Relative perturbation theory for diagonally dominant matrices

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based on joint works with Megan Dailey (Indiana, USA), Plamen Koev (San José, Cal, USA), and Qiang Ye (Kentucky, USA)

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Outline

1. Introduction
2. Looking at diagonally dominant matrices with other eyes
3. Perturbation theory for the inverse
4. Perturbation theory for linear systems
5. Perturbation theory for LDU factorization
6. Perturbation theory for eigenvalues of symmetric matrices
7. Perturbation theory for singular values
8. Perturbation of simple eigenvalues of nonsymmetric matrices
9. Conclusions

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Definition (Lévy (1881). DD matrices)

The matrix $A \in \mathbb{R}^{n \times n}$ is row diagonally dominant (rdd) if

$$\sum_{j \neq i} |a_{ij}| \leq |a_{ii}|, \quad i = 1, 2, \ldots, n.$$

A is column diagonally dominant (cdd) if A^T is row diagonally dominant.

DD matrices arise in many applications:

- Numerical methods for PDEs, Collocation matrices in cubic splines for certain boundary conditions, Markov chains, Graph Laplacians, ...
- In social sciences, biology, economy, physics, engineering...
- Sparse Symmetric DD linear systems have received considerable attention in the last years from the point of view of randomized algorithms for computing their solutions in "Nearly-Linear" time, via graph preconditioners,...(D. Spielman, S. H. Teng)

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- best error bounds that Q. Ye proved after a direct error analysis that requires considerable efforts are

$$\frac{\|L-\widehat{L}\|_\infty}{\|L\|_\infty} \leq 6\,n\,8^{(n-1)}\epsilon, \quad \frac{\|U-\widehat{U}\|_\infty}{\|U\|_\infty} \leq 6\cdot 8^{(n-1)}\epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 5\cdot 8^{(n-1)}\epsilon,$$

  where $n \times n$ is the size of the matrix and ϵ the unit roundoff.

- $\epsilon = 2^{-53} \approx 10^{-16}$ in double precision, **so the bounds are > 1 for $n > 20$...**
- However, there are no condition numbers in the bounds and numerical experiments indicated accuracy.
- **Can we prove better bounds?**

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- **Using a structured perturbation theory of LDU factorization of Diagonally Dominant matrices** and intricate error analysis, we proved *(D. and Koev, Numer. Math, 2011)*

$$\frac{\|L-\widehat{L}\|_M}{\|L\|_M} \leq 14\, n^3 \epsilon, \quad \frac{\|U-\widehat{U}\|_M}{\|U\|_M} \leq 14\, n^3 \epsilon, \quad \frac{|d_{ii}-\widehat{d}_{ii}|}{|d_{ii}|} \leq 14\, n^3 \epsilon \ \ \forall i$$

for the errors of Q. Ye's algorithm (here $\|A\|_M = \max_{ij} |a_{ij}|$).

My motivation to study diagonally dominant matrices (IV)

- **Fundamental consequences:** Q. Ye's algorithm + other existing implicit algorithms for factorized matrices allow us **to compute for Diagonally Dominant matrices with guaranteed high relative accuracy**

  1. solutions of linear systems for most right-hand-sides *(D and Molera, IMA Journal of Numerical Analysis, 2012)*,
  2. solutions of least square problems for most right-hand-sides *(Castro, Ceballos, D, Molera, SIMAX, 2013)*,
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- No algorithms, no error analysis!!!
- **To present a family of relative perturbation bounds under certain structured perturbations** for several magnitudes corresponding to **Diagonally Dominant matrices**.
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References

All the presented perturbation results are included in

1. D and Koev, *Perturbation theory for the LDU factorization and accurate computations for diagonally dominant matrices*, Numerische Mathematik, 119 (2011), pp. 337-371.
2. Dailey, D, and Ye, *A new perturbation bound for the LDU factorization of diagonally dominant matrices*, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 904-930.
3. Dailey, D, and Ye, *Relative perturbation theory for diagonally dominant matrices*, SIAM Journal on Matrix Analysis and Applications, 35 (2014), pp. 1303-1328.
4. Q. Ye, *Relative perturbation bounds for eigenvalues of symmetric positive definite diagonally dominant matrices*, SIAM Journal on Matrix Analysis and Applications, 31 (2009), pp. 11-17.

