Polynomial Zigzag Matrices, Dual Minimal Bases, and Applications

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joint work with
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Outline

1. Preliminary concepts: Minimal Indices and Bases
2. Forney’s theorem on dual minimal bases: Goal of the talk
3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications
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1. Preliminary concepts: Minimal Indices and Bases
2. Forney’s theorem on dual minimal bases: Goal of the talk
3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications
Theorem (Kronecker Canonical Form (KCF))

For any matrix pencil $A - \lambda B$, $A, B \in \mathbb{C}^{m \times n}$, there exist nonsingular matrices $U$ and $V$ such that

$$U (A - \lambda B) V = L_{\epsilon_1} \oplus \cdots \oplus L_{\epsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T \oplus J_{k_1} (\lambda - \lambda_1) \oplus \cdots \oplus J_{k_f} (\lambda - \lambda_f) \oplus N_{\ell_1} (\lambda) \oplus \cdots \oplus N_{\ell_s} (\lambda),$$

where

$$L_{\epsilon} = \begin{bmatrix} 1 & \lambda \\ & \ddots \\ & & 1 & \lambda \end{bmatrix}_{\epsilon \times (\epsilon + 1)}, \quad L_{\eta}^T = \begin{bmatrix} 1 & & & \\ & \lambda & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}_{(\eta + 1) \times \eta},$$

$$J_{k} (\lambda - \lambda_i) = \begin{bmatrix} \lambda - \lambda_i & 1 \\ & \ddots \\ & & \ddots \\ & & & 1 \end{bmatrix}_{k \times k}, \quad N_{\ell} (\lambda) = \begin{bmatrix} 1 & \lambda \\ & \ddots \\ & & \ddots \\ & & & 1 \end{bmatrix}_{\ell \times \ell}.$$
Minimal indices of pencils (I)

Theorem (Kronecker Canonical Form (KCF))

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\]

\[
\oplus J_{k_1} (\lambda - \lambda_1) \oplus \cdots \oplus J_{k_f} (\lambda - \lambda_f) \oplus N_{\ell_1} (\lambda) \oplus \cdots \oplus N_{\ell_s} (\lambda),
\]

where

- \( L_{\epsilon} = \begin{bmatrix} 1 & \lambda \\ & \ddots & \ddots \\ & \ddots & 1 & \lambda \\ & & & \end{bmatrix}_{\epsilon \times (\epsilon+1)} \)
- \( L_{\eta}^T = \begin{bmatrix} 1 \\ \lambda \\ & \ddots \\ & & \ddots & 1 \\ & & & \lambda \\ \end{bmatrix}_{(\eta+1) \times \eta} \)
- \( J_k (\lambda - \lambda_i) = \begin{bmatrix} \lambda - \lambda_i & 1 \\ & \ddots & \ddots \\ & \ddots & 1 \\ & & \lambda - \lambda_i \end{bmatrix}_{k \times k} \)
- \( N_{\ell} (\lambda) = \begin{bmatrix} 1 & \lambda \\ & \ddots \\ & & \ddots \\ & & & 1 \end{bmatrix}_{\ell \times \ell} \)
Definition (Minimal Indices of a Matrix Pencil)

Let $A - \lambda B$ be a matrix pencil with KCF

$$U(A - \lambda B)V = L_{\varepsilon_1} \oplus \cdots \oplus L_{\varepsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T$$

$$\oplus J_{k_1}(\lambda - \lambda_1) \oplus \cdots \oplus J_{k_f}(\lambda - \lambda_f) \oplus N_{\ell_1}(\lambda) \oplus \cdots \oplus N_{\ell_s}(\lambda),$$

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$$L_{\varepsilon} = \begin{bmatrix} 1 & \lambda & \cdots & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & & & \vdots \\ 1 & \cdots & \cdots & \cdots & \cdots & \lambda \end{bmatrix}_{\varepsilon \times (\varepsilon + 1)}$$

$$L_{\eta}^T = \begin{bmatrix} 1 & \lambda & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \lambda \end{bmatrix}_{(\eta + 1) \times \eta}$$

Then

- the numbers $\varepsilon_1, \ldots, \varepsilon_p$ are the right minimal indices of $A - \lambda B$,
- the numbers $\eta_1, \ldots, \eta_q$ are the left minimal indices of $A - \lambda B$. 
Definition (Minimal Indices of a Matrix Pencil)

Let $A - \lambda B$ be a matrix pencil with KCF

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\end{bmatrix}_{\varepsilon \times (\varepsilon+1)}, \quad L_T^{\eta} = \begin{bmatrix}
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\lambda \\
\vdots \\
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\end{bmatrix}_{(\eta+1) \times \eta}.$$

Then

- the numbers $\varepsilon_1, \ldots, \varepsilon_p$ are the **right minimal indices** of $A - \lambda B$,
- the numbers $\eta_1, \ldots, \eta_q$ are the **left minimal indices** of $A - \lambda B$. 

