Diagonally dominant matrices: Surprising recent results on a classical type of matrices

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based on joint works with Megan Dailey (Indiana, USA), Plamen Koev (San José, Cal, USA), and Qiang Ye (Kentucky, USA)

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Outline

1. Introduction
2. My motivation to study diagonally dominant matrices
3. Looking at DD matrices with other eyes!!!
4. Perturbation theory for the inverse
5. Perturbation theory for linear systems
6. Perturbation theory for LDU factorization
7. Perturbation theory for eigenvalues of symmetric matrices
8. Perturbation theory for singular values
9. Perturbation of simple eigenvalues of nonsymmetric matrices
10. Conclusions and open problems

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Definition (Lévy (1881)...)

The matrix $A \in \mathbb{R}^{n\times n}$ is ROW DIAGONALLY DOMINANT (rdd) if

$$\sum_{j\neq i} |a_{ij}| \leq |a_{ii}|, \quad i = 1, 2, \ldots, n.$$

$A \in \mathbb{R}^{n\times n}$ is COLUMN DIAGONALLY DOMINANT (cdd) if A^T is row diagonally dominant.

Example

$$A = \begin{bmatrix} -4 & 2 & 2 \\ 1 & 6 & 4 \\ 1 & -2 & 5 \end{bmatrix} \quad (rdd), \qquad B = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ -2 & 1 & 5 \end{bmatrix} \quad (cdd).$$

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Famous Example (I): Second difference matrix

$$K_n = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

- This matrix arises by discretizing one-dimensional boundary value problems (second derivatives).
- Numerical methods for solving elliptic PDEs are a source of many linear systems of equations whose coefficients form diagonally dominant matrices.

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Famous Example (II): Collocation matrices in cubic splines

To compute the cubic spline (with parabolic boundary conditions) of a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \quad \text{with} \quad x_1 < x_2 < \cdots < x_n\,,$$

one needs to solve a system of equations whose coefficient matrix is

$$\begin{bmatrix} 1 & 1 & & & & \\ h_2 & 2(h_1 + h_2) & h_1 & & & \\ & h_3 & 2(h_2 + h_3) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-2} + h_{n-1}) & h_{n-2} \\ & & & & 1 & 1 \end{bmatrix},$$

where $h_k = x_{k+1} - x_k > 0$.

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- Markov chains.
- Graph Laplacians.
- Applications include:
  - Social sciences,
  - Biology,
  - Economy,
  - Physics,
  - Engineering ...
- Sparse Symmetric dd linear systems have received considerable attention in the last years from the point of view of randomized algorithms for computing their solution in "Nearly-Linear" time, via graph preconditioners,...(D. Spielman, S. H. Teng)

DD matrices are a very important class of matrices from many points of view: theory, algorithms, and applications

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Selected results for diagonally dominant matrices (I)

Theorem (Lévy-Desplanques Theorem, 1881-1886)

Let the matrix $A \in \mathbb{R}^{n\times n}$ be **strictly** *row diagonally dominant, that is,*

$$\sum_{j\neq i} |a_{ij}| < |a_{ii}|, \quad i = 1, 2, \ldots, n.$$

Then A is nonsingular.

Theorem

Let the matrix $A \in \mathbb{R}^{n\times n}$ be **strictly** *row diagonally dominant. Then the number of eigenvalues of A with positive (resp. negative) real part is equal to the number of positive (resp. negative) diagonal entries of A*

Example

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix} \quad (rdd), \quad \text{eigenvalues} = \{-4.2702, 5.6351 \pm 1.8363\, i\}$$

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Let $A \in \mathbb{R}^{n\times n}$ be row or column diagonally dominant. Then **all the Schur complements of A have the same kind of diagonal dominance as A**.

