satisfy this property, and this makes other Fiedler matrices more difficult.
- However, one can prove that every Fiedler matrix can be expressed as a sum of a permutation matrix plus a matrix whose rank varies from 1 to $\lfloor (n+1)/2 \rfloor$,
- but this is not enough to find explicit expressions of their singular values, although yes for simplifying their computations.

- These formulas, that can be easily proved, are linked to the fact that C_1 is the sum of a unitary (permutation) matrix plus a matrix of rank one.

$$\begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 - 1 \end{bmatrix}$$

- No other Fiedler companion matrices satisfy this property, and this makes other Fiedler matrices more difficult.
- However, one can prove that every Fiedler matrix can be expressed as a sum of a permutation matrix plus a matrix whose rank varies from 1 to $\lfloor (n+1)/2 \rfloor$,
- but this is not enough to find explicit expressions of their singular values, although yes for simplifying their computations.

Singular values of Frobenius companion matrices (II)

- These formulas, that can be easily proved, are linked to the fact that C_1 is the sum of a unitary (permutation) matrix plus a matrix of rank one.

$$\begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 - 1 \end{bmatrix}$$

- No other Fiedler companion matrices satisfy this property, and this makes other Fiedler matrices more difficult.
- However, one can prove that every Fiedler matrix can be expressed as a sum of a permutation matrix plus a matrix whose rank varies from 1 to $\lfloor (n+1)/2 \rfloor$,
- but this is not enough to find explicit expressions of their singular values, although yes for simplifying their computations.

Example: Fiedler Matrices as Permutation + low-rank (I)

$$M_5 M_4 M_0 M_1 M_2 M_3 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & -a_4 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & -a_0 - 1 & 0 & 0 & 0 \end{bmatrix}$$

$= \boxed{\textbf{permutation} + \textbf{rank-2}}$

By eliminating rows and columns identically zero, the **rank-2 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (I)

$$M_5 M_4 M_0 M_1 M_2 M_3 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & -a_4 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & -a_0 - 1 & 0 & 0 & 0 \end{bmatrix}$$

$=$ **permutation** $+$ **rank-2**

By eliminating rows and columns identically zero, the **rank-2 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (I)

$$M_5 M_4 M_0 M_1 M_2 M_3 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & -a_4 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & -a_0 - 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boxed{\textbf{permutation} + \textbf{rank-2}}$$

By eliminating rows and columns identically zero, the **rank-2 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (I)

$$M_5 M_4 M_0 M_1 M_2 M_3 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & -a_4 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & -a_0 - 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boxed{\text{permutation} + \text{rank-2}}$$

By eliminating rows and columns identically zero, the **rank-2 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (I)

$$M_5M_4M_0M_1M_2M_3 = \begin{bmatrix} -a_5 & -a_4 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 1 \\ 0 & 0 & -a_0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & -a_4 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & 0 & 0 \\ 0 & 0 & -a_0 - 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boxed{\textbf{permutation} + \textbf{rank-2}}$$

By eliminating rows and columns identically zero, the **rank-2 matrix** becomes

$$\begin{bmatrix} -a_5 & -a_4 & -a_3 \\ 0 & 0 & -a_2 \\ 0 & 0 & -a_1 \\ 0 & 0 & -a_0 - 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \times & \times & \times \\ & & \times \\ & & \times \\ & & \times \end{bmatrix}$$

Example: Fiedler Matrices as Permutation + low-rank (II)

$$M_0 M_2 M_4 M_1 M_3 M_5 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 - 1 & 0 \end{bmatrix}$$

$= \boxed{\textbf{permutation} + \textbf{rank-3}}$

By eliminating rows and columns identically zero, the **rank-3 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (II)

$$M_0M_2M_4M_1M_3M_5 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0-1 & 0 \end{bmatrix}$$

$=$ **permutation $+$ rank-3**

By eliminating rows and columns identically zero, the **rank-3 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (II)

$$M_0 M_2 M_4 M_1 M_3 M_5 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 - 1 & 0 \end{bmatrix}$$

$$= \boxed{\textbf{permutation} + \textbf{rank-3}}$$

By eliminating rows and columns identically zero, the **rank-3 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (II)

$$M_0M_2M_4M_1M_3M_5 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0-1 & 0 \end{bmatrix}$$

$$= \textbf{permutation} + \textbf{rank-3}$$

By eliminating rows and columns identically zero, the **rank-3 matrix** becomes

Example: Fiedler Matrices as Permutation + low-rank (II)

$$M_0 M_2 M_4 M_1 M_3 M_5 = \begin{bmatrix} -a_5 & 1 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} -a_5 & 0 & 0 & 0 & 0 & 0 \\ -a_4 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -a_0 - 1 & 0 \end{bmatrix}$$

$$= \textbf{permutation} + \textbf{rank-3}$$

By eliminating rows and columns identically zero, the **rank-3 matrix** becomes

$$\begin{bmatrix} -a_5 & 0 & 0 \\ -a_4 & -a_3 & 0 \\ 0 & -a_2 & -a_1 \\ 0 & 0 & -a_0 - 1 \end{bmatrix} \longrightarrow \begin{bmatrix} \times & 0 & 0 \\ \times & \times & 0 \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

The pattern of the low-rank term

The examples above show a property that holds for every Fiedler matrix.