F. M. Dopico (U. Carlos III, Madrid)

"Polynomial Zigzag matrices"

IWOTA, July 6-10, 2015
Rational vector spaces and subspaces

In this talk:

- \( \mathbb{F} \) is an arbitrary field.
- \( \mathbb{F}[\lambda] \) is the ring of polynomials with coefficients in \( \mathbb{F} \).
- \( \mathbb{F}(\lambda) \) is the field of rational functions over \( \mathbb{F} \).
- \( \mathbb{F}(\lambda)^n \) is the vector space over the field \( \mathbb{F}(\lambda) \) of \( n \)-tuples with entries in \( \mathbb{F}(\lambda) \).

Example:

\[
\begin{bmatrix}
\frac{\lambda + 2}{\lambda^2} \\
\frac{1}{\lambda + 1}^3
\end{bmatrix} \in \mathbb{R}(\lambda)^2
\]

\( \mathbb{F}(\lambda)^n \) is known as a rational vector space and its subspaces as rational vector subspaces. (Wolovich-1974, Forney-1975)
Rational vector spaces and subspaces

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$\mathbb{F}(\lambda)^n$ is known as a rational vector space and its subspaces as rational vector subspaces. (Wolovich-1974, Forney-1975)
Minimal bases of rational vector subspaces

- Any rational subspace $\mathcal{V} \subseteq \mathbb{F}(\lambda)^n$ has bases consisting entirely of vector polynomials.

- **Example:**

  $\begin{bmatrix}
  \frac{\lambda + 2}{\lambda^2} \\
  \frac{1}{(\lambda + 1)^3}
  \end{bmatrix} \in \mathcal{V} \implies \lambda^2 (\lambda + 1)^3

  \begin{bmatrix}
  \frac{\lambda + 2}{\lambda^2} \\
  \frac{1}{(\lambda + 1)^3}
  \end{bmatrix} =

  \begin{bmatrix}
  (\lambda + 2)(\lambda + 1)^3 \\
  \lambda^2
  \end{bmatrix} \in \mathcal{V}$

**Definition (Minimal basis)**

A minimal basis of the rational subspace $\mathcal{V} \in \mathbb{F}(\lambda)^n$ is a basis consisting of vector polynomials whose sum of degrees is minimal among all bases of $\mathcal{V}$ consisting of vector polynomials.

- Introduced by Plemelj-1908, Muskhelishvili and Vekua-1943, but Forney-1975 made this concept very important in Multivariable Linear System Theory, then appeared in the book by Kailath-1980, ...
Minimal bases of rational vector subspaces

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\begin{bmatrix}
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A **minimal basis** of the rational subspace $\mathcal{V} \in \mathbb{F}(\lambda)^n$ is a basis

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  \begin{bmatrix}
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A **minimal basis** of the rational subspace \( \mathcal{V} \in \mathbb{F}(\lambda)^n \) is a basis

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There are many (infinite) minimal bases of a rational subspace $V \subseteq \mathbb{F}(\lambda)^n$, but...

**Theorem (Forney, 1975...probably known before)**

The ordered list of degrees of the vector polynomials in any minimal basis of $V \subseteq \mathbb{F}(\lambda)^n$ is always the same.

**Definition**

These degrees are called the **minimal indices** of $V \subseteq \mathbb{F}(\lambda)^n$. 
There are many (infinite) minimal bases of a rational subspace \( \mathcal{V} \subseteq \mathbb{F}(\lambda)^n \), but...

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These degrees are called the **minimal indices** of \( \mathcal{V} \subseteq \mathbb{F}(\lambda)^n \).
Connection to minimal indices of pencils (I)

**Definition (Minimal Indices of a Matrix Pencil)**

Let \( A - \lambda B \) be an \( m \times n \) matrix pencil with KCF

\[
U(A - \lambda B)V = L_{\varepsilon_1} \oplus \cdots \oplus L_{\varepsilon_p} \oplus L_{\eta_1}^T \oplus \cdots \oplus L_{\eta_q}^T \oplus \text{regular blocks}
\]

where

\[
L_{\varepsilon} = \begin{bmatrix}
1 & \lambda \\
& \ddots & \ddots \\
& & 1 & \lambda \\
\end{bmatrix}_{\varepsilon \times (\varepsilon + 1)}, \quad L_{\eta}^T = \begin{bmatrix}
1 & \ddots \\
& \ddots & 1 \\
& & \lambda \\
\end{bmatrix}_{(\eta + 1) \times \eta}
\]

Then

- the numbers \( \varepsilon_1, \ldots, \varepsilon_p \) are the **right minimal indices** of \( A - \lambda B \),
- the numbers \( \eta_1, \ldots, \eta_q \) are the **left minimal indices** of \( A - \lambda B \).
Connection to minimal indices of pencils (II)

Proposition

Let $A - \lambda B$ be an $m \times n$ matrix pencil with entries in $\mathbb{F}$. Then:

- The **right minimal indices** of $A - \lambda B$ are the minimal indices of the rational right NULL space of $A - \lambda B$, i.e.,

  \[
  \mathcal{N}_r(A - \lambda B) := \left\{ x(\lambda) \in \mathbb{F}(\lambda)^{n \times 1} : (A - \lambda B)x(\lambda) \equiv 0 \right\}.
  \]

- The **left minimal indices** of $A - \lambda B$ are the minimal indices of the rational left NULL space of $A - \lambda B$, i.e.,

  \[
  \mathcal{N}_\ell(A - \lambda B) := \left\{ y(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times m} : y(\lambda)^T(A - \lambda B) \equiv 0^T \right\}.
  \]
We use for brevity the following definition.