Outline

- We assume in the first part that **$A \in \mathbb{R}^{n \times n}$ satisfies $a_{ii} \geq 0$** for all i. (**No restriction** for inverses, linear systems, least square problems, SVD, but **yes** for eigenvalues and then we will remove this assumption).
- Define the diagonally dominant parts of A and store them in a column vector $v = (v_1, v_2, \ldots, v_n)^T$ where

$$v_i := a_{ii} - \sum_{j \neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $$A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$$
- The pair (A_D, v) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through pairs of this type. A matrix A parameterized is this way will be denoted as

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Parameterizing row diagonally dominant matrices (Q. Ye, 2008)

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To compute the cubic spline (with parabolic b. c.) of a set of points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n), \quad \text{with} \quad x_1 < x_2 < \cdots < x_n \, ,$$

one needs to solve a system of equations whose coefficient matrix is

$$\begin{bmatrix} 1 & 1 & & & & \\ h_2 & 2(h_1+h_2) & h_1 & & & \\ & h_3 & 2(h_2+h_3) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-1} & 2(h_{n-2}+h_{n-1}) & h_{n-2} \\ & & & & 1 & 1 \end{bmatrix},$$

where $h_k = x_{k+1} - x_k > 0$.

- $v_1 = 0, v_2 = h_1 + h_2, \ldots, v_{n-1} = h_{n-2} + h_{n-1}, v_n = 0$
- Diagonally dominant parts can be computed accurately (without the entries) directly from the parameters defining the problem!!
- This happens in many other applications.

Diagonally dominant parts of collocation matrices in cubic splines

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Key features of Q. Ye's (2008) algorithm for LDU of DD matrices

- INPUT: $\mathcal{D}(A_D, v)$ with $v \geq 0$ *(not the entries of the matrix A)!!!*
- It performs Gaussian elimination with complete pivoting or column diagonal dominance pivoting.
- If we denote $A^{(1)} := A$ and $A^{(k)}$ is the matrix obtained after $k-1$ steps of Gaussian elimination are performed, then the algorithm iterates

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- $v^{(k+1)}$ **is obtained from $\mathcal{D}(A_D^{(k)}, v^{(k)})$ as a sum of nonnegative terms. There are no cancellation errors in this part!!**
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Example: Two small relative componentwise perturbations of a row diag. dominant matrix A. **Both preserve the diag. dominant structure.**

$$A = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1.5 & 1.5 \\ -1 & 2.001 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3.0015 & -1.5015 & 1.5 \\ -1 & 2.002002 & 1 \\ 2 & 0.5 & 2.5 \end{bmatrix}$$

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Singular values of A, B and C

	A	B	C
σ_1	4.641	4.640	4.642
σ_2	2.910	2.909	2.910
σ_3	$6.663 \cdot 10^{-4}$	$3.332 \cdot 10^{-4}$	$6.673 \cdot 10^{-4}$

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Only one main point...

"Every lecture should make only one main point."

From Gian-Carlo Rota, *"Ten lessons I wish I had been taught"*, Notices of the AMS, 44 (1997) 22-25.

Outline

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n \times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n \times n}$ *be row diagonally dominant matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Then

- A *is nonsingular if and only if* $\widetilde{A}$ *is nonsingular.*
- $\boxed{\dfrac{\|\widetilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \dfrac{n(3n-2)\delta}{1-2n\delta}}$*, for* 1*-,* 2*-,* ∞*-norms.*

To be compared with

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with $\|b - \widetilde{b}\| \leq \delta \|b\|$. *If* $2n\delta < 1$, *then in the* 1-, 2-, *and* ∞-*norms,*

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For most vectors b, $\|A^{-1}\|\,\|b\|/\|x\|$ is a moderate number and for A ill-conditioned,

$$\|A^{-1}\|\,\|b\|/\|x\| \ll \kappa(A)$$

Outline

Pivoting in row diagonally dominant matrices

- The DD structure is preserved by performing the same permutations in rows and columns, i.e., by any diagonal pivoting strategy.
- The entry with largest absolute value in a DD matrix is on the diagonal $\Longrightarrow$ complete pivoting in DD matrices is a diagonal pivoting strategy.
- Every row DD matrix has at least one column which is diagonally dominant.

$$A = \begin{bmatrix} -4 & 1 & 3 \\ 0 & 6 & 5 \\ 1 & -2 & 7 \end{bmatrix} \quad (rdd).$$

- The column diagonal dominance pivoting strategy chooses at each step of GE the diagonal entry with largest absolute value among those corresponding to the columns which are diagonally dominant.