Theorem (Fiedler Matrices as permutation plus special low-rank)

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with the permutation σ of $(0, 1, \ldots, n-1)$. Then

$$M_\sigma = U_\sigma + V_\sigma, \qquad \text{where}$$

- *U_σ is a permutation matrix.*
- *V_σ is such that if all zero rows and columns are removed, then a matrix $\widetilde{V}_\sigma$ with* **staircase nonzero pattern** *is obtained, i.e., pattern of the type*

$$\begin{bmatrix} \times & \times & \times & & & & & \\ & & \times & \times & & & & \\ & & & \times & \times & \times & & \\ & & & & & \times & & \\ & & & & & \times & \times & \times \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \times & \times & \times & & & \\ & & \times & \times & & \\ & & & \times & \times & \times \\ & & & & & \times \\ & & & & & \times \end{bmatrix},$$

that can be constructed by placing $\times$ in the $(1,1)$ entry and then in the right-neighbor or in the down-neighbor entry up to the right-lowest entry.

The pattern of the low-rank term

The examples above show a property that holds for every Fiedler matrix.

Theorem (Fiedler Matrices as permutation plus special low-rank)

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with the permutation σ of $(0, 1, \ldots, n-1)$. Then

$$M_\sigma = U_\sigma + V_\sigma, \qquad \textit{where}$$

- *U_σ is a permutation matrix.*
- *V_σ is such that if all zero rows and columns are removed, then a matrix $\widetilde{V}_\sigma$ with* **staircase nonzero pattern** *is obtained, i.e., pattern of the type*

$$\begin{bmatrix} \times & \times & \times & & & & & \\ & & \times & \times & & & & \\ & & & \times & \times & \times & & \\ & & & & & \times & & \\ & & & & & \times & \times & \times \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} \times & \times & \times & & & \\ & & \times & \times & & \\ & & & \times & \times & \times \\ & & & & & \times \\ & & & & & \times \end{bmatrix},$$

that can be constructed by placing $\times$ in the $(1,1)$ entry and then in the right-neighbor or in the down-neighbor entry up to the right-lowest entry.

The pattern of the low-rank term

The examples above show a property that holds for every Fiedler matrix.

Theorem (Fiedler Matrices as permutation plus special low-rank)

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with the permutation σ of $(0, 1, \ldots, n-1)$. Then

$$M_\sigma = U_\sigma + V_\sigma, \qquad \textit{where}$$

- *U_σ is a permutation matrix.*
- *V_σ is such that if all zero rows and columns are removed, then a matrix $\widetilde{V}_\sigma$ with* **staircase nonzero pattern** *is obtained, i.e., pattern of the type*

$$\begin{bmatrix} \times & \times & \times & & & & & \\ & & \times & \times & & & & \\ & & & \times & \times & \times & & \\ & & & & & \times & & \\ & & & & & \times & \times & \times \end{bmatrix} \quad \textit{or} \quad \begin{bmatrix} \times & \times & \times & & & \\ & & \times & \times & & \\ & & & \times & \times & \times \\ & & & & & \times \\ & & & & & \times \end{bmatrix},$$

that can be constructed by placing $\times$ in the $(1,1)$ entry and then in the right-neighbor or in the down-neighbor entry up to the right-lowest entry.

$$\boxed{M_\sigma = U_\sigma + V_\sigma \,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple **nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ** makes possible:

- To determine the (generic) rank of V_σ, denoted by r_σ.
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n\times r_\sigma}, \; R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma) \times (2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

Consequences of the pattern of the low-rank term

$$\boxed{M_\sigma = U_\sigma + V_\sigma \,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple **nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ** makes possible:

- To determine the (generic) rank of V_σ, denoted by r_σ.
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n\times r_\sigma}, \; R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma) \times (2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

$$\boxed{M_\sigma = U_\sigma + V_\sigma \,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple **nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ** makes possible:

- To determine the (generic) rank of V_σ, denoted by r_σ.
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n \times r_\sigma}, \; R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma) \times (2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

$$\boxed{M_\sigma = U_\sigma + V_\sigma\,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple **nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ** makes possible:

- To determine the (generic) rank of V_σ, denoted by r_σ.
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n\times r_\sigma},\ R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma) \times (2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

Consequences of the pattern of the low-rank term

$$\boxed{M_\sigma = U_\sigma + V_\sigma\,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple **nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ** makes possible:

- To determine the (generic) rank of V_σ, denoted by r_σ.
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n\times r_\sigma},\ R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma)\times(2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

Consequences of the pattern of the low-rank term

$$\boxed{M_\sigma = U_\sigma + V_\sigma\,,}$$

- where U_σ is a permutation and V_σ is such that if all zero rows-columns are removed, then a matrix $\widetilde{V}_\sigma$ with **staircase nonzero pattern** is obtained.

The simple nonzero-pattern structure of V_σ together with the sequence of consecutions-inversions of M_σ makes possible:

- **To determine the (generic) rank of V_σ, denoted by r_σ.**
- To find (via easy symbolic algorithms) a rank revealing factorization of V_σ

$$V_\sigma = L_\sigma R_\sigma, \quad \text{with} \quad L_\sigma \in \mathbb{R}^{n\times r_\sigma},\ R_\sigma \in \mathbb{R}^{r_\sigma \times n}$$

- To construct a $(2r_\sigma) \times (2r_\sigma)$ matrix such that if $2r_\sigma < n$, then the square roots of its eigenvalues are the singular values of M_σ different from 1.

More on consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \dots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the **reduced consecution-inversion structured sequence** of M_σ, denoted by RCISS(σ) as follows:

- If M_σ has a consecution at 0, then

$$\text{RCISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \dots),$$

  where M_σ has $\mathfrak{c}_0$ consecutive consecutions at $0, 1, \dots, \mathfrak{c}_0 - 1$, $\mathfrak{i}_0$ consecutive inversions at $\mathfrak{c}_0, \mathfrak{c}_0 + 1, \dots, \mathfrak{c}_0 + \mathfrak{i}_0 - 1$, and so on...

- If M_σ has an inversion at 0, then

$$\text{RCISS}(\sigma) = (\mathfrak{i}_0, \mathfrak{c}_0, \mathfrak{i}_1, \mathfrak{c}_1, \dots),$$

  where M_σ has $\mathfrak{i}_0$ consecutive inversions at $0, 1, \dots, \mathfrak{i}_0 - 1$, $\mathfrak{c}_0$ consecutive consecutions at $\mathfrak{i}_0, \mathfrak{i}_0 + 1, \dots, \mathfrak{i}_0 + \mathfrak{c}_0 - 1$, and so on...

More on consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the **reduced consecution-inversion structured sequence** of M_σ, denoted by RCISS(σ) as follows:

- If M_σ has a consecution at 0, then

$$\text{RCISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots),$$

  where M_σ has $\mathfrak{c}_0$ consecutive consecutions at $0, 1, \ldots, \mathfrak{c}_0 - 1$, $\mathfrak{i}_0$ consecutive inversions at $\mathfrak{c}_0, \mathfrak{c}_0 + 1, \ldots, \mathfrak{c}_0 + \mathfrak{i}_0 - 1$, and so on...

- If M_σ has an inversion at 0, then

$$\text{RCISS}(\sigma) = (\mathfrak{i}_0, \mathfrak{c}_0, \mathfrak{i}_1, \mathfrak{c}_1, \ldots),$$

  where M_σ has $\mathfrak{i}_0$ consecutive inversions at $0, 1, \ldots, \mathfrak{i}_0 - 1$, $\mathfrak{c}_0$ consecutive consecutions at $\mathfrak{i}_0, \mathfrak{i}_0 + 1, \ldots, \mathfrak{i}_0 + \mathfrak{c}_0 - 1$, and so on...

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the **reduced consecution-inversion structured sequence** of M_σ, denoted by $\mathrm{RCISS}(\sigma)$ as follows:

- If M_σ has a consecution at 0, then

  $$\mathrm{RCISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots),$$

  where M_σ has $\mathfrak{c}_0$ consecutive consecutions at $0, 1, \ldots, \mathfrak{c}_0 - 1$, $\mathfrak{i}_0$ consecutive inversions at $\mathfrak{c}_0, \mathfrak{c}_0 + 1, \ldots, \mathfrak{c}_0 + \mathfrak{i}_0 - 1$, and so on...