**Definition**

Let $A - \lambda B$ be an $m \times n$ matrix pencil with entries in $\mathbb{F}$. Then:

- A **right minimal basis** of $A - \lambda B$ is a minimal basis of the rational right NULL space of $A - \lambda B$, i.e.,

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Example of minimal basis and minimal indices of a pencil

\[ A - \lambda B = \begin{bmatrix} 1 & \lambda \\ 1 & \lambda \\ 1 & \lambda \end{bmatrix} \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{3 \times 5} \]

Right minimal indices of \( A - \lambda B = \{1, 2\} \) and no left minimal indices.

\[ \mathcal{N}_r(A - \lambda B) = \text{Span}\{ \begin{bmatrix} -\lambda \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \lambda^2 \\ -\lambda \\ 1 \end{bmatrix} \} = \text{Span}\{ \begin{bmatrix} -\lambda^3 \\ \lambda^2 \\ \lambda^3 \\ -\lambda^2 \\ \lambda \end{bmatrix}, \begin{bmatrix} \lambda^5 \\ -\lambda^4 \\ \lambda^2 \\ -\lambda \\ 1 \end{bmatrix} \} \]

Sum of degrees of \( \{u_1, u_2\} = 1 + 2 = 3 \) (right minimal bases of \( A - \lambda B \))

Sum of degrees of \( \{w_1, w_2\} = 3 + 5 = 8 \)
Example of minimal basis and minimal indices of a pencil

\[ A - \lambda B = \begin{bmatrix} 1 & \lambda \\ \lambda & 1 \\ 1 & \lambda \\ \lambda & 1 \end{bmatrix} \in \mathbb{F}[\lambda]^{3 \times 5} \]

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Sum of degrees of $\{u_1, u_2\} = 1 + 2 = 3$ (right minimal bases of $A - \lambda B$)

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Example of minimal basis and minimal indices of a pencil

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A - \lambda B = \begin{bmatrix}
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\[
\mathcal{N}_r(A - \lambda B) = \text{Span}\left\{ \begin{bmatrix}
-\lambda \\
1 \\
0
\end{bmatrix} , \begin{bmatrix}
0 \\
0 \\
\lambda^2
\end{bmatrix} \right\} = \text{Span}\left\{ \begin{bmatrix}
-\lambda^3 \\
\lambda^2 \\
-\lambda
\end{bmatrix} , \begin{bmatrix}
\lambda^5 \\
-\lambda^4 \\
\lambda^2
\end{bmatrix} \right\}
\]

Sum of degrees of \( \{u_1, u_2\} = 1 + 2 = 3 \) \hspace{1cm} (right minimal bases of \( A - \lambda B \))

Sum of degrees of \( \{w_1, w_2\} = 3 + 5 = 8 \)
Every polynomial matrix has left and right minimal bases and indices

Remark

They are defined through the null spaces of the polynomial matrix, since for polynomial matrices of degree larger than 1, there is NO a “KCF”

Example:

\[ P(\lambda) = \begin{bmatrix} 1 & \lambda^3 \\ 1 & \lambda \\ 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{3 \times 5} \]

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Sum of degrees of \( \{u_1, u_2\} = 3 + 2 = 5 \) (right minimal bases of \( P(\lambda) \))

Sum of degrees of \( \{w_1, w_2\} = 3 + 5 = 8 \)

Right minimal indices of \( P(\lambda) = \{2, 3\} \)
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Sum of degrees of $$\{u_1, u_2\} = 3 + 2 = 5$$  
(right minimal bases of $$P(\lambda)$$)  
Sum of degrees of $$\{w_1, w_2\} = 3 + 5 = 8$$

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Sum of degrees of \( \{u_1, u_2\} = 3 + 2 = 5 \) \hspace{1cm} \text{(right minimal bases of} \  P(\lambda) \text{)}

Sum of degrees of \( \{w_1, w_2\} = 3 + 5 = 8 \)

Right minimal indices of \( P(\lambda) = \{2, 3\} \)
Every polynomial matrix has left and right minimal bases and indices

Remark

They are defined through the null spaces of the polynomial matrix, since for polynomial matrices of degree larger than 1, there is NO a “KCF”

Example:

\[ P(\lambda) = \begin{bmatrix} 1 & \lambda^3 \\ 1 & \lambda \\ 1 & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{3\times5} \]

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**Right minimal indices of** \( P(\lambda) = \{2, 3\} \)
REMARK: In the rest of the talk, we arrange all (minimal) bases as the rows of matrices and often call “basis” to the matrix.

