# Paul Van Dooren's Index Sum Theorem: To Infinity and Beyond

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Colloquium in honor of Paul Van Dooren
on becoming Emeritus Professor

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Louvain-la-Neuve, Belgium
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# What is this? Where can we find it?

INT. J. CONTROL, 1979, VOL. 30, NO. 2, 235–243

## Properties of the system matrix of a generalized state-space system†

G. VERGHESE‡, P. VAN DOOREN§ and T. KAILATH‡

For an irreducible polynomial system matrix $P(s)=\begin{bmatrix} T(s) & -U(s) \\ V(s) & W(s) \end{bmatrix}$, Rosenbrock (1970, p. 111) has shown that the *polar* structure of the associated transfer function $R(s)=V(s)T^{-1}(s)U(s)$ at any finite frequency is isomorphic to the zero structure of $T(s)$ at that frequency, while the *zero* structure of $R(s)$ at any finite frequency is isomorphic to that of $P(s)$ at the same frequency. In this paper we obtain the appropriate extensions for the structure *at infinite frequencies* in the particular case of systems for which $T(s)=sE-A$ (with $E$ possibly singular), $U(s)=B$, $V(s)=C$, and $W(s)=D$, under a strengthened irreducibility condition. We term such systems *generalized state-space systems*, and note that *any* rational $R(s)$ may be realized in this form. We also demonstrate in this case that a minimal basis (in the sense of Forney (1975) for the left or right null space of $P(s)$ directly generates one with the same minimal indices for the corresponding null space of $R(s)$, and vice versa. These results also enable us to identify the pole-zero excess of $R(s)$ as being equal to the sum of the minimal indices of its null spaces. Connections with Kronecker's theory of matrix pencils are made.

# Why do I call this result "Paul's" Index Sum Theorem?

The following theorem†, whose proof we merely outline for lack of space, demonstrates an important consequence of the preceding two theorems.

*Theorem* 3

Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite *and* infinite) respectively of an arbitrary rational matrix $R(s)$, and let $\alpha(R)$ denote the sum of the minimal indices of the left and right null spaces of $R(s)$. Then

$$\delta_p(R) = \delta_z(R) + \alpha(R) \tag{21}$$

...

† First obtained, in a slightly different way, by Van Dooren, in earlier unpublished research.

G. Verghese, P. Van Dooren, T. Kailath, Int. J. Control, 1979, Vol 30, page 241.

# ...whose proof we merely outline for lack of space...

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$$\delta_p(R) = \delta_z(R) + \alpha(R) \tag{21}$$

## ...outline for lack of space...



(We refer the reader to Verghese *et al.* (1980) for further details.) A consequence of our results is that the pencil (26) may be used to obtain structural information on $R(s)$. The process of obtaining a pencil that contains this information on $R(s)$ may be termed *linearization* of $R(s)$.

ACKNOWLEDGMENTS

The authors would like to thank Professor S. Kung of University of Southern California for several useful discussions.

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FORNEY, G. D., 1975, *SIAM J. Control*, 13, 493.
GANTMACHER, F. R., 1960, *The Theory of Matrices* (New York : Chelsea).
KUNG, S., and KAILATH, T., 1979, *I.E.E.E. Conference on Decision and Control*, San Diego, U.S.A.
McMILLAN, B., 1952, *Bell Syst. tech. J.*, 31, 541.
ROSENBROCK, H. H., 1970, *State-Space and Multivariable Theory* (New York : Wiley); 1974, *Int. J. Control*, 20, 191.
THORP, J. S., 1973, *Int. J. Control*, 18, 577.
VAN DOOREN, P., 1977, Report TW34, Division of Applied Mathematics and Computer Science, Catholic University, Lenven, Belgium.
VERGHESE, G., LEVY, B., and KAILATH, T., 1980, *I.E.E.E. Trans. autom. Control*, (to be published).
VERGHESE, G., 1978, Ph.D. Dissertation, Electrical Engineering Department, Stanford University, California, U.S.A.

This is the last page of G. Verghese, P. Van Dooren, T. Kailath, Int. J. Control, 1979, Vol 30, page 243.

There is still half a page available!!

