Volker Mehrmann and modern factorizations of symplectic matrices

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Conference in Honor of Volker Mehrmann
on the Occasion of his 60th Birthday

Technische Universität Berlin. 6 May 2015

Why this "symplectic" topic? (I)

- I would like to talk on how Volker's research has influenced my own research, and
- also on a topic where Volker's achievements have had a strong impact.
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$S \in \mathbb{R}^{2n\times 2n}$ is **symplectic** if

$$S^T J S = J,$$

where

$$J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

- All partitions we consider have 2×2 blocks of size $n \times n$

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad \text{with} \quad S_{ij} \in \mathbb{R}^{n\times n}$$

- From the definition, it is obvious that for every symplectic matrix

$$\det(S) = \pm 1$$

- We consider only real matrices in this talk.

Basic definitions and notations

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009

A SYMPLECTIC ORTHOGONAL METHOD FOR SINGLE INPUT OR SINGLE OUTPUT DISCRETE TIME OPTIMAL QUADRATIC CONTROL PROBLEMS*

VOLKER MEHRMANN†

Abstract. A new, numerically stable, structure preserving method for the discrete linear quadratic control problem with single input or single output is introduced, which is similar to Byers' method in the continuous case and faster than the general QZ-algorithm approach of Pappas, Laub, and Sandell.

Proposition 2.36 in 1988-SIMAX-paper by Volker

Another important tool in the study of symplectic pencils/matrices is the following.

PROPOSITION 2.36. *Let*

$$S=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in S_{2n},$$

and suppose that S_{22}^{-1} exists. Then S can be factored into the following product of three symplectic factors:

(2.37) $$S=\begin{bmatrix} I & S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} S_{22}^{-*} & 0 \\ 0 & S_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ S_{22}^{-1}S_{21} & I \end{bmatrix}.$$

Note, that if S_{11} is invertible, then we obtain the analogous factorization

(2.38) $$S=\begin{bmatrix} I & 0 \\ S_{21}S_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} S_{11} & 0 \\ 0 & S_{11}^{-*} \end{bmatrix}\begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}.$$

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Restatement and consequence of Proposition 2.36-1988

Block LDU of symplectic matrices (Prop. 2.36 , Mehrmann, SIMAX, 1988)

Let

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

be symplectic and S_{11} be nonsingular. Then

$$S = \begin{bmatrix} I & 0 \\ S_{21}S_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & S_{11}^{-T} \end{bmatrix} \begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix},$$

where the three factors are symplectic, equivalently,

where $S_{21}S_{11}^{-1}$ and $S_{11}^{-1}S_{12}$ are symmetric matrices.

Proof. Easy.

Corollary

If $S \in \mathbb{R}^{2n \times 2n}$ is symplectic and S_{11} is nonsingular, then

$$\det(S) = +1$$

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$$S = \begin{bmatrix} I & 0 \\ S_{21}S_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} S_{11} & 0 \\ 0 & S_{11}^{-T} \end{bmatrix} \begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix},$$

where the three factors are symplectic, equivalently,

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Proof. Easy.

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If $S \in \mathbb{R}^{2n\times 2n}$ is symplectic and S_{11} is nonsingular, then

$$\det(S) = +1$$

Restatement and consequence of Proposition 2.36-1988

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And if S_{11} is singular??

The Complementary Bases Theorem (D. and Johnson, LAA, 2006)

Let $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$ be symplectic,

rank $S_{11} = k < n$, $\alpha \subseteq \{1, \ldots, n\}$ with $|\alpha| = k$, and $\alpha' \cup \alpha = \{1, \ldots, n\}$.

Assume that

$$\text{rank}\, S_{11}(\alpha, :) = k$$

Then

$$\begin{bmatrix} S_{11}(\alpha, :) \\ S_{21}(\alpha', :) \end{bmatrix} \in \mathbb{R}^{n \times n} \quad \text{is invertible}$$

Example:

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

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Symplectic interchanges (used since the 70's or before...)

Definition

Let $1 \leq j \leq n$. The symplectic interchange matrix

$$\Pi_j \in \mathbb{R}^{2n \times 2n}$$

is the matrix obtained

- by interchanging the rows j and $j+n$ of I_{2n}
- and multiplying the $(j+n)$th row by -1.

Example with $n = 2$ and $j = 2$:

$$\Pi_2 = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

Remarks

- $\det(\Pi_j) = +1$ and Π_j is symplectic.

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Block LDU parametrization of the set of symplectic matrices

Combining Prop. 2.36-Volker-1988 and Complementary Bases Thm. :

Theorem (D. and Johnson, SIMAX, 2009)

The set of $2n \times 2n$ real symplectic matrices is

$$\mathcal{S} = \left\{ Q \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & G^{-T} \end{bmatrix} \begin{bmatrix} I & E \\ 0 & I \end{bmatrix} : \begin{array}{c} G \in \mathbb{R}^{n\times n} \text{ nonsingular} \\ C = C^T \ , \ E = E^T \\ Q \text{ product of symp. interch.} \end{array} \right\},$$

and the **four factors are symplectic with determinant $+1$**. Also

$$\mathcal{S} = \left\{ Q \begin{bmatrix} G & GE \\ CG & G^{-T} + CGE \end{bmatrix} : \begin{array}{c} G \in \mathbb{R}^{n\times n} \text{ nonsingular} \\ C = C^T \ , \ E = E^T \\ Q \text{ product of symp. interch.} \end{array} \right\}$$

The number of needed symplectic interchanges ranges from 0 to n.

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Outline

Symplectic-orthogonal matrices

- **A main line in** Mackey, Mackey, Mehrmann, unpublished, 2006 **(MMM-2006) is**
- **to present very simple and elegant proofs of**

Proposition

If $Q \in \mathbb{R}^{2n \times 2n}$ is symplectic and orthogonal, then $\det Q = +1$.

- (one of these proofs was presented previously in Bunse-Gerstner, Byers, Mehrmann, SIMAX, 1992)
- **and**, then, to present and prove (sometimes via new algorithmic ways) a number of
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- **Symplectic QR-like Factorization.** Any symplectic matrix $S \in \mathbb{R}^{2n \times 2n}$ can be factored as

$$S = Q \underbrace{\begin{bmatrix} R & Z \\ 0 & R^{-T} \end{bmatrix}}_{\text{symplectic}}, \quad \text{with} \left\{ \begin{array}{l} Q \text{ symplectic orthogonal} \\ R \in \mathbb{R}^{n \times n} \text{ upper triangular} \end{array} \right.$$

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Symplectic-Orthogonal factorizations of symplectics in MMM-2006

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(Xu, LAA, 2003)

Outline

- A considerable part of MMM-2006 is devoted to symplectic analogues of Householder reflectors, called **symplectic reflectors**.
- **Symplectic reflectors were used in Volker's Master Thesis (1979)** to construct an **SR-algorithm** for the eigenproblem of general matrices.
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Symplectic reflectors and Symplectic-Triangular-form

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Conclusions

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