In plain words, **all matrices arising if we apply Gaussian elimination to A (without pivoting) have the same kind of diagonal dominance as A.**

Example

$$A = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & -4 & 2 & -10 \end{bmatrix} \quad (rdd) \quad \sim \quad \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & -2.5 & 2.75 & -10.75 \end{bmatrix}$$

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Parenthesis: Errors in Gaussian elimination (GE) (I)

Theorem (Wilkinson, 1961)

Let $B \in \mathbb{R}^{n\times n}$ be **ANY nonsingular matrix***, let $b \in \mathbb{R}^n$, and let*

$$\widehat{x}$$

be the approximate solution of

$$Bx = b$$

computed by GE in a computer in **double precision***. Then*

$$(B + \Delta B)\widehat{x} = b, \quad \frac{\|\Delta B\|_\infty}{\|B\|_\infty} \leq 6 \cdot n^3 \cdot 10^{-16} \cdot \boldsymbol{\rho}_n,$$

where

$$\boldsymbol{\rho}_n = \frac{\max_{ijk} |a_{ij}^{(k)}|}{\max_{ij} |a_{ij}|},$$

is the **growth factor of Gaussian elimination***. Here $A^{(1)} := A, A^{(2)}, \ldots, A^{(n)}$ are the matrices appearing in the Gaussian elimination process.*

Parenthesis: Errors in Gaussian elimination (GE) (II)

Example (Growth factor)

$$A = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & -4 & 2 & -10 \end{bmatrix} \sim \quad A^{(2)} = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & -2.5 & 2.75 & -10.75 \end{bmatrix} \sim$$

$$A^{(3)} = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & 3.62 & -11.62 \end{bmatrix} \sim A^{(4)} = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & 0 & -11.76 \end{bmatrix}$$

$$\rho = \frac{11.76}{10} = 1.1760$$

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Selected results for diagonally dominant matrices (III)

$$(B + \Delta B)\widehat{x} = b, \quad \frac{\|\Delta B\|_\infty}{\|B\|_\infty} \leq 6 \cdot n^3 \cdot 10^{-16} \cdot \boldsymbol{\rho}_n,$$

Wilkinson (1961)- Wendroff (1966) proved

Class of matrix	Method	Bound on $\boldsymbol{\rho}_n$
General	GE without pivoting	unbounded
General	GE with partial pivoting	2^{n-1} (huge, but usually small)

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diag. dominant	GE **without** pivoting	2

Selected results for diagonally dominant matrices (IV)

Theorem

If $A \in \mathbb{R}^{n \times n}$ *is row or column diagonally dominant, then the Gaussian elimination algorithm* **without pivoting** *for solving* $Ax = b$ *is* **backward stable**. *More precisely, the computed solution* $\widehat{x}$ *satisfies*

$$(A + \Delta A)\widehat{x} = b, \qquad \frac{\|\Delta A\|_\infty}{\|A\|_\infty} \leq 12 \cdot n^3 \cdot 10^{-16}$$

Remark

Very important for preserving simultaneously structures and backward stab.

Example

$$\begin{bmatrix} 2 & -1 & & \\ -4 & 5 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \sim \text{(only one row operation)} \sim \begin{bmatrix} 2 & -1 & & \\ & 3 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix}$$

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only implies

$$\frac{\|x - \widehat{x}\|_\infty}{\|x\|_\infty} \leq \kappa(A) \cdot 12 \cdot n^3 \cdot 10^{-16}, \quad \text{where } \kappa(A) = \|A\|_\infty \|A^{-1}\|_\infty$$

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$$A = \begin{bmatrix} 10^{16} & -10^8/5 & 1/10 \\ 10^{16}/3 & 10^8 & -1/10 \\ 10^{16}/3 & -10^8/5 & 1 \end{bmatrix} \quad \text{and} \quad b = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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Last basic concept on GE of dd matrices: LU (LDU) factorization (I)

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ -0.25 & -0.23 & 1 & 0 \\ -0.75 & -0.38 & 0.63 & 1 \end{bmatrix} \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & 0 & -11.76 \end{bmatrix} \equiv LU$$

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Remark

- A rdd $\Longrightarrow U$ rdd.
- A cdd $\Longrightarrow L$ cdd.

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$$A = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ 3 & -4 & 2 & -10 \end{bmatrix} \sim A^{(4)} = \begin{bmatrix} -4 & 2 & 1 & -1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & 0 & -11.76 \end{bmatrix}$$

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Remark

- A rdd $\Longrightarrow U$ rdd.
- A cdd $\Longrightarrow L$ cdd.