# Linearizations of rational matrices are mentioned in...



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VERGHESE, G., 1978, Ph.D. Dissertation, Electrical Engineering Department, Stanford University, California, U.S.A.

...the last page of G. Verghese, P. Van Dooren, T. Kailath, Int. J. Control, 1979, Vol 30, page 243.

ILAS 2016-Talk. Wednesday, July 13. 12:00-12:30. Strong linearizations of rational matrices. **S. Marcaida** (joint work with A. Amparán, F. Dopico, I. Zaballa). MS24 Matrix Polynomials, Matrix Functions and Applications, organized by I. Zaballa and P. Psarrakos.

# Paul's Index Sum Theorem is also in his PhD Thesis

<u>Proposition 5.10</u>

The polar and zero degree of a rational matrix and the minimal orders of its left and right null spaces satisfy the equality

$$\boxed{\delta_p(R) = \delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R)}$$

<u>Proof</u>

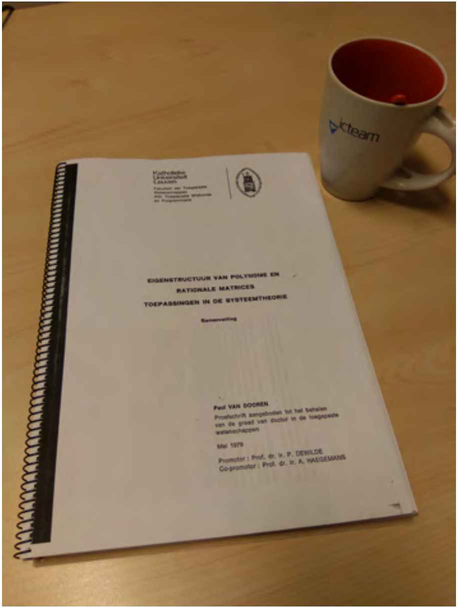
From the above remarks and theorem 3.8 it follows that ($\lambda E - A$ being regular) :

$$\delta_p(R) = \delta_z(\lambda E-A) = \delta_p(\lambda E-A)$$

$$\delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R) = \delta_z(\lambda\hat{E}-\hat{A}) + \hat{\varepsilon}(\lambda\hat{E}-\hat{A}) + \hat{\eta}(\lambda\hat{E}-\hat{A}) = \delta_p(\lambda\hat{E}-\hat{A})$$

Since $\delta_p(\lambda E-A) = \text{rank } E$ and $\delta_p(\lambda\hat{E}-\hat{A}) = \text{rank } \hat{E}$ (see theorem 3.8), we have that $\delta_p(\lambda E-A) = \delta_p(\lambda\hat{E}-\hat{A})$, which completes the proof.

▽▽▽



# ...but the proof is still shorter and the page 3.10 that should include Theorem 3.8 is missing in Paul's PhD Thesis

<u>Proposition 5.10</u>

The polar and zero degree of a rational matrix and the minimal orders of its left and right null spaces satisfy the equality

$$\boxed{\delta_p(R) = \delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R)}$$

<u>Proof</u>

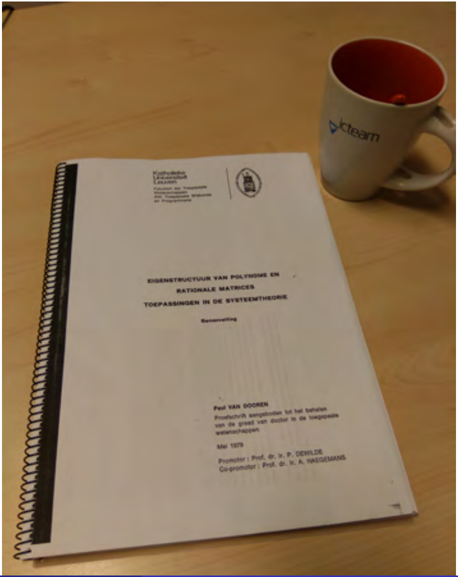
From the above remarks and theorem 3.8 it follows that ($\lambda E-A$ being regular) :

$$\delta_p(R) = \delta_z(\lambda E-A) = \delta_p(\lambda E-A)$$

$$\delta_z(R) + \hat{\varepsilon}(R) + \hat{\eta}(R) = \delta_z(\lambda\hat{E}-\hat{A}) + \hat{\varepsilon}(\lambda\hat{E}-\hat{A}) + \hat{\eta}(\lambda\hat{E}-\hat{A}) = \delta_p(\lambda\hat{E}-\hat{A})$$

