Strong linearizations of rational matrices: theory and explicit constructions

Froilán M. Dopico

joint work with **Agurtzane Amparan** (U. País Vasco, Spain),
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- Given a nonsingular rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times p}$ (in practice $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) the rational eigenvalue problem (REP) consists in computing numbers $\lambda_0 \in \mathbb{F}$ and vectors $x_0 \in \mathbb{F}^p$ such that

$$G(\lambda_0)x_0 = 0$$

- REPs appear in different applications. Examples can be found for instance in
  1. Mehrmann & Voss. GAMM-Reports, 2004,
  2. Su & Bai. SIMAX, 2011,
  3. Mohammadi & Voss, submitted, 2016.
- Example from Mehrmann & Voss, 2004: Damped vibration of a structure.

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{\sigma_i}{\lambda + \sigma_i} L_i L_i^T,$$

$M, K \in \mathbb{R}^{p\times p}$ symmetric positive definite, $L_i \in \mathbb{R}^{p\times r_i}$, $r_i \ll p$ (rational part with low rank common in applications), $\sigma_i > 0$.

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Setting (I): Rational eigenvalue problems (REPs)

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Su & Bai, SIMAX 2011

1. Su & Bai (2011) write (via easy manipulations) rational matrices as the one in previous slide as

$$G(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 + C(\lambda E - A)^{-1}B \ \in \mathbb{F}(\lambda)^{p\times p},$$

with $E \in \mathbb{F}^{n\times n}$ nonsingular.

2. Then, they construct

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda E - A & 0 & 0 & \ldots & 0 & B \\ \hline -C & \lambda D_q + D_{q-1} & D_{q-2} & \cdots & D_1 & D_0 \\ 0 & -I_p & \lambda I_p & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & -I_p & \lambda I_p \end{array}\right],$$

3. and compute the eigenvalues of $G(\lambda)$ as the eigenvalues of the pencil $L(\lambda)$. They can also recover eigenvectors (no eigenvectors in this talk).

Setting (II): Solution via Linearization + QZ (or other method)
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Setting (III): Open problems suggested by Su & Bai's paper (SIMAX 2011)

- Su & Bai's paper is a pioneer contribution that introduces a new, robust, and clear way to compute eigenvalues of REPs, but
- the provided theory is not complete (although works well in most practical scenarios). More precisely:
- due to the lack of a key technical assumption on $C(\lambda E - A)^{-1}B$, it is not guaranteed that all (finite) eigenpairs of the rational matrix $G(\lambda)$ can be obtained from the (finite) eigenpairs of the linearization $L(\lambda)$;
- in case of multiple eigenvalues, it is not proved that they have the same partial multiplicities in the rational matrix $G(\lambda)$ and in the linearization $L(\lambda)$;
- only linearizations without eigenvalues at ∞ are considered, and no relation is established with the structure at infinite of the rational matrix $G(\lambda)$;
- no rigorous definition is provided for "linearization" of a rational matrix and/or the properties it must satisfy;
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Setting (IV): Contribution by Alam & Behera (SIMAX 2016) (Behera's PhD Thesis 2014)

- These authors take care of many of the open problems suggested by Su & Bai's paper.
- They provide a clear definition of when a pencil, i.e., a linear matrix polynomial, is a linearization of a square rational matrix that may be regular or singular.
- Their definition guarantees that the complete structures of finite zeros and finite poles of the rational matrix are inside the linearization, which allows us to get from the linearization the finite eigenvalues (those finite zeros that are not poles) including partial multiplicities.
- They provide many other examples of Fiedler-like linearizations of rational matrices, apart from the unique one presented by Su & Bai.

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Despite the very important advances made by Alam & Behera some fundamental issues remain unsolved:

1. No connection is established at all between the structure at infinity of the rational matrix and the one of the linearizations proposed so far, and the available definition does not seem amenable for getting this.
2. Rectangular rational matrices have not been considered.
3. The available definition does not guarantee that the transfer function of the linearization is "equivalent" to the original rational matrix. So, though the eigenvalues are in the linearization, other interesting properties can be missed.