Last basic concept on GE dd matrices: LU (LDU) factorization (II)

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- Q. Ye, in *Computing singular values of diagonally dominant matrices to high relative accuracy*, *Math. Comp. (2008)*,
- developed a very ingenuous algorithm for **computing accurately? in $2n^3$ flops** the LDU factorization (Gaussian Elimination) with complete pivoting or column diagonal dominance pivoting of row diagonally dominant matrices
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Motivation (II)

- best error bounds that Q. Ye proved after a direct error analysis that requires considerable efforts are

$$\frac{\|L-\widehat{L}\|_\infty}{\|L\|_\infty} \leq 6\,n\,8^{(n-1)}\epsilon, \quad \frac{\|U-\widehat{U}\|_\infty}{\|U\|_\infty} \leq 6\cdot 8^{(n-1)}\epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 5\cdot 8^{(n-1)}\epsilon,$$

  where $n \times n$ is the size of the matrix and ϵ the unit roundoff.

- $\epsilon = 2^{-53} \approx 10^{-16}$ in double precision, **so the bounds are > 1 for** $n > 20$...

- However, there are no condition numbers in the bounds and numerical experiments indicated accuracy.

- **Can we prove better bounds?**

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- **Fundamental consequences:** Q. Ye's algorithm + other existing implicit algorithms for factorized matrices allow us **to compute for Diagonally Dominant matrices with guaranteed high relative accuracy**

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Goal of the talk

- No algorithms, no error analysis!!! (difficult)
- **To present a family of new perturbation bounds under certain structured perturbations** for several magnitudes corresponding to **Diagonally Dominant matrices**: inverses, solutions of linear systems, LDU factorization, singular values, eigenvalues.
- Common key point in (almost all) these perturbation bounds: they are always tiny for tiny structured perturbations in this class, even for extremely ill conditioned matrices (independent of traditional condition numbers).

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Example

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix} \Longrightarrow B = \begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 4 & -2 & -1 \\ 1 & 6 & 2 \\ 1 & -2 & 5 \end{bmatrix}$$

Parameterizing row diagonally dominant matrices (Q. Ye) (I)

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- Define the diagonally dominant parts of A and store them in a column vector $v = (v_1, v_2, \ldots, v_n)^T$ where

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $$A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$$
- The pair (A_D, v) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through pairs of this type. A matrix A parameterized is this way will be denoted as

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To compute the cubic spline (with parabolic b. c.) of a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \quad \text{with} \quad x_1 < x_2 < \cdots < x_n\,,$$

one needs to solve a system of equations whose coefficient matrix is

$$\begin{bmatrix} 1 & 1 & & & & \\ h_2 & 2(h_1+h_2) & h_1 & & & \\ & h_3 & 2(h_2+h_3) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-2}+h_{n-1}) & h_{n-2} \\ & & & & 1 & 1 \end{bmatrix},$$

where $h_k = x_{k+1} - x_k > 0$.

- $$v_1 = 0, v_2 = h_1 + h_2, \ldots, v_{n-1} = h_{n-2} + h_{n-1}, v_n = 0$$
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Key features of Q. Ye's algorithm for LDU of dd matrices

- INPUT: $\mathcal{D}(A_D, v)$ with $v \geq 0$ *(not the entries of the matrix A)!!!*.
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- $A_D^{(k+1)}$ is computed from $\mathcal{D}(A_D^{(k)}, v^{(k)})$ by applying the usual Gaussian elimination process. So cancellation errors may appear but they are bounded in an absolute sense.

Fundamental point: updating diagonally dominant parts

$$v_i^{(k)} := a_{ii}^{(k)} - \sum_{j \neq i} |a_{ij}^{(k)}|$$

Lemma (Q. Ye, 2008)