Since $\delta_p(\lambda E-A)$ = rank $E$ and $\delta_p(\lambda\hat{E}-\hat{A})$ = rank $\hat{E}$ (see theorem 3.8), we have that $\delta_p(\lambda E-A) = \delta_p(\lambda\hat{E}-\hat{A})$, which completes the proof.

∇∇∇



## But, please, do not get into panic!

- **The situation is not as in "Fermat's Last Theorem"** ☺,
- since we can conjecture the statement of the "lost Theorem 3.8" in Paul's PhD Thesis,
- and even to prove such conjectured theorem.
- **Conjectured Theorem 3.8 in Paul's Thesis**. *Let* $\delta_p(\lambda B - A)$ *and* $\delta_z(\lambda B - A)$ *denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary matrix pencil* $\lambda B - A$*, and let* $\alpha(\lambda B - A)$ *denote the sum of its left and right minimal indices. Then*

$$\delta_p(\lambda B - A) = \delta_z(\lambda B - A) + \alpha(\lambda B - A)\,.$$

- In plain words, this states that Paul's Index Sum Theorem for arbitrary rational matrices holds, in particular, for matrix pencils, which can be easily proved via the Kronecker Canonical Form of matrix pencils.
- Then the proof for any rational matrix follows from Möbius transformations and the properties of "strongly irreducible generalized state-space polynomial system pencils" for proper rational matrices.

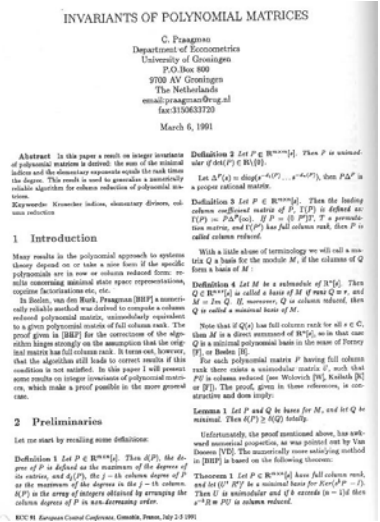
## and also Paul's tender and family personality ...

> I dedicate this thesis to Maggie, for continuously encouraging me in my work, for cheering me up in difficult moments, for her careful typing of this manuscript, for telling Valérie that her daddy is too busy to play with her right now and for so many other things; in short, for being my loving wife.

I have highlighted the most important part of this paragraph, but observe

- "for her careful typing of this manuscript"
- Would Paul's PhD Thesis exist without **Maggie?**
- Open problem...

- at least up to my limited knowledge.
- In fact, I conjecture that it remained dormant (forgotten??) even in Paul's mind.
- (**Parenthesis**: References on Paul's Index Sum Theorem, its use, and its applications are most welcome since I would like to write a "historical report" on it, because it is a fundamental result that in the last few years has been applied to the solution of some important problems for matrix polynomials and it would be surprising for me if it was not used before.)
- In 1991 the index sum theorem appears again but **only for matrix polynomials** and written in such a form that **nobody established a connection between Paul's general result** for arbitrary rational matrices and the "new theorem" by
- C. Praagman. *Invariants of polynomial matrices*. Proceedings of the First European Control Conference, Grenoble 1991. (I. Landau, Ed.) INRIA, 1274-1277, 1991.
- W. H. L. Neven and C. Praagman. *Column reduction of polynomial matrices*. Linear Algebra Appl., 188/189:569–589, 1993.