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Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be row DD matrices, and $A = LDU$ and $\widetilde{A} = \widetilde{L}\,\widetilde{D}\,\widetilde{U}$ be their factorizations. If

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with} \quad \delta < 1,$$

then

- For any diagonal pivoting strategy,

$$|\widetilde{d}_{ii} - d_{ii}| \leq \frac{2n\delta}{1-2n\delta}\,|d_{ii}|, \quad i = 1:n, \quad \text{and} \quad \frac{\|\widetilde{U} - U\|_\infty}{\|U\|_\infty} \leq 3n^2\delta\,.$$

- For complete and column diagonal dominance pivoting,

$$\frac{\|\widetilde{L} - L\|_\infty}{\|L\|_\infty} \leq \frac{n\delta}{1-n\delta}\left(3 + \frac{2n\delta}{1-n\delta}\right) \quad \text{and} \quad \frac{\|\widetilde{L} - L\|_1}{\|L\|_1} \leq \frac{n(8n-2)\,\delta}{1-(12\,n+1)\delta}\,.$$

Outline

Theorem (Q. Ye, SIMAX, 2009)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ be diagonally dominant symmetric matrices with nonnegative diagonal entries such that

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ and $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n \geq 0$ be, respectively, the eigenvalues of $A = \mathcal{D}(A_D, v)$ and $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$. Then

$$\boxed{|\widetilde{\lambda}_i - \lambda_i| \leq \delta\, |\lambda_i|, \qquad i = 1, \ldots, n.}$$

To be compared with

$$|\widetilde{\lambda}_i - \lambda_i| \leq (\|A\|_2 \|A^{-1}\|_2) \frac{\|\widetilde{A} - A\|_2}{\|A\|_2} |\lambda_i|$$

Theorem (Q. Ye, SIMAX, 2009)

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be diagonally dominant symmetric matrices with nonnegative diagonal entries such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ *and* $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n \geq 0$ *be, respectively, the eigenvalues of* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *. Then*

$$\boxed{|\widetilde{\lambda}_i - \lambda_i| \leq \delta\, |\lambda_i|, \qquad i = 1, \ldots, n.}$$

To be compared with

$$|\widetilde{\lambda}_i - \lambda_i| \leq \left(\|A\|_2 \|A^{-1}\|_2\right) \frac{\|\widetilde{A} - A\|_2}{\|A\|_2} \; |\lambda_i|$$

Parameterizing row DD matrices with diagonal entries of any sign

- Let $A \in \mathbb{R}^{n\times n}$.
- Define $v = (v_1, v_2, \dots, v_n)$ where

$$v_i := |a_{ii}| - \sum_{j\neq i} |a_{ij}|$$

- A is row diagonally dominant if and only if $v_i \geq 0$ for all i.
- $A_D := \begin{cases} 0 & \text{for } i = j \\ a_{ij} & \text{for } i \neq j \end{cases}$
- Define $\boxed{S = \text{diag}(\text{sign}(a_{11}), \dots, \text{sign}(a_{nn}))}$ (sign$(0) := 1$).
- The triplet (A_D, v, S) allows us to recover the matrix A and we parameterize the set of $n \times n$ matrices through triplets of this type. Any matrix A parameterized is this way will be denoted as

$$\boxed{A = \mathcal{D}(A_D, v, S)}$$

Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S) \in \mathbb{R}^{n\times n}$ *be diagonally dominant symmetric matrices such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \textit{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \textit{with } \delta < 1.$$

Let $\lambda_1 \geq \cdots \geq \lambda_n$ *and* $\widetilde{\lambda}_1 \geq \cdots \geq \widetilde{\lambda}_n$ *be, respectively, the eigenvalues of* $A = \mathcal{D}(A_D, v, S)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}, S)$ *.*

Assume $2n^2(n+2)\delta < 1$ *and define* $\nu := \dfrac{2n^2(n+1)\delta}{1 - n\delta}$.