For $k+1 \leq i \leq n$*,*

$$v_i^{(k+1)} = v_i^{(k)} + \sum_{j=k+1,\, j\neq i}^{n} (1 - s_{ij}^{(k)})\, |a_{ij}^{(k)}| + \frac{|a_{ik}^{(k)}|}{|a_{kk}^{(k)}|} \left(v_k^{(k)} + \sum_{j=k+1}^{n} (1 - t_{ij}^{(k)}) |a_{kj}^{(k)}| \right),$$

where $s_{ij}^{(k)} = \mathrm{sign}\left(a_{ij}^{(k+1)} a_{ij}^{(k)}\right)$ *and* $t_{ij}^{(k)} = \begin{cases} -\mathrm{sign}\left(a_{ij}^{(k+1)}\, a_{ik}^{(k)}\, a_{kj}^{(k)}\right), & i \neq j \\ \mathrm{sign}\left(a_{ik}^{(k)}\, a_{ki}^{(k)}\right), & i = j \end{cases}$

- Sum of positive terms and $v_i^{(k+1)} \geq v_i^{(k)}$

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Fundamental contribution: updating diagonal dominances

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Example of updating diagonally dominant parts

$$A^{(1)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 1 & 6 & 2 & -2 \\ 1 & -2 & 5 & 1 \\ -3 & 4 & -2 & 10 \end{bmatrix}, \quad v^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \sim$$

$$A^{(2)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & -1.5 & 5.25 & 0.75 \\ 0 & 2.5 & -2.75 & 10.75 \end{bmatrix}, \quad v^{(2)} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 5.5 \end{bmatrix} \sim$$

$$A^{(3)} = \begin{bmatrix} 4 & -2 & -1 & 1 \\ 0 & 6.5 & 2.25 & -2.25 \\ 0 & 0 & 5.77 & 0.23 \\ 0 & 0 & -3.62 & 11.62 \end{bmatrix}, \quad v^{(3)} = \begin{bmatrix} 0 \\ 2 \\ 5.54 \\ 8 \end{bmatrix} \sim$$

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What happens if the vector v in $\mathcal{D}(A_D, v)$ is not known?

- If only the entries of the starting matrix A are known, then one can compute with the usual *recursive summation* method

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}| \qquad \text{for all } i,$$

  but it may produce large relative cancellation errors if $a_{ii} \approx \sum_{j \neq i} |a_{ij}|$ and this would spoil the accuracy of the whole computation.

- In case of severe cancellation, one can compute the v_i with *doubly compensated summation* (Priest, 1992) that computes the sum of n numbers with relative error $2 \cdot 10^{-16}$ with cost of $10(n-1)$ flops.

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Example: Two small relative componentwise perturbations of a row diag. dominant matrix A. **Both preserve the diag. dominant structure.**

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

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Singular values of A, B and C

	A	B	C
σ_1	4.641	4.640	4.642
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“Every lecture should make only one main point.”

From Gian-Carlo Rota, *“Ten lessons I wish I had been taught”*, Notices of the AMS, 44 (1997) 22-25.

Outline

1. Introduction
2. My motivation to study diagonally dominant matrices
3. Looking at DD matrices with other eyes!!!
4. **Perturbation theory for the inverse**
5. Perturbation theory for linear systems
6. Perturbation theory for LDU factorization
7. Perturbation theory for eigenvalues of symmetric matrices
8. Perturbation theory for singular values
9. Perturbation of simple eigenvalues of nonsymmetric matrices
10. Conclusions and open problems

Bounds for the inverse under structured perturbations

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be row diagonally dominant matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Then

- A *is nonsingular if and only if* $\widetilde{A}$ *is nonsingular.*
- $\boxed{\dfrac{\|\widetilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \dfrac{n(3n-2)\delta}{1-2n\delta}}$, *for* 1-, 2-, ∞-*norms.*

To be compared with

$$\frac{\|\widetilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \lesssim (\|A\|\|A^{-1}\|)\, \frac{\|\widetilde{A} - A\|}{\|A\|}$$

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$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Then

- *A is nonsingular if and only if $\widetilde{A}$ is nonsingular.*
- $\boxed{\dfrac{\|\widetilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \dfrac{n(3n-2)\delta}{1-2n\delta}}$, *for 1-, 2-, ∞-norms.*

To be compared with

$$\frac{\|\widetilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \lesssim \left(\|A\|\|A^{-1}\|\right) \frac{\|\widetilde{A} - A\|}{\|A\|}$$