INVARIANTS OF POLYNOMIAL MATRICES

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March 6, 1991

1 Introduction

2 Preliminaries

ECC 91 European Control Conference, Grenoble, France, July 2-5 1991

**Theorem 3** *Let $P \in \mathbb{R}^{m \times n}[s]$ be a polynomial matrix of rank $r$ and degree $d$. Then the sum of its structure indices equals $rd$.*

**Proof.** It can be deduced immediately from the Kronecker normal form displayed above that the theorem holds for matrix polynomials of degree 1. The rank of $L^P$ equals $m(d-1)+r$, hence its number of left minimal indices (and that of $P$) is $m-r$.
From theorem 2 we conclude that the sum of the structure indices of $P$ equals the sum of the structure indices of $L^P$ minus $(m-r)(d-1)$, hence equals
$md - m + r - md + m + rd - r = rd$.

- **It looks very different that Paul's result**, since the rank and the degree do not appear at all in Paul's original statement.
- They appear here because **the definition of "structural indices at $\infty$" is different** that in Paul's Thesis.
- Connections with Paul's result are not mentioned.

# Same comments apply to Neven and Praagman's 1993 LAA paper

**Column Reduction of Polynomial Matrices**

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and

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Submitted by Paul Van Dooren

THEOREM 3. *Let $P \in \mathbf{R}^{m \times n}[s]$ be a polynomial matrix. Then the sum of its structural indices equals $r(P)d(P)$.*

# Next clues about Index Sum Theorems: Vera Kublanovskaya 1999 (1)

*Journal of Mathematical Sciences, Vol. 96, No. 3, 1999*

METHODS AND ALGORITHMS OF SOLVING SPECTRAL PROBLEMS FOR POLYNOMIAL AND RATIONAL MATRICES

V. N. Kublanovskaya

UDC 519

Dedicated to the memory of my son Alexander

- It is a 203-pages long survey paper (almost a book),
- which includes many results without proofs, just stated,
- among them in page 3093

*The following balance relations connecting scalar spectral characteristics of a $\lambda$-matrix hold:*

(a)

$$\gamma_p[R] = \gamma_z[R] + \varepsilon[R] + \eta[R] \qquad (1.1.21)$$

*for a rational $m \times n$ matrix $R(\lambda)$ of rank $\rho$;*

(b)

$$\beta_c[D] + \beta_\infty[D] + \varepsilon[D] + \eta[D] = \rho s \qquad (1.1.22)$$

*for a polynomial $m \times n$ matrix $D(\lambda)$ of rank $\rho$ and degree $s$.*

Here, $\gamma_p[R]$ is the sum of negative structural indices of all singular points of $R(\lambda)$; $\gamma_z[R]$ is the sum of positive structural indices of $R(\lambda)$; $\beta_c[D]$ is the sum of all finite elementary divisors of $D(\lambda)$; $\beta_\infty[D]$ is the sum of all infinite elementary divisors of $D(\lambda)$; $\varepsilon[R]$ and $\varepsilon[D]$ are the sums of all right minimal indices of the matrices $R(\lambda)$ and $D(\lambda)$, respectively; $\eta[R]$ and $\eta[D]$ are the sums of all left minimal indices of the matrices $R(\lambda)$ and $D(\lambda)$, respectively.

- Both versions of the Index Sum Theorem are stated one after the other: Paul's 1979 for arbitrary rational matrices and Praagman's 1991 for polynomial matrices,
- but **they are stated as independent results, without establishing any connection between them!!, and without proofs (surprising)**.
- The references provided (in a very vague way) are: Khazanov's PhD Thesis (1983), Paul's PhD Thesis (1979), IEEE Trans. Aut. Control. by Paul (1981)???, Wimmer Proc. AMS (1979) (NO RELATION)????
- This survey defines in a confusing way the structure at infinity.

## I do not have more news to tell about Index Sum Theorems until...

- I met Steve Mackey in a “cafetería” in Madrid in June 2009 and we figured out the “Index Sum Theorem” for matrix polynomials
- based on some hand-written notes that Fernando De Terán and myself had sent previously to Steve
- for proving “Non-existence of structured companion forms for many classes of structured matrix polynomials of even degrees”.
- At that time, we did not know Praagman’s result and even less Paul’s result, since we were strongly focused on solving polynomial eigenvalue problems via linearizations with particular emphasis on preserving relevant structures, a very active area of research in the last decade.
- So, essentially, rational matrices did not exist for us.
- During Steve’s visit in June 2009 to Madrid, we proved the Index Sum Theorem for polynomial matrices in arbitrary fields (Praagman’s for reals, Paul’s for complex), we discussed about a name for it (rank theorem, extended rank-nullity theorem,...), and later (probably in 2011) we gave such result the name of the “Index Sum Theorem”.