- If M_σ has an inversion at 0, then

  $$\mathrm{RCISS}(\sigma) = (\mathfrak{i}_0, \mathfrak{c}_0, \mathfrak{i}_1, \mathfrak{c}_1, \ldots),$$

  where M_σ has $\mathfrak{i}_0$ consecutive inversions at $0, 1, \ldots, \mathfrak{i}_0 - 1$, $\mathfrak{c}_0$ consecutive consecutions at $\mathfrak{i}_0, \mathfrak{i}_0 + 1, \ldots, \mathfrak{i}_0 + \mathfrak{c}_0 - 1$, and so on...

More on consecutions and inversions (I)

Let M_σ be the Fiedler matrix associated with permutation σ of $(0, 1, \ldots, n-1)$:

$$M_\sigma = M_{i_0} M_{i_1} \cdots M_{i_{n-1}}.$$

We define the **reduced consecution-inversion structured sequence** of M_σ, denoted by RCISS(σ) as follows:

- If M_σ has a consecution at 0, then

$$\mathrm{RCISS}(\sigma) = (\mathfrak{c}_0, \mathfrak{i}_0, \mathfrak{c}_1, \mathfrak{i}_1, \ldots),$$

  where M_σ has $\mathfrak{c}_0$ consecutive consecutions at $0, 1, \ldots, \mathfrak{c}_0 - 1$, $\mathfrak{i}_0$ consecutive inversions at $\mathfrak{c}_0, \mathfrak{c}_0 + 1, \ldots, \mathfrak{c}_0 + \mathfrak{i}_0 - 1$, and so on...

- If M_σ has an inversion at 0, then

$$\mathrm{RCISS}(\sigma) = (\mathfrak{i}_0, \mathfrak{c}_0, \mathfrak{i}_1, \mathfrak{c}_1, \ldots),$$

  where M_σ has $\mathfrak{i}_0$ consecutive inversions at $0, 1, \ldots, \mathfrak{i}_0 - 1$, $\mathfrak{c}_0$ consecutive consecutions at $\mathfrak{i}_0, \mathfrak{i}_0 + 1, \ldots, \mathfrak{i}_0 + \mathfrak{c}_0 - 1$, and so on...

More on consecutions and inversions (II)

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13}M_{12}M_{10}M_6M_4M_1M_0M_2M_3M_5M_7M_8M_9M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2, 1, 1, 1, 3, 1, 1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3, 2).$$

This is the **rank-determining list** of M_σ.

More on consecutions and inversions (II)

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13} M_{12} M_{10} M_6 M_4 M_1 M_0 M_2 M_3 M_5 M_7 M_8 M_9 M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2, 1, 1, 1, 3, 1, 1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3, 2).$$

This is the **rank-determining list** of M_σ.

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13}M_{12}M_{10}M_6M_4M_1M_0M_2M_3M_5M_7M_8M_9M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1,2,1,1,1,3,1,1,2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2,1,1,1,3,1,1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3,2).$$

This is the **rank-determining list** of M_σ.

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13} M_{12} M_{10} M_6 M_4 M_1 M_0 M_2 M_3 M_5 M_7 M_8 M_9 M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2, 1, 1, 1, 3, 1, 1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3, 2).$$

This is the **rank-determining list** of M_σ.

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13}M_{12}M_{10}M_6M_4M_1M_0M_2M_3M_5M_7M_8M_9M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2, 1, 1, 1, 3, 1, 1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3, 2).$$

This is the **rank-determining list** of M_σ.

More on consecutions and inversions (II)

Example of RCISS(σ): Consider the next Fiedler matrix of $p(z) = z^{14} + \sum_{i=0}^{13} a_k z^k$

$$M_\sigma = M_{13}M_{12}M_{10}M_6M_4M_1M_0M_2M_3M_5M_7M_8M_9M_{11}$$

Then

$$\text{RCISS}(\sigma) = (1,2,1,1,1,3,1,1,2)$$

The **rank-determining list** of M_σ is obtained from RCISS(σ) in two steps:

1. Remove from RCISS(σ) the first and last entry

$$(2,1,1,1,3,1,1)$$

2. Construct a list with the lengths of the strings of consecutive ones

$$\mathcal{L}(\sigma) = (3,2).$$

This is the **rank-determining list** of M_σ.