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Consider the systems

$$A\,x = b \quad \text{and} \quad \widetilde{A}\,\widetilde{x} = \widetilde{b}$$

with $\|b - \widetilde{b}\| \leq \delta \|b\|$. *If* $2n\delta < 1$, *then in the* 1-, 2-, *and* ∞-*norms,*

$$\frac{\|\widetilde{x} - x\|}{\|x\|} \leq \left(\frac{(3n^2 - 2n + 1)\delta + (3n^2 - 4n)\delta^2}{1 - 2n\delta} \right) \frac{\|A^{-1}\| \, \|b\|}{\|x\|}$$

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$$\frac{\|\widetilde{x} - x\|}{\|x\|} \lesssim (\|A\| \|A^{-1}\|) \left(\frac{\|\widetilde{A} - A\|}{\|A\|} + \frac{\|\widetilde{b} - b\|}{\|b\|} \right)$$

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For most vectors b, $\|A^{-1}\|\,\|b\|/\|x\|$ is a moderate number and for A ill-conditioned,

$$\|A^{-1}\|\,\|b\|/\|x\| \ll \kappa(A)$$

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Pivoting in row dd matrices (I)

- The dd structure is preserved by performing the same permutations in rows and columns, i.e., by any diagonal pivoting strategy.
- Other pivoting strategies destroy the dd structure.
- The element with largest absolute value in a dd matrix is on the diagonal.
- Complete pivoting in dd matrices is a diagonal pivoting strategy.
- Every row dd matrix has at least one column which is diagonally dominant.

$$A = \begin{bmatrix} -4 & 1 & 3 \\ 0 & 6 & 5 \\ 1 & -2 & 7 \end{bmatrix} \quad (rdd).$$

- The column diagonal dominance pivoting strategy chooses at each step of GE the entry with largest absolute value among those corresponding to the columns which are diagonally dominant.

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- Both complete and column diagonal dominance pivoting give a **row diagonally dominant factor** U in $A = LDU$, and so
- $\kappa_\infty(U) := \|U\|_\infty \|U^{-1}\|_\infty \le 2n$.
- Column diagonal dominance pivoting also produce a **column diagonally dominant factor** L, and so
- $\kappa_1(L) \le 2n$.
- To have both condition numbers bounded is relevant for computations with guaranteed high relative accuracy via rank-revealing decompositions.

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Bounds for the LDU factors under structured perturbations

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be row diagonally dominant matrices, and $A = LDU$ and $\widetilde{A} = \widetilde{L}\,\widetilde{D}\,\widetilde{U}$ be their factorizations. If

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1,$$

then

- For $i = 1 : n$ and any pivoting strategy,

$$|\widetilde{d}_{ii} - d_{ii}| \leq \frac{2n\delta}{1 - 2n\delta}\, |d_{ii}|$$

- For any pivoting strategy,

$$\frac{\|\widetilde{U} - U\|_\infty}{\|U\|_\infty} \leq 3n^2\delta$$

Bounds for the L factor

Theorem (continuation)

- For complete pivoting,

$$\frac{\|\widetilde{L} - L\|_\infty}{\|L\|_\infty} \leq \frac{n\delta}{1 - n\delta}\left(3 + \frac{2n\delta}{1 - n\delta}\right)$$

- For column diagonal dominance pivoting,

$$\frac{\|\widetilde{L} - L\|_1}{\|L\|_1} \leq \frac{n(8n - 2)\,\delta}{1 - (12\,n + 1)\delta}\,.$$

Complete or column diag pivoting are essential for good behavior of L: Example

Matrix ordered according to a pivoting strategy designed to make the **factor L column diagonally dominant** and as much as possible.

$$A = \begin{bmatrix} 1000 & 100 & 500 \\ 0 & 0.1 & 0.05 \\ 100 & 10 & 120 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 400 \\ 0.05 \\ 10 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0.1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1000 & & \\ & 0.1 & \\ & & 70 \end{bmatrix} \begin{bmatrix} 1 & 0.1 & 0.5 \\ & 1 & 0.5 \\ & & 1 \end{bmatrix}$$

Example: $\delta \approx 10^{-2}$ perturbation in $\mathcal{D}(A_D, v)$.