In this scenario, **our goals are...**

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Goals of the talk

- To provide a definition of strong linearization of an arbitrary rational matrix that guarantees that the complete structures of finite and infinite zeros and poles of the rational matrix are inside the linearization.
- To emphasize that such definition guarantees that the "transfer" function of any strong linearization is "equivalent" (finite and at infinity) to the given rational matrix.
- To present infinitely many examples of such strong linearizations immediately constructible if the rational matrix is given in the form mentioned before, i.e.,

$$G(\lambda) = D_q\lambda^q + D_{q-1}\lambda^{q-1} + \cdots + D_0 + C(\lambda E - A)^{-1}B \ \in \mathbb{F}(\lambda)^{p\times m},$$

or even if the polynomial part is expressed in some other important different bases

$$G(\lambda) = D_q b_q(\lambda) + D_{q-1} b_{q-1}(\lambda) + \cdots + D_0 + C(\lambda E - A)^{-1}B,$$

whenever $C(\lambda E - A)^{-1}B$ is a **minimal order state-space realization**.

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Always in my mind: strong linearizations of polynomial matrices

- Very active area of research in the last decade: closely related to numerical algorithms for polynomial eigenproblems,
- even in the large-scale setting via Arnoldi methods for such problems: SOAR (Bai & Su, 2005), Q-Arnoldi (Meerbergen, 2008), TOAR (Su & Bai & Lu, 2008, 2016), Chebyshev basis (Kressner & Roman, 2014), CORK (Van Beeumen & Meerbergen & Michiels, 2015),...
- A linearization for $D(\lambda) = D_d \lambda^d + \cdots + D_1\lambda + D_0$ is a **matrix pencil** $\mathcal{L}(\lambda)$, such that,

$$U(\lambda)\,\mathcal{L}(\lambda)\,V(\lambda) = \begin{bmatrix} I_s & \\ & D(\lambda) \end{bmatrix} \qquad (U(\lambda), V(\lambda) \text{ unimodular}).$$

- $\mathcal{L}(\lambda)$ is a "strong linearization" if, **in addition**, $\mathrm{rev}\,\mathcal{L}(\lambda)$ is a linearization for $\mathrm{rev}\,P(\lambda)$, where $\mathrm{rev}\,D(\lambda) := D_0\,\lambda^d + \cdots + D_{d-1}\,\lambda + D_d = \lambda^d\,D(1/\lambda)$.

$D(\lambda)$ and $\mathcal{L}(\lambda)$ have the same finite and infinite elementary divisors.

- Our definition of strong linearization for rational matrices is motivated by and collapses to the one for polynomial matrices.

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Outline

1. Basics on rational matrices with emphasis on structure at infinity
2. Definition of strong linearizations of rational matrices
3. Equivalent characterizations of strong linearizations
4. Explicit constructions of many strong linearizations

Outline

1. **Basics on rational matrices with emphasis on structure at infinity**
2. Definition of strong linearizations of rational matrices
3. Equivalent characterizations of strong linearizations
4. Explicit constructions of many strong linearizations

Polynomial and strictly proper parts of a rational matrix

- A rational matrix $G(\lambda)$ is a matrix whose entries are rational functions with coefficients in $\mathbb{F}$.
- **Any rational matrix $G(\lambda)$ can be uniquely expressed as**

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda),$$

where

1. $D(\lambda)$ is a polynomial matrix (polynomial part), and
2. the rational matrix $G_{sp}(\lambda)$ is **strictly proper** (strictly proper part), i.e., $\lim_{\lambda\to\infty} G_{sp}(\lambda) = 0$.

- This decomposition is often immediately available in applications (Merhmann & Voss, 2004):

$$G(\lambda) = \lambda^2 M + K - \sum_{i=1}^{k} \frac{\sigma_i}{\lambda + \sigma_i} L_i L_i^T,$$

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Definition

The **Smith-McMillan form** of a rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ is the following diagonal matrix obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)G(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(m-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(p-r)\times r} & & 0_{(p-r)\times(m-r)} \end{array}\right].$$

- $\varepsilon_1(\lambda), \ldots, \varepsilon_r(\lambda), \psi_1(\lambda), \ldots, \psi_r(\lambda)$ are monic polynomials,
- the so-called **invariant fractions** $\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)}$ are irreducible and **unique**,
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Definition (finite zeros, finite poles, finite eigenvalues)