## But, fortunately, we delayed the publication for some reasons

- The "Index Sum Theorem" was not really our goal, we were involved in a much larger project.
- We presented talks on related results in the ILAS Conferences in Pisa (2010) and Braunschweig (2011), in particular, results related to "spectral equivalences of matrix polynomials", and, as a consequence,
- Stavros Vologiannidis recommended us to read his joint paper with N. Karampetakis, *Infinite elementary divisor structure-preserving transformations for polynomial matrices*, Int. J. Appl. Math. Comput. Sci., 2003,
- where Praagman's Index Sum Theorem for polynomial matrices is stated and Praagman's 1991 proceedings paper is cited as the source of such result (together with a 1998 paper in Kybernetika by E. Antoniou, A. Vardulakis, and N. Karampetakis).
- Again, connections with Paul's 1979 result are completely missing in Karampetakis and Vologiannidis paper,
- as well as in ...

# The following very long paper in LAA (2014)

Linear Algebra and its Applications 459 (2014) 264–333

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

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## Spectral equivalence of matrix polynomials and the Index Sum Theorem

Fernando De Terán [a,1], Froilán M. Dopico [b,*,1], D. Steven Mackey [c,2]

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[c] Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA

**Theorem 6.5** *(Index Sum Theorem for Matrix Polynomials). Suppose $P(\lambda)$ is an arbitrary $m \times n$ matrix polynomial over an arbitrary field. Then*

$$\delta_{\text{fin}}(P) + \delta_{\infty}(P) + \mu(P) = \text{grade}(P) \cdot \text{rank}(P). \tag{6.4}$$

- In between F. De Terán, D. S. Mackey, and myself rediscovered the "Index Sum Theorem" for matrix polynomials in 2009 and we submitted it in 2013, there was one remarkable activity on this result.
- In the 2011-Householder Symposium, Stefan Johansson presented the poster *Stratification of Full Normal Rank Polynomial Matrices*, joint work with Bo Kågström and Paul Van Dooren.
- In that poster a version of the Index Sum Theorem **valid only** for full rank polynomial matrices was stated and was fundamental in the proof of the stratification results presented there (although many other nontrivial results were also needed).
- At that time, we already knew the general version of the Index Sum Theorem for arbitrary matrix polynomials (not that Praagman obtained it in 1991) and we commented this fact to Stefan, which triggered a meeting of
- Stefan Johansson, Bo Kågström, Paul Van Dooren, Fernando De Terán, Steve Mackey and myself.

- Paul was willing to believe "our" Index Sum Theorem for arbitrary matrix polynomials and, much later, he told me that he liked it very much.
- (private comment during his visit to my Department from November 2013 to July 2014)
- However, at that time, Stefan Johansson, Bo Kågström, and Paul did not have a clear idea on how to use it for getting a stratification hierarchy for arbitrary (i.e., non full rank) matrix polynomials and
- the output of the meeting was that each group would cite the results of the other (as we indeed did).
- (Related comment: stratification problem for general matrix polynomials recently solved in A. Dmytryshyn, S. Johansson, B. Kågström, and P. Van Dooren, Geometry of spaces for matrix polynomial Fiedler linearizations, in preparation).

# Index sum theorem in Johansson, Kågström, Paul's paper in LAA (2013)

Linear Algebra and its Applications 439 (2013) 1062–1090

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

## Stratification of full rank polynomial matrices☆

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**Theorem 5.2.** *An $m \times n$ polynomial matrix $P(s)$ of exact degree $d$ and normal-rank $m$ has $m$ finite elementary divisors $(s - \lambda_i)^{h_j^{(i)}}$, $j = 1, \ldots, m$, for each zero $\lambda_i$, $i = 1, \ldots, q$, $m$ infinite elementary divisors $1/s^{h_j^{(\infty)}}$, and $n - m$ right minimal indices $\epsilon_j$, $j = 1, \ldots, n - m$ (some of these indices can be trivially zero) satisfying*

$$\sum_{i=1}^{q}\sum_{j=1}^{m} h_j^{(i)} + \sum_{j=1}^{m} h_j^{(\infty)} + \sum_{j=1}^{n-m} \epsilon_j = dm. \tag{14}$$