Theorem (Forney 1975...probably known before)

The rows of a polynomial matrix $N(\lambda)$ over a field $\mathbb{F}$ are a minimal basis of the subspace they span if and only if

(a) $N(\lambda_0)$ has full row rank for all $\lambda_0 \in \overline{\mathbb{F}}$, and

(b) the highest-row-degree coefficient matrix of $N(\lambda)$ has also full row rank.

Example (of minimal basis)

$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$
Practical characterization of minimal bases

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$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

- $N(\lambda)$ satisfies (a) by the 1’s.
- $N(\lambda)$ satisfies (b) since its highest-row-degree coefficient matrix is
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One more example

**Theorem (Forney 1975...probably known before)**

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(b) the highest-row-degree coefficient matrix of $N(\lambda)$ has also full row rank.

**Example (NOT minimal basis)**

$$N(\lambda) = \begin{bmatrix} -\lambda^3 & 1 & \lambda^3 & -\lambda^2 & \lambda \\ -\lambda^5 & \lambda^2 & \lambda^2 & -\lambda & 1 \end{bmatrix}$$

- This $N(\lambda)$ DOES NOT satisfy (a) because

$$N(1) = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}$$

is rank deficient.
Outline

1 Preliminary concepts: Minimal Indices and Bases

2 Forney's theorem on dual minimal bases: Goal of the talk

3 Polynomial Zigzag matrices

4 Solving the inverse problem for dual minimal bases

5 Conclusions and Applications
Dual Minimal Bases

Definition (Dual Minimal Bases)

Polynomial matrices \( M(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \) and \( N(\lambda) \in \mathbb{F}[\lambda]^{k \times n} \) are said to be dual minimal bases if

(a) both are minimal bases,

(b) \( m + k = n \),

(c) and \( M(\lambda) N(\lambda)^T = 0 \).

Remark

- The “name” is not standard. Forney (1975) uses the “rational subspaces spanned by the rows of \( M(\lambda) \) and \( N(\lambda) \) are dual subspaces of \( \mathbb{F}(\lambda)^n \)”.

- Dual minimal bases have classical applications in Linear System Theory for constructing left and right coprime factorizations of transfer functions, also for solving certain matrix polynomial equations, and, modern applications, for constructing strong linearizations and \( \ell \)-ifications of polynomials matrices, solving inverse problems...
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Dual Minimal Bases: Example and comments

Example \((M(\lambda)N(\lambda)^T = 0)\)

\[M(\lambda) = \begin{bmatrix} 1 & \lambda & \lambda & \lambda \\ 1 & 1 & \lambda & \lambda \end{bmatrix} \in \mathbb{F}[\lambda]^{3\times4}\]

\[N(\lambda) = \begin{bmatrix} \lambda^3 & -\lambda^2 & \lambda & -1 \end{bmatrix} \in \mathbb{F}[\lambda]^{1\times4}\]

Remarks

In general, for dual minimal bases \(M(\lambda)N(\lambda)^T = 0\) viewed as polynomial matrices:

- \(M(\lambda)\) is a left minimal basis of \(N(\lambda)^T\),
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Forney’s theorem and main goal of the talk

Theorem (Forney 1975...probably known before)

Let $M(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ be dual minimal bases with row degrees $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$, respectively. Then

\[ \sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j. \]

GOAL: Solve the corresponding INVERSE PROBLEM

Given two lists of nonnegative integers $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases having these numbers as their row degrees?
- can we explicitly construct dual minimal bases having any lists of prescribed row degrees with the same sum?

Our main new tool will be...

F. M. Dopico  (U. Carlos III, Madrid)  Polynomial Zigzag matrices  
IWOTA, July 6-10, 2015  19 / 39
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3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications
Definition of Forward-Zigzag Polynomial Matrices

Example of forward Zigzag matrix:

\[ Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 \\
\lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 \\
\lambda^{15} & 1 & \lambda^2 & \lambda^3
\end{bmatrix} \]

Definition

\[ Z(\lambda) \in \mathbb{F}^{m \times n} \text{ with } m < n \text{ is a forward-zigzag matrix, if} \]

(a) each row of \( Z(\lambda) \) is of the form

\[ \begin{bmatrix}
0 & \ldots & 0 & 1 & \lambda^{p_1} & \lambda^{p_2} & \ldots & \lambda^{p_k} & 0 & \ldots & 0
\end{bmatrix}, \]

with \( 0 < p_1 < p_2 < \cdots < p_k \) and \( k \geq 1 \).

(b) \( Z(\lambda) \) is in a double-echelon form: the last nonzero entry of each row and the first nonzero entry of the row just below are in the same column.