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Definition (structural indices)

Given any $c \in \overline{\mathbb{F}}$, one can write for each $i = 1, \ldots, r$,

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Then, the sequence of structural indices of $G(\lambda)$ at c is

$$S(G, c) = (\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c)).$$

Finite zeros, finite poles, and finite eigenvalues of a Rational Matrix

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The matrix

$$G(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

has the Smith-McMillan form

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and the sequences of structural indices are (rank$(G) = 5$)

- $S(G,1) = (-1,-1,0,0,2)$,
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Example: sequences of structural indices at finite values

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Definition

Let $G(\lambda)$ be a rational matrix. Then, the pole-zero structure of $G(\lambda)$ at $\lambda = \infty$ is the pole-zero structure of $G(1/\lambda)$ at $\lambda = 0$.

More precisely, the sequence of structural indices of $G(\lambda)$ at $\lambda = \infty$ is the sequence of structural indices of $G(1/\lambda)$ at $\lambda = 0$.

Proposition

Let us express the rational matrix $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ as

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda), \quad \text{where}$$

$D(\lambda)$ is its polynomial part and $G_{sp}(\lambda)$ is its strictly proper part.

1. If $D(\lambda) \neq 0$, then $-\deg(D)$ is the smallest structural index of $G(\lambda)$ at infinity.
2. If $D(\lambda) = 0$, then the smallest structural index of $G(\lambda)$ at infinity is positive.

KEY Remark

This proposition has an **important impact on how to define strong linearizations of rational matrices** since **rational matrices with polynomial parts of different degrees cannot have the same structure at infinity**.

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KEY PROPERTY on the smallest structural index at infinity

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The change of variable $G(\lambda) \to G(1/\lambda)$ is not needed: biproper matrices

Definition (Biproper matrices)

A square rational matrix is biproper if

- for all its entries, the degree of the numerator is smaller than or equal to the degree of the denominator (that is, the entries are proper rational functions), and
- its determinant is a nonzero rational function whose numerator and denominator have the same degree.

Theorem (Vardulakis, 1991; Amparan, Marcaica, Zaballa, 2015)

Let $G(\lambda), R(\lambda) \in \mathbb{F}(\lambda)^{p \times m}$ be two rational matrices. Then the following statements are equivalent:

1. *$G(\lambda)$ and $R(\lambda)$ have the same structural indices at ∞.*
2. *There exist two biproper matrices $B_1(\lambda)$ and $B_2(\lambda)$ such that*

$$G(\lambda) = B_1(\lambda)\, R(\lambda)\, B_2(\lambda)\,.$$

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Definition

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$, let

$$g = \begin{cases} -\text{degree of polynomial part of } G(\lambda), \\ 0 \ \ \text{if } G(\lambda) \text{ has not polynomial part}, \end{cases}$$

and let

$$n = \text{least order of strictly proper part of } G(\lambda).$$

A **strong linearization of $G(\lambda)$ is a matrix pencil**

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))}$$

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such that the following conditions hold:

(a) if $n > 0$ then $\det(A_1\lambda + A_0) \neq 0$, and

(b) if $\widehat{G}(\lambda) = (D_1\lambda + D_0) + (C_1\lambda + C_0)(A_1\lambda + A_0)^{-1}(B_1\lambda + B_0)$ and $\widehat{g}$ is the corresponding quantity of $\widehat{G}(\lambda)$ then:

- **(i)** there exist unimodular matrices $U_1(\lambda)$, $U_2(\lambda)$ such that

$$U_1(\lambda)\ \mathrm{diag}(G(\lambda), I_s)\, U_2(\lambda) = \widehat{G}(\lambda), \quad \text{and}$$

- **(ii)** there exist biproper matrices $B_1(\lambda)$, $B_2(\lambda)$ such that

$$B_1(\lambda)\ \mathrm{diag}(\lambda^g\, G(\lambda), I_s)\, B_2(\lambda) = \lambda^{\widehat{g}}\, \widehat{G}(\lambda).$$

Comment on condition (ii) of previous definition

A completely equivalent definition is obtained if condition **(ii)** in previous slide is replaced by

(equivalent definition)

(ii)' there exist unimodular matrices $W_1(\lambda)$, $W_2(\lambda)$ such that

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This resembles the definition of strong linearizations of rational matrices through “reversals”.