The rank of the low-rank term

Theorem (Fiedler Matrices as permutation plus special low-rank)

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with the permutation σ of $(0, 1, \ldots, n-1)$,

- *let t* **be the number of entries of RCISS(σ)**, *and*
- *let $\mathcal{L}(\sigma) = (\ell_1, \ldots, \ell_q)$ be the* **rank-determining list** *of M_σ.*

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and (generically)

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil$$

Example

RCISS$(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2) \Longrightarrow t = 9$ and $\mathcal{L}(\sigma) = (3, 2)$. Then

$$\operatorname{rank} V_\sigma = 9 - \lceil 3/2 \rceil - \lceil 2/2 \rceil = 6 .$$

The rank of the low-rank term

Theorem (Fiedler Matrices as permutation plus special low-rank)

Let M_σ be the Fiedler matrix of $p(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ associated with the permutation σ of $(0, 1, \ldots, n-1)$,

- *let t* **be the number of entries of RCISS(σ)**, *and*
- *let $\mathcal{L}(\sigma) = (\ell_1, \ldots, \ell_q)$ be the* **rank-determining list** *of M_σ.*

Then $M_\sigma = U_\sigma + V_\sigma$, where U_σ is a permutation matrix and (generically)

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil$$

Example

RCISS$(\sigma) = (1, 2, 1, 1, 1, 3, 1, 1, 2) \Longrightarrow t = 9$ and $\mathcal{L}(\sigma) = (3, 2)$. Then

$$\operatorname{rank} V_\sigma = 9 - \lceil 3/2 \rceil - \lceil 2/2 \rceil = 6 \, .$$

Some intuitions on the rank of the low-rank term

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

- It is a complicated function of the consecutions/inversions.
- Smaller numbers of transitions from series of consecutions to series of inversions give smaller t, and this encourages smaller ranks,
- but high numbers of consecutive transitions give larger ℓ_j, and this also encourages smaller ranks.
- Pentadiagonal matrices corresponds (almost always) to the maximal value of $\operatorname{rank} V_\sigma$ and these M_σ have not singular values equal to one.

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

- It is a complicated function of the consecutions/inversions.
- Smaller numbers of transitions from series of consecutions to series of inversions give smaller t, and this encourages smaller ranks,
- but high numbers of consecutive transitions give larger ℓ_j, and this also encourages smaller ranks.
- Pentadiagonal matrices corresponds (almost always) to the maximal value of $\operatorname{rank} V_\sigma$ and these M_σ have not singular values equal to one.

Some intuitions on the rank of the low-rank term

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

- It is a complicated function of the consecutions/inversions.
- Smaller numbers of transitions from series of consecutions to series of inversions give smaller t, and this encourages smaller ranks,
- but high numbers of consecutive transitions give larger ℓ_j, and this also encourages smaller ranks.
- Pentadiagonal matrices corresponds (almost always) to the maximal value of $\operatorname{rank} V_\sigma$ and these M_σ have not singular values equal to one.

Some intuitions on the rank of the low-rank term

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

- It is a complicated function of the consecutions/inversions.
- Smaller numbers of transitions from series of consecutions to series of inversions give smaller t, and this encourages smaller ranks,
- but high numbers of consecutive transitions give larger ℓ_j, and this also encourages smaller ranks.
- Pentadiagonal matrices corresponds (almost always) to the maximal value of $\operatorname{rank} V_\sigma$ and these M_σ have not singular values equal to one.

Some intuitions on the rank of the low-rank term

Then $\boxed{M_\sigma = U_\sigma + V_\sigma,}$ where U_σ is a permutation matrix and

$$\operatorname{rank} V_\sigma = t - \sum_{j=1}^{q} \left\lceil \frac{\ell_j}{2} \right\rceil \leq \left\lfloor \frac{n+1}{2} \right\rfloor$$

- It is a complicated function of the consecutions/inversions.
- Smaller numbers of transitions from series of consecutions to series of inversions give smaller t, and this encourages smaller ranks,
- but high numbers of consecutive transitions give larger ℓ_j, and this also encourages smaller ranks.
- Pentadiagonal matrices corresponds (almost always) to the maximal value of $\operatorname{rank} V_\sigma$ and these M_σ have not singular values equal to one.

Outline

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.

Conclusion and future work

- Fiedler companion matrices may behave very differently than classical Frobenius companion matrices, although the basic algebraic properties are the same.
- From the point of view of condition numbers for inversion Frobenius companion forms should not be used if $|p(0)| < 1$,
- since they are the Fiedler matrices closest to be singular.
- If $|p(0)| < 1$ use, instead, any Fiedler matrix having a number of initial consecutions or inversions equal to 1.
- One of these is being considered by Tisseur and coworkers to replace Frobenius companion matrices in MATLAB's command `polyeig`.
- Ongoing work in my group: study of eigenvalue condition numbers and backward errors of eigenvalues,
- but a general treatment valid for all Fiedler matrices is very difficult and we need to focus on particular relevant cases.
- Many problems remain open on Fiedler companion matrices and combinatorial properties of consecutions/inversions will play a relevant role in their solution.