(c) \( Z(\lambda) \) has no zero columns.
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\[
\begin{bmatrix}
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\vdots \\
1 \lambda^{p_1} \lambda^{p_2} \cdots \lambda^{p_k} \\
\vdots \\
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- Every zigzag matrix is a minimal basis.
- I like to say that zigzag matrices generalize to degrees larger than 1 right singular blocks of the KCF of pencils, which are simple examples of forward zigzag matrices.
- For any two adjacent columns of a zigzag matrix, there is a unique row having two nonzero entries in those columns.
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- **Unit column sequence of a forward-zigzag matrix.** In the example
  \[U,N,N,U,U,N,N,N,U,N,N\]
  number of Us = number of rows

- **Degree-gap sequence of a forward-zigzag matrix.** In the example
  \[2, 5, 1, 3, 1, 3, 4, 7, 2, 1\]

- **Structure sequence of a forward-zigzag matrix.** In the example
  \[U \ 2 \ N \ 5 \ N \ 1 \ U \ 3 \ U \ 1 \ N \ 3 \ N \ 4 \ N \ 7 \ U \ 2 \ N \ 1 \ N\]

From the structure sequence we can construct \(Z(\lambda)\).
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Example of backward Zigzag matrix:

\[ \hat{Z}(\lambda) = \begin{bmatrix}
\lambda^2 & 1 & \lambda^5 & 1 & \lambda^5 & \lambda^4 & \lambda & 1 \\
\lambda^5 & 1 & \lambda^5 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 \\
\lambda^4 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 & \lambda & 1
\end{bmatrix} \]

- Backward Zigzag matrices are obtained by reversing the order of rows and columns of forward Zigzag matrices.
- Similar properties and tools as for forward Zigzag matrices.
- In the example, the structure sequence is

\[ [N \ 2 \ U \ 5 \ U \ 1 \ N \ 3 \ N \ 1 \ U \ 3 \ U \ 4 \ U \ 7 \ N \ 2 \ U \ 1 \ U]. \]
Example of backward Zigzag matrix:

\[
\hat{Z}(\lambda) = \begin{bmatrix}
\lambda^2 & 1 \\
\lambda^5 & 1 \\
\lambda^5 & \lambda^4 & \lambda & 1 \\
\lambda^3 & 1 \\
\lambda^4 & 1 \\
\lambda^9 & \lambda^2 & 1 \\
\lambda & 1
\end{bmatrix}
\]

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\[\begin{bmatrix}
N & 2 & U & 5 & U & 1 & N & 3 & N & 1 & U & 3 & U & 4 & U & 7 & N & 2 & U & 1 & U
\end{bmatrix}].\]
Backward-Zigzag Polynomial Matrices

Example of backward Zigzag matrix:

\[
\hat{Z}(\lambda) = \begin{bmatrix}
\lambda^2 & 1 & \lambda^5 & 1 & \lambda^5 & 1 & \lambda^4 & 1 \\
\lambda^5 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 \\
\lambda^9 & \lambda^2 & 1 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 \\
\end{bmatrix}
\]

- Backward Zigzag matrices are obtained by reversing the order of rows and columns of forward Zigzag matrices.
- Similar properties and tools as for forward Zigzag matrices.
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\begin{bmatrix}
N & 2 & U & 5 & U & 1 & N & 3 & N & 1 & U & 3 & U & 4 & U & 7 & N & 2 & U & 1 & U
\end{bmatrix}
\]
Dual Zigzag Matrices

Definition

Suppose

1. \( Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \) is a forward-zigzag matrix and
2. \( Z^{\diamond}(\lambda) \in \mathbb{F}[\lambda]^{k \times n} \) is a backward-zigzag matrix

with the same number of columns. Then \( Z(\lambda) \) and \( Z^{\diamond}(\lambda) \) are said to be dual zigzag matrices, if they have

(a) the same degree-gap sequence, but
(b) complementary unit column sequences, where \( U \) and \( N \) are each other’s complement.

Corollary

\#\text{rows of } Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \text{ plus } \#\text{rows of } Z^{\diamond}(\lambda) \in \mathbb{F}[\lambda]^{k \times n} \text{ is equal to the number of columns, i.e.,}

\[ m + k = n \]
Dual Zigzag Matrices

**Definition**
Suppose
1. $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ is a forward-zigzag matrix and
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**Corollary**

#rows of $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ plus #rows of $Z(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ is equal to the number of columns, i.e.,

$$m + k = n$$
Example of Dual Zigzag Matrices

\[
Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 & \\
\lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & \\
1 & \lambda & 1 & \lambda^2 & \lambda^3 & \end{bmatrix}
\]

\[
Z^{\diamond}(\lambda) = \begin{bmatrix}
\lambda^2 & 1 & \lambda^5 & 1 & \\
\lambda^5 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \\
\lambda^5 & \lambda^4 & \lambda & 1 & \lambda^3 & \\
\lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 & \\
\lambda & 1 & & & & 
\end{bmatrix}
\]