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Strong linearizations contain the complete zero/pole structure

Theorem (Spectral characterization of strong linearizations)

Let $G(\lambda) \in \mathbb{F}(\lambda)^{p\times m}$ and n be the least order of the strictly proper part of $G(\lambda)$. Let

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))},$$

with A_1 invertible. Then $L(\lambda)$ **is a strong linearization of $G(\lambda)$ if and only if** *the following two conditions hold:*

(I) *$G(\lambda)$ and $L(\lambda)$ have the same number of left and the same number of right minimal indices, and*

(II) *$L(\lambda)$ preserves the finite and infinite structures of poles and zeros of $G(\lambda)$.*

For simplicity, we assume that $G(\lambda)$ has (non-zero) polynomial part, and that $D_1 + C_1 A_1^{-1} B_1 \neq 0$ (these assumptions are not essential).

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \quad \text{versus} \quad G(\lambda)$$

1. The nontrivial invariant polynomials of $A_1\lambda + A_0$ are the nontrivial denominators of the Smith-McMillan form of $G(\lambda)$ (eigenvalues of $A_1\lambda + A_0 \equiv$ finite poles of $G(\lambda)$).
2. The nontrivial invariant polynomials of $L(\lambda)$ are the nontrivial numerators of the Smith-McMillan form of $G(\lambda)$ (eigenvalues of $L(\lambda) \equiv$ finite zeros of $G(\lambda)$).
3. If $r = \operatorname{rank}(G(\lambda))$, then $n + s + r = \operatorname{rank}(L(\lambda))$. If $0 \leq e_1 \leq \cdots \leq e_{n+s+r}$ are the partial multiplicities of $L(\lambda)$ at infinity, then $e_i = 0$ for $i = 1, \ldots, n+s$ and

$$e_{n+s+i} - \text{degree polynomial part of } G(\lambda), \quad i = 1, \ldots, r,$$

are the structural indices at infinity of $G(\lambda)$ (∞**-structs. related by shift**).

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The meaning of (II): "...preserves the finite and infinite structures..."

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For simplicity, we assume that $G(\lambda)$ has (non-zero) polynomial part, and that $D_1 + C_1 A_1^{-1} B_1 \neq 0$ (these assumptions are not essential).

$$L(\lambda) = \begin{bmatrix} A_1\lambda + A_0 & B_1\lambda + B_0 \\ -(C_1\lambda + C_0) & D_1\lambda + D_0 \end{bmatrix} \in \mathbb{F}[\lambda]^{(n+(p+s))\times(n+(m+s))} \quad \text{versus} \quad G(\lambda)$$

1. The nontrivial invariant polynomials of $A_1\lambda + A_0$ are the nontrivial denominators of the Smith-McMillan form of $G(\lambda)$ (eigenvalues of $A_1\lambda + A_0 \equiv$ finite poles of $G(\lambda)$).
2. The nontrivial invariant polynomials of $L(\lambda)$ are the nontrivial numerators of the Smith-McMillan form of $G(\lambda)$ (eigenvalues of $L(\lambda) \equiv$ finite zeros of $G(\lambda)$).
3. If $r = \text{rank}(G(\lambda))$, then $n + s + r = \text{rank}(L(\lambda))$. If $0 \leq e_1 \leq \cdots \leq e_{n+s+r}$ are the partial multiplicities of $L(\lambda)$ at infinity, then $e_i = 0$ for $i = 1, \ldots, n + s$ and

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Outline

1. Basics on rational matrices with emphasis on structure at infinity
2. Definition of strong linearizations of rational matrices
3. **Equivalent characterizations of strong linearizations**
4. Explicit constructions of many strong linearizations