$$\widetilde{A} = \begin{bmatrix} 1000 & 101 & 500 \\ 0 & 0.1 & 0.05 \\ 100 & 10 & 120 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 399 \\ 0.05 \\ 10 \end{bmatrix}$$

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Theorem (Q. Ye, SIMAX, 2009)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be diagonally dominant symmetric matrices with nonnegative diagonal entries such that

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n \geq 0$ be, respectively, the eigenvalues of $A = \mathcal{D}(A_D, v)$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$. Then

$$\boxed{|\widetilde{\lambda}_i - \lambda_i| \leq \delta\, |\lambda_i|, \qquad i = 1, \ldots, n.}$$

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$$|\widetilde{\lambda}_i - \lambda_i| \leq (\|A\|_2 \|A^{-1}\|_2)\, \frac{\|\widetilde{A} - A\|_2}{\|A\|_2}\, |\lambda_i|$$

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Parameterizing row dd matrices with diagonal entries of any sign

- Let $A \in \mathbb{R}^{n \times n}$.
- Define $v = (v_1, v_2, \ldots, v_n)$ where

$$v_i := |a_{ii}| - \sum_{j \neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$
- Define $\boxed{S = \text{diag}(\text{sign}(a_{11}), \ldots, \text{sign}(a_{nn}))}$ $(\text{sign}(0) := 1)$.
- The triplet (A_D, v, S) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through triplets of this type. Any matrix A parameterized is this way will be denoted as

$$\boxed{A = \mathcal{D}(A_D, v, S)}$$

Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S) \in \mathbb{R}^{n\times n}$ *be diagonally dominant symmetric matrices such that*

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Let $\lambda_1 \geq \cdots \geq \lambda_n$ *and* $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n$ *be, respectively, the eigenvalues of* $A = \mathcal{D}(A_D, v, S)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S)$ *.*

Assume $2n^2(n+2)\delta < 1$ *and define* $\nu := \dfrac{2n^2(n+1)\delta}{1 - n\delta}$.

Then

$$\begin{aligned} |\widetilde{\lambda}_i - \lambda_i| &\leq (2\nu + \nu^2)\, |\lambda_i| \\ &\approx (\, 4n^3\delta + O(\delta^2)\,)\, |\lambda_i|, \qquad i = 1, \ldots, n \end{aligned}$$

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Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ *and* $\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_n \geq 0$ *be, respectively, the singular values of* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *.*
Define

$$\nu := \frac{4n^3 \sqrt{2(n+1)}}{1 - (12n+1)\,\delta}\, \delta\, .$$

If $0 \leq \nu < 1$, *then*

$$\boxed{|\widetilde{\sigma}_i - \sigma_i| \leq \ (2\nu + \nu^2)\, \sigma_i, \qquad i = 1, \ldots, n.}$$

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Let λ be a simple eigenvalue of $A \in \mathbb{R}^{n\times n}$ with right and left eigenvectors x and y. Then $\tilde{A} = A + E$ has an eigenvalue $\tilde{\lambda}$ such that

$$\tilde{\lambda} - \lambda = \frac{y^* E x}{y^* x} + \mathcal{O}\left(\|E\|_2^2\right)$$

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(y,x)\|E\|_2 + \mathcal{O}\left(\|E\|_2^2\right),$$

where $\sec\theta(y,x) = \dfrac{\|y\|_2\|x\|_2}{|y^* x|}$. The relative perturbation bound is:

$$\frac{|\tilde{\lambda} - \lambda|}{|\lambda|} \leq \left(\sec\theta(y,x)\,\frac{\|A\|_2}{|\lambda|}\right)\frac{\|E\|_2}{\|A\|_2} + \mathcal{O}\left(\|E\|_2^2\right).$$

It can be large as a consequence of two facts:

- $\|A\|_2/|\lambda|$ can be large and/or
- $\sec\theta(y,x)$ can be large.

For parameterized perturb of rdd matriced, we will remove $\|A\|_2/|\lambda|$.