Then

$$\begin{aligned} |\widetilde{\lambda}_i - \lambda_i| &\leq (2\nu + \nu^2)\, |\lambda_i| \\ &\approx (\, 4n^3\delta + O(\delta^2)\,)\, |\lambda_i|, \qquad i = 1, \ldots, n \end{aligned}$$

Outline

Theorem

Let $A = \mathcal{D}(A_D, v) \in \mathbb{R}^{n\times n}$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v}) \in \mathbb{R}^{n\times n}$ *be row diagonally dominant matrices with nonnegative diagonal entries such that*

$$|\widetilde{v} - v| \leq \delta\, v \quad \text{and} \quad |\widetilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{with } \delta < 1.$$

Let $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ *and* $\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_n \geq 0$ *be, respectively, the singular values of* $A = \mathcal{D}(A_D, v)$ *and* $\widetilde{A} = \mathcal{D}(\widetilde{A}_D, \widetilde{v})$ *.*
Define

$$\nu := \frac{4n^3 \sqrt{2(n+1)}}{1 - (12n+1)\,\delta}\, \delta\, .$$

If $0 \leq \nu < 1$, *then*

$$\boxed{|\widetilde{\sigma}_i - \sigma_i| \leq (2\nu + \nu^2)\, \sigma_i, \qquad i = 1, \ldots, n.}$$

Outline

Comments for general matrices

Let λ be a simple eigenvalue of $A \in \mathbb{R}^{n \times n}$ with right and left eigenvectors x and y. Then $\tilde{A} = A + E$ has an eigenvalue $\tilde{\lambda}$ such that

$$\frac{|\tilde{\lambda} - \lambda|}{|\lambda|} \leq \left(\sec\theta(y,x) \, \frac{\|A\|_2}{|\lambda|} \right) \frac{\|E\|_2}{\|A\|_2} + \mathcal{O}\left(\|E\|_2^2\right),$$

where $\sec\theta(y,x) = \dfrac{\|y\|_2 \|x\|_2}{|y^* x|}$.

This bound can be large as a consequence of two facts:

- $\|A\|_2/|\lambda|$ can be large and/or
- $\sec\theta(y,x)$ can be large.

For parameterized perturbations of row diagonally dominant matrices, we will remove $\|A\|_2/|\lambda|$.

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Theorem

Let $A = \mathcal{D}(A_D, v, S) \in \mathbb{R}^{n \times n}$ *be s.t.* $v \geq 0$ *and let* λ *be an eigenvalue of* A *with a right eigenvector* x. *Let* $\tilde{A} = \mathcal{D}(\tilde{A}_D, \tilde{v}, S) \in \mathbb{R}^{n \times n}$ *be such that*

$$|\tilde{v} - v| \leq \delta v \quad \text{and} \quad |\tilde{A}_D - A_D| \leq \delta |A_D|, \quad \text{for some } 0 \leq \delta < 1,$$

and let $\tilde{\lambda}$ *be an eigenvalue of* $\tilde{A}$ *with a left eigenvector* $\tilde{y}$ *such that* $\tilde{y}^* x \neq 0$. *If* $(13n + 7n^3 \sec\theta(\tilde{y}, x))\,\delta < 1$, *then*

$$|\tilde{\lambda} - \lambda| \leq \frac{8n^{7/2} + 7n^3}{1 - (13n + 7n^3 \sec\theta(\tilde{y}, x))\,\delta} \sec\theta(\tilde{y}, x)\,\delta\,|\lambda|,$$

where

$$\sec\theta(\tilde{y}, x) = \frac{\|\tilde{y}\|_2 \|x\|_2}{|\tilde{y}^* x|}$$

Outline

Conclusions

- To work with matrices only through their entries may be not convenient, even in the case we force the preservation of structures.
- To parameterize important classes of matrices in order to preserve explicitly their structure may be important both in theory and in applications.
- A good set of parameters should have better perturbation properties than the entries.
- A good set of parameters should allow us to work numerically with the matrices, that is, to construct algorithms based on these parameters for the fundamental tasks of Numerical Linear Algebra.
- This is easy to say, but it might be hard to do, but in the particular case of DD matrices, this idea has led to a very satisfactory perturbation theory and to highly accurate algorithms.

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