- Our definition of strong linearization of a rational matrix is based on requiring the “equivalence” (finite and at infinity) between the original rational matrix and the “transfer function” of the pencil called “strong linearization”.
- This approach guarantees that all the information of the original rational problem, including the finite and infinite zero/pole structures, is recorded in the “strong linearization”.
- However, it is not easy to work directly with this definition.
- Therefore, we have developed equivalent characterizations of strong linearizations based on
  1. polynomial system matrices of rational matrices and
  2. two new classes of equivalence relations between them: transfer system equivalence and transfer system equivalence at infinity.
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Characterizations in terms of polynomial system matrices

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Outline

1 Basics on rational matrices with emphasis on structure at infinity

2 Definition of strong linearizations of rational matrices

3 Equivalent characterizations of strong linearizations

4 **Explicit constructions of many strong linearizations**

(1) Polynomial ($D(\lambda)$) and strictly proper parts ($G_{sp}(\lambda)$) of the rational matrix.

$$G(\lambda) = D(\lambda) + G_{sp}(\lambda) \in \mathbb{F}(\lambda)^{p\times m}\,.$$

Given in many applications of REPs.

(2) A minimal order state-space realization of $G_{sp}(\lambda)$:

$$G_{sp}(\lambda) = C(\lambda I_n - A)^{-1}B\,.$$

That is to say:

$$\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \cdots A^{n-1}B \end{bmatrix} = n, \quad \operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

"Almost" given in many applications of REPs where $n \ll \min\{p, m\}$ and $\operatorname{rank} B = n$ and $\operatorname{rank} C = n$. (If not, use algorithms: Rosenbrock's method (1970) stabilized by Van Dooren (1979, 1981) implemented in SLICOT (1999). There are more...)

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(3) A strong block minimal bases linearization of the polynomial part $D(\lambda)$ (D., Lawrence, Pérez, Van Dooren, 2016)

$$\mathcal{L}(\lambda) = \begin{bmatrix} M(\lambda) & K_2(\lambda)^T \\ K_1(\lambda) & 0 \end{bmatrix}$$

There are infinitely many very easily constructible: Paul's Talk, Robol & Vandebril & Van Dooren (2016), Lawrence & Pérez (2016), Fassbender & Pérez & Shayanfar (2016),...

Some "easy" constant matrices $\widehat{K}_1$ and $\widehat{K}_2$ related to $\mathcal{L}(\lambda)$ are also needed.

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Starting data for our constructions (II)

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Theorem

With the notation and hypotheses of previous slides, for any nonsingular constant matrices $X, Y \in \mathbb{F}^{n \times n}$ the linear polynomial matrix

$$L(\lambda) = \left[\begin{array}{c|cc} X(\lambda I_n - A)Y & XB\widehat{K}_1 & 0 \\ \hline -\widehat{K}_2^T CY & M(\lambda) & K_2(\lambda)^T \\ 0 & K_1(\lambda) & 0 \end{array} \right]$$

is a strong linearization of $G(\lambda)$.

Example 1. Strong linearization based on Frobenius companion linearization for polynomials

- Given rational matrix:

$$G(\lambda) = D_d\lambda^d + \cdots + D_1\lambda + D_0 + C(\lambda I_n - A)^{-1}B \in \mathbb{F}(\lambda)^{p\times m}.$$

- Strong linearization (Su & Bai (SIMAX, 2011) with minimal order state-space requirement):

$$L(\lambda) = \left[\begin{array}{c|ccccc} \lambda I_n - A & 0 & 0 & \cdots & 0 & B \\ \hline -C & \lambda D_d + D_{d-1} & D_{d-2} & \cdots & D_1 & D_0 \\ 0 & -I_m & \lambda I_m & & & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \lambda I_m & \\ 0 & & & & -I_m & \lambda I_m \end{array}\right]$$

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Example 2. Strong linearization based on Chebyshev colleague linearization for polynomials

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with polynomial part expressed in Chebyshev basis of the second kind.

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Example 3. Strong linearization based on another block Kronecker pencil

- Given rational matrix:

$$G(\lambda) = \lambda^5 D_5 + \lambda^4 D_4 + \lambda^3 D_3 + \lambda^2 D_2 + \lambda D_1 + D_0 + C(\lambda I_n - A)^{-1} B \in \mathbb{F}(\lambda)^{p \times m}$$

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