Both have the same degree-gap sequence: 2, 5, 1, 3, 1, 3, 4, 7, 2, 1.
Example of Dual Zigzag Matrices

\[
Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 & 1 & \lambda^3 & 1 & \lambda & 1 & \lambda^4 & \lambda^8 & \lambda^{15} & 1 & \lambda^2 & \lambda^3
\end{bmatrix}
\]

\[
Z^\diamond (\lambda) = \begin{bmatrix}
\lambda^2 & 1 & \lambda^5 & 1 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 & \lambda & 1
\end{bmatrix}
\]

Both have the same degree-gap sequence:

\[2, 5, 1, 3, 1, 3, 4, 7, 2, 1\]
Example of Dual Zigzag Matrices

\[
Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 & 1 & \lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & 1 & \lambda^2 & \lambda^3 \\
\lambda^2 & 1 & \lambda^5 & 1 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 \\
\lambda^5 & 1 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 & \lambda & 1
\end{bmatrix}
\]

\[
Z^\diamond(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 & 1 & \lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & 1 & \lambda^2 & \lambda^3 \\
\lambda^2 & 1 & \lambda^5 & 1 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 \\
\lambda^5 & 1 & \lambda^4 & \lambda & 1 & \lambda^3 & 1 & \lambda^4 & 1 & \lambda^9 & \lambda^2 & 1 & \lambda & 1
\end{bmatrix}
\]

Both have the same degree-gap sequence:

\[2, 5, 1, 3, 1, 3, 4, 7, 2, 1\]
Suppose
\[ Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \text{ is any forward-zigzag matrix,} \]
\[ Z^{\diamond}(\lambda) \in \mathbb{F}[\lambda]^{(n-m) \times n} \text{ is its dual backward zigzag matrix, and} \]
\[ \Sigma_n := \text{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1}). \]

Then
\[ Z(\lambda) \cdot (Z^{\diamond}(\lambda) \cdot \Sigma_n)^T = 0, \]

i.e., \( Z(\lambda) \) and \( (Z^{\diamond}(\lambda) \cdot \Sigma_n) \) are dual minimal bases.
From dual Zigzag matrices to dual minimal bases

\[ Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 & 1 & \lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} & 1 & \lambda^2 & \lambda^3 \\
\lambda^2 & -1 & \lambda^5 & -1 & 1 & \lambda^5 & -\lambda^4 & \lambda & -1 & \lambda^4 & 1 & \lambda^4 & -1 & \lambda^4 \\
-\lambda^5 & 1 & -\lambda^4 & \lambda & -\lambda^3 & 1 & -\lambda^3 & -1 & \lambda^4 & -\lambda^9 & \lambda^2 & -1 & -\lambda & 1
\end{bmatrix} \]

Both have the same degree-gap sequence:

2, 5, 1, 3, 1, 3, 4, 7, 2, 1
Our strategy

- Dual Zigzag matrices are Dual minimal bases (up to signs of columns),
- but very simple ones.
- So, it seems possible to use them for solving our inverse problem...
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- Dual Zigzag matrices are Dual minimal bases (up to signs of columns),
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Our inverse problem

**Theorem (Forney 1975...probably known before)**

Let $M(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $N(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ be dual minimal bases with row degrees $(\eta_1, \ldots, \eta_m)$ and $(\varepsilon_1, \ldots, \varepsilon_k)$, respectively. Then

$$
\sum_{i=1}^{m} \eta_i = \sum_{j=1}^{k} \varepsilon_j.
$$

**GOAL: Solve the corresponding INVERSE PROBLEM**

Given two lists of nonnegative integers $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ that have the same sum:

- do there exist dual minimal bases having these numbers as their row degrees?
- can we explicitly construct dual minimal bases having any lists of prescribed row degrees with the same sum?
Outline

1. Preliminary concepts: Minimal Indices and Bases
2. Forney’s theorem on dual minimal bases: Goal of the talk
3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications
Lemma

Suppose \( Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n} \) is a forward-zigzag matrix with structure sequence

\[ S = \begin{bmatrix} s_1 & \delta_1 & s_2 & \delta_2 & \ldots & s_{n-1} & \delta_{n-1} & s_n \end{bmatrix}. \]

Then:

(a) \( Z(\lambda) \) has row degrees equal to the partial sums of degree gaps between any two consecutive \( U \)'s and after the last \( U \).

(b) \( Z^\dagger(\lambda) \) has row degrees equal to the partial sums of degree gaps before the first \( N \) and between any two consecutive \( N \)'s.

(The row degrees are ordered from top to bottom.)
Example: Row degrees of a zigzag matrix and its dual

Example of forward Zigzag matrix:

\[ Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 \\
1 & \lambda^3 \\
1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} \\
1 & \lambda^2 & \lambda^3
\end{bmatrix} \]

\[ S = \begin{bmatrix}
\end{bmatrix} \]
Example: Row degrees of a zigzag matrix and its dual

Example of forward Zigzag matrix:

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Z(\lambda) = \begin{bmatrix}
1 & \lambda^2 & \lambda^7 & \lambda^8 \\
& 1 & \lambda^3 \\
& & 1 & \lambda & \lambda^4 & \lambda^8 & \lambda^{15} \\
& & & 1 & \lambda^2 & \lambda^3
\end{bmatrix}
\]

\[
S = \begin{bmatrix}
\text{U} & 2 & \text{N} & 5 & \text{N} & 1 & \text{U} & 3 & \text{U} & 1 & \text{N} & 3 & \text{N} & 4 & \text{N} & 7 & \text{U} & 2 & \text{N} & 1 & \text{N}
\end{bmatrix}
\]

Row degrees \( Z[\lambda] \)

\[
(2 + 5 + 1, 3, 1 + 3 + 4 + 7, 2 + 1) = (8, 3, 15, 3).
\]
Example: Row degrees of a zigzag matrix and its dual

Example of forward Zigzag matrix:

$$Z(\lambda) = \begin{bmatrix} 1 & \lambda^2 & \lambda^7 & \lambda^8 \\ \lambda^3 & 1 & \lambda & \lambda^4 & \lambda^8 \\ 1 & \lambda & \lambda^2 & \lambda^3 \\ \end{bmatrix}$$

$$S = \begin{bmatrix} U & 2 & N & 5 & N & 1 & U & 3 & U & 1 & N & 3 & N & 4 & N & 7 & U & 2 & N & 1 & N \end{bmatrix}$$

Row degrees $Z[\lambda]$

$$(2 + 5 + 1, 3, 1 + 3 + 4 + 7, 2 + 1) = (8, 3, 15, 3).$$

Row degrees $Z^{\diamond}[\lambda]$

$$(2, 5, 1 + 3 + 1, 3, 4, 7 + 2, 1) = (2, 5, 5, 3, 4, 9, 1).$$
The “partial” sums of the row degrees of Zigzag matrices

The partial sums of the row degrees of $Z(\lambda)$ and $Z^\diamond(\lambda)$ can also be computed easily from the previous recipe and we get...

Corollary (Partial row degree sums of dual zigzag matrices)

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $Z^\diamond(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ are dual zigzag matrices with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively. Then:

$$\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i$$

whenever $(\alpha, \beta) \neq (m, k)$, $1 \leq \alpha \leq m$ and $1 \leq \beta \leq k$; that is, a leading partial sum of row degrees of a zigzag matrix is never equal to a leading partial sum of row degrees of its dual.

- But once this “extra” constraint is assumed,
- it is easy to solve the inverse problem for dual Zigzag matrices.
The “partial” sums of the row degrees of Zigzag matrices

The partial sums of the row degrees of $Z(\lambda)$ and $Z^\diamond(\lambda)$ can also be computed easily from the previous recipe and we get...

**Corollary (Partial row degree sums of dual zigzag matrices)**

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**Corollary (Partial row degree sums of dual zigzag matrices)**

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $Z^{\diamond}(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ are dual zigzag matrices with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively. Then:

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The “partial” sums of the row degrees of Zigzag matrices

The partial sums of the row degrees of $Z(\lambda)$ and $Z^{\Diamond}(\lambda)$ can also be computed easily from the previous recipe and we get...

**Corollary (Partial row degree sums of dual zigzag matrices)**

Suppose $Z(\lambda) \in \mathbb{F}[\lambda]^{m \times n}$ and $Z^{\Diamond}(\lambda) \in \mathbb{F}[\lambda]^{k \times n}$ are dual zigzag matrices with row degrees $(\eta_1, \eta_2, \ldots, \eta_m)$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, respectively. Then:

$$\sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i$$

whenever $(\alpha, \beta) \neq (m, k)$, $1 \leq \alpha \leq m$ and $1 \leq \beta \leq k$; that is, a leading partial sum of row degrees of a zigzag matrix is never equal to a leading partial sum of row degrees of its dual.

- But once this “extra” constraint is assumed,
- it is easy to solve the inverse problem for dual Zigzag matrices.
Theorem

Let \((\eta_1, \ldots, \eta_m)\) and \((\varepsilon_1, \ldots, \varepsilon_k)\) be two lists of positive integers such that

\[
\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i \neq \sum_{i=1}^{\beta} \varepsilon_i,
\]

for \((\alpha, \beta) \neq (m, k), \ 1 \leq \alpha \leq m \ \text{and} \ 1 \leq \beta \leq k.\)

Then

- there exists a unique forward-zigzag matrix \(Z(\lambda) \in \mathbb{F}[\lambda]^{m \times (m+k)}\) with row degrees \((\eta_1, \ldots, \eta_m)\) such that,

- its dual backward-zigzag matrix \(Z^\diamond(\lambda) \in \mathbb{F}[\lambda]^{k \times (m+k)}\) has row degrees \((\varepsilon_1, \ldots, \varepsilon_k)\).

In addition, the structure sequence of \(Z(\lambda)\) is constructed via the following simple procedure:
Let \((\eta_1, \ldots, \eta_m)\) and \((\varepsilon_1, \ldots, \varepsilon_k)\) be two lists of positive integers such that

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\]

for \((\alpha, \beta) \neq (m, k), 1 \leq \alpha \leq m \text{ and } 1 \leq \beta \leq k\).