*All structures satisfying the constraints (14) are possible for such a polynomial matrix.*

**Connections with Paul's 1979 result for rational matrices are not established.**

# The connection was finally established in 2015 (2014) in...

SIAM J. MATRIX ANAL. APPL.
Vol. 36, No. 1, pp. 302–328



MATRIX POLYNOMIALS WITH COMPLETELY PRESCRIBED EIGENSTRUCTURE*

FERNANDO DE TERÁN†, FROILÁN M. DOPICO†, AND PAUL VAN DOOREN‡

THEOREM 3.1 (index sum theorem). *Let $P(\lambda)$ be an $m \times n$ matrix polynomial of degree $d$ and rank $r$ having the following eigenstructure:*

- *$r$ invariant polynomials $p_j(\lambda)$ of degrees $\delta_j$, for $j = 1, \ldots, r$,*
- *$r$ infinite partial multiplicities $\gamma_1, \ldots, \gamma_r$,*
- *$n - r$ right minimal indices $\varepsilon_1, \ldots, \varepsilon_{n-r}$, and*
- *$m - r$ left minimal indices $\eta_1, \ldots, \eta_{m-r}$,*

*where some of the degrees, partial multiplicities, or indices can be zero, and/or one or both of the lists of minimal indices can be empty. Then*

$$\sum_{j=1}^{r} \delta_j + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr. \tag{3.1}$$

*Remark* 3.2. A very interesting remark pointed out by an anonymous referee is that the index sum theorem for matrix polynomials can be obtained as an easy corollary of a more general result valid for arbitrary rational matrices, which is much older than reference [28]. This result is [36, Theorem 3], which can also be found in [18, Theorem 6.5-11]. Using the notion of structural indices at $\alpha$ introduced in

[28] is Praagman's 1991 paper; [36] Verghese, Van Dooren, Kailath's 1979 paper; [18] Kailath's 1980 book.

**Paragraph of Referee's report:**

> *"It turns out that Theorem 3.1 of the paper (the index sum theorem) can be directly derived from Theorem 6.5-11 (in Kailath's 1980 book) by using appropriately the structure (zeros and poles) of matrix polynomials at infinity and Corollary 4.41 of [A. I. G. Vardulakis, Linear multivariable control, John Wiley and Sons, New York, 1991]. Although this corollary covers only the case of non-singular matrix polynomials (and, from my point of view, its proof is not completely satisfactory), it can be rigorously proved to be true for any matrix polynomial. It relates the poles and zeros at infinity of a matrix polynomial to its degree and elementary divisors at infinity. In summary, the index sum theorem is the "index sum theorem described in [33]" (Verghese, Van Dooren, Kailath's 1979 paper) when applied to matrix polynomials as special instances of matrices of rational functions."*

**Note:** the words in blue added by F. Dopico.

**we will show how and why**

**Theorem (Paul's Index Sum Theorem for Rational Matrices)**

*Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then*

$$\delta_p(R) = \delta_z(R) + \alpha(R)\,.$$

**implies**

**Theorem (Index Sum Theorem for Polynomial Matrices)**

*Let $\delta(P)$ be the sum of the degrees of all the elementary divisors (finite and infinite) of an arbitrary polynomial matrix $P(\lambda)$, and let $\alpha(P)$ denote the sum of its left and right minimal indices. Then*

$$\delta(P) + \alpha(P) = \textit{degree}(P) \cdot \textit{rank}(P).$$

**and we will discuss some problems related to the Index Sum Theorem.**

## Outline

# Outline

## Rational matrices and polynomial matrices

- A rational matrix $R(\lambda)$ is a matrix whose entries are rational functions with coefficients in $\mathbb{C}$ (as in Paul's Thesis).
- A polynomial matrix $P(\lambda)$ is a matrix whose entries are polynomials with coefficients in $\mathbb{C}$.
- Any scalar rational function $r(\lambda) = \frac{n(\lambda)}{d(\lambda)}$ can be uniquely written via Euclidean division of $n(\lambda)$ by $d(\lambda)$ as

$$r(\lambda) = p(\lambda) + r_{sp}(\lambda), \quad \text{where}$$

  - $p(\lambda)$ is a polynomial, and
  - the rational function $r_{sp}(\lambda)$ is **strictly proper**, i.e., $\lim_{\lambda\to\infty} r_{sp}(\lambda) = 0$.
- Thus, **any rational matrix $R(\lambda)$ can be uniquely expressed as**