Comments for general matrices

Let λ be a simple eigenvalue of $A \in \mathbb{R}^{n\times n}$ with right and left eigenvectors x and y. Then $\tilde{A} = A + E$ has an eigenvalue $\tilde{\lambda}$ such that

$$\tilde{\lambda} - \lambda = \frac{y^* E x}{y^* x} + \mathcal{O}\left(\|E\|_2^2\right)$$

$$|\tilde{\lambda} - \lambda| \leq \sec\theta(y,x)\|E\|_2 + \mathcal{O}\left(\|E\|_2^2\right),$$

where $\sec\theta(y,x) = \dfrac{\|y\|_2\|x\|_2}{|y^* x|}$. The relative perturbation bound is:

$$\frac{|\tilde{\lambda} - \lambda|}{|\lambda|} \leq \left(\sec\theta(y,x)\,\frac{\|A\|_2}{|\lambda|}\right)\frac{\|E\|_2}{\|A\|_2} + \mathcal{O}\left(\|E\|_2^2\right).$$

It can be large as a consequence of two facts:

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Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n \times n}$ *be s.t.* $v \geq 0$ *and let* λ *be an eigenvalue of* A *with a right eigenvector* x. *Let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S) \in \mathbb{R}^{n \times n}$ *be such that*

$$|\tilde{v} - v| \leq \delta v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{for some } 0 \leq \delta < 1,$$

and let $\tilde{\lambda}$ *be an eigenvalue of* $\tilde{A}$ *with a left eigenvector* $\tilde{y}$ *such that* $\tilde{y}^* x \neq 0$. *If* $(13n + 7n^3 \sec\theta(\tilde{y}, x))\,\delta < 1$, *then*

$$|\tilde{\lambda} - \lambda| \leq \frac{8n^{7/2} + 7n^3}{1 - (13n + 7n^3 \sec\theta(\tilde{y}, x))\,\delta} \sec\theta(\tilde{y}, x)\,\delta\,|\lambda|\,,$$

where

$$\sec\theta(\tilde{y}, x) = \frac{\|\tilde{y}\|_2 \|x\|_2}{|\tilde{y}^* x|}$$

Example: Good perturbation properties of eigenvalues of nonsymmetric row diagonally dominant matrices

Two types of small ($\approx 10^{-3}$) relative componentwise perturbations of a rDD matrix A:

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(A) = \begin{bmatrix} 0 \\ 0.002 \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(B) = \begin{bmatrix} 0 \\ 0.001 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 3 & 1.5 & 4.5 \end{bmatrix}, \qquad v(C) = \begin{bmatrix} 0 \\ 0.002002 \\ 0 \end{bmatrix}$$

Eigenvalues and condition numbers:

	A	B	C	$\sec\theta(y,x)$
λ_1	$8.5686 \cdot 10^{-4}$	$4.2850 \cdot 10^{-4}$	$8.5803 \cdot 10^{-4}$	1.035
λ_2	3.5011	3.5006	3.5023	1.080
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Outline

1. Introduction
2. My motivation to study diagonally dominant matrices
3. Looking at DD matrices with other eyes!!!
4. Perturbation theory for the inverse
5. Perturbation theory for linear systems
6. Perturbation theory for LDU factorization
7. Perturbation theory for eigenvalues of symmetric matrices
8. Perturbation theory for singular values
9. Perturbation of simple eigenvalues of nonsymmetric matrices
10. **Conclusions and open problems**

General advices

- To work with matrices only through their entries may be not convenient, even in the case we force the preservation of structures.
- To parameterize important classes of matrices in order to preserve explicitly their structure may be important both in theory and in applications.
- A good set of parameters should have better perturbation properties than the entries.
- A good set of parameters should allow us to work numerically with the matrices, that is, to construct algorithms based on these parameters for the fundamental tasks of Numerical Linear Algebra.
- This is easy to say, but it might be hard to do.

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Conclusions and open problems

- We have presented structured perturbation results of rDD matrices for many basic problems in Numerical Linear Algebra, **except least square problems and eigenvectors**.

- Except in the case of eigenvalues of nonsymmetric matrices, **the perturbation bounds that we have obtained are rigorous and we have proved that are always tiny for tiny perturbations**.

- **Numerical methods to perform accurate and efficient dense Numerical Linear Algebra with parameterized rDD matrices are available**, **except in the case of eigenvalues of nonsymmetric matrices (open problem!!)**.