Then

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In addition, the structure sequence of \(Z(\lambda)\) is constructed via the following simple procedure:
Solving the inverse problem for dual Zigzag matrices: Construction

**Example:** \((\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3), (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)\).

1. Define the partial sums \(l_0 := 0,\)

\[
\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i, \quad \alpha = 1, 2, 3, \quad \text{and} \quad r_\beta := \sum_{i=1}^{\beta} \varepsilon_i, \quad \beta = 1, \ldots, 7.
\]

2. Order them in two lists

\[
\begin{bmatrix}
\ell_0 & \ell_1 & \ell_2 & \ell_3 \\
0 & 8 & 11 & 26
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\
2 & 7 & 12 & 15 & 19 & 28 & 29
\end{bmatrix}.
\]

3. Merge both lists in one ordered list

\[
\begin{bmatrix}
l_0 & r_1 & r_2 & l_1 & l_2 & r_3 & r_4 & r_5 & l_3 & r_6 & r_7 \\
0 & 2 & 7 & 8 & 11 & 12 & 15 & 19 & 26 & 28 & 29
\end{bmatrix}.
\]

4. Replacements \(l_i \rightarrow U, \quad r_j \rightarrow N\) gives unit column sequence of \(Z(\lambda)\):

\[
U, N, N, U, U, N, N, N, U, N, N
\]

5. Differences of consecutive terms gives the degree gap sequence of \(Z(\lambda)\):

\[
2, 5, 1, 3, 1, 3, 4, 7, 2, 1
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Example: \((\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3), (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)\).  

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Solving the inverse problem for dual Zigzag matrices: Construction

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\[2, 5, 1, 3, 1, 3, 4, 7, 2, 1\]
Solving the inverse problem for dual Zigzag matrices: Construction

Example: \((\eta_1, \eta_2, \eta_3, \eta_4) = (8, 3, 15, 3), (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_7) = (2, 5, 5, 3, 4, 9, 1)\).

1. Define the partial sums \(\ell_0 := 0\),

\[
\ell_\alpha := \sum_{i=1}^{\alpha} \eta_i, \quad \alpha = 1, 2, 3, \quad \text{and} \quad r_\beta := \sum_{i=1}^{\beta} \varepsilon_i, \quad \beta = 1, \ldots, 7.
\]

2. Order them in two lists

\[
\begin{bmatrix}
\ell_0 & \ell_1 & \ell_2 & \ell_3 \\
0 & 8 & 11 & 26
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 \\
2 & 7 & 12 & 15 & 19 & 28 & 29
\end{bmatrix}.
\]

3. Merge both lists in one ordered list

\[
\begin{bmatrix}
\ell_0 & r_1 & r_2 & \ell_1 & \ell_2 & r_3 & r_4 & r_5 & \ell_3 & r_6 & r_7 \\
0 & 2 & 7 & 8 & 11 & 12 & 15 & 19 & 26 & 28 & 29
\end{bmatrix}.
\]

4. Replacements \(\ell_i \rightarrow U, \quad r_j \rightarrow N\) gives unit column sequence of \(Z(\lambda)\):

\[U,N,N,U,U,N,N,N,U,N,N\]

5. Differences of consecutive terms gives the degree gap sequence of \(Z(\lambda)\):

\[2, 5, 1, 3, 1, 3, 4, 7, 2, 1\]
The general inverse problem for dual minimal bases

- If the prescribed lists \((\eta_1, \ldots, \eta_m)\) and \((\varepsilon_1, \ldots, \varepsilon_k)\) of degrees satisfy

\[
\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \sum_{i=1}^{\alpha} \eta_i = \sum_{i=1}^{\beta} \varepsilon_i,
\]

for at least one \((\alpha, \beta) \neq (m, k)\),

- then the problem has to be solved by splitting it into smaller subproblems,

- such that each of them is solved via dual Zigzag matrices,

- and the general solution is the direct sum of the smaller Zigzag solutions.

- Zero row degrees (if any) are easily incorporated by placing them in the first positions of the lists and by using \([I \ 0]\) and \([0 \ I]\) matrices in the direct sums.
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\sum_{i=1}^{m} \eta_i = \sum_{i=1}^{k} \varepsilon_i \quad \text{and} \quad \alpha \sum_{i=1}^{m} \eta_i = \beta \sum_{i=1}^{k} \varepsilon_i,
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Outline

1. Preliminary concepts: Minimal Indices and Bases
2. Forney's theorem on dual minimal bases: Goal of the talk
3. Polynomial Zigzag matrices
4. Solving the inverse problem for dual minimal bases
5. Conclusions and Applications
We have found an explicit simple solution of the inverse row degree problem for dual minimal bases via the new class of Zigzag matrices. This solution has been used (or is being used) by us and others (Lawrence, Van Barel, ...) for:

- constructing strong linearizations and \( \ell \)-ifications of matrix polynomials with certain desired properties,

- in backward error analyses of numerical algorithms for solving polynomial eigenvalue problems via linearizations, and

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