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \quad \text{where}$$

  - $P(\lambda)$ is a polynomial matrix (polynomial part), and
  - the rational matrix $R_{sp}(\lambda)$ is **strictly proper** (strictly proper part), i.e., $\lim_{\lambda\to\infty} R_{sp}(\lambda) = 0$.
- **Unimodular matrices** are square polynomial matrices with constant nonzero determinant.

### Definition

The **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ is the following diagonal matrix obtained under unimodular transformations $U(\lambda)$ and $V(\lambda)$:

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\varepsilon_1(\lambda), \cdots, \varepsilon_r(\lambda), \psi_1(\lambda), \cdots, \psi_r(\lambda)$ are monic polynomials,
- the fractions $\dfrac{\varepsilon_i(\lambda)}{\psi_i(\lambda)}$ are irreducible,
- $\varepsilon_j(\lambda)$ divides $\varepsilon_{j+1}(\lambda)$, for $j = 1, \ldots, r-1$,
- $\psi_{j+1}(\lambda)$ divides $\psi_j(\lambda)$, for $j = 1, \ldots, r-1$,

**Definition**

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$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \dots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$ are unique and
- are called the **FINITE invariant rational functions** of $R(\lambda)$.
- $r = \text{rank}\, G(\lambda)$.

## Definition (finite zeros and finite poles)

Given the **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$:

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right].$$

- The **finite zeros** of $R(\lambda)$ are the roots of $\varepsilon_r(\lambda)$.
- The **finite poles** of $R(\lambda)$ are the roots of $\psi_1(\lambda)$.

## Remark

Given any $c \in \mathbb{C}$, one can write for each $i = 1, \ldots, r$

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with } \widetilde{\varepsilon}_i(c) \neq 0,\ \widetilde{\psi}_i(c) \neq 0.$$

## Definition (Structural indices at $c \in \mathbb{C}$)

Let the **Smith-McMillan form** of a rational matrix $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be

$$U(\lambda)R(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)} & & & \\ & \ddots & & 0_{r\times(n-r)} \\ & & \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)} & \\ \hline & 0_{(m-r)\times r} & & 0_{(m-r)\times(n-r)} \end{array}\right],$$

and let $c \in \mathbb{C}$ be a number. Then, the sequence of structural indices of $R(\lambda)$ at $c$ is the sequence of $r = \operatorname{rank} R(\lambda)$ integers

$$S(R, c) = (\sigma_1(c), \sigma_2(c), \dots, \sigma_r(c))$$

where

$$\frac{\varepsilon_i(\lambda)}{\psi_i(\lambda)} = (\lambda - c)^{\sigma_i(c)} \, \frac{\widetilde{\varepsilon}_i(\lambda)}{\widetilde{\psi}_i(\lambda)}, \qquad \text{with} \quad \widetilde{\varepsilon}_i(c) \neq 0, \; \widetilde{\psi}_i(c) \neq 0.$$

**Proposition (properties of structural indices at $c \in \mathbb{C}$)**

Let $R(\lambda) \in \mathbb{C}(\lambda)^{m\times n}$ be a rational matrix, let $c \in \mathbb{C}$, and let

$$S(R,c) = (\sigma_1(c), \sigma_2(c), \dots, \sigma_r(c))$$

be the sequence of structural indices of $R(\lambda)$ at $c$. Then

- $\sigma_1(c) \leq \sigma_2(c) \leq \cdots \leq \sigma_r(c)$.
- If $S(R,c) = (0, \dots, 0)$, then $c$ is not a finite pole nor a finite zero.
- If $\sigma_1(c) < 0$, then $c$ is a finite pole of $R(\lambda)$.
- If $\sigma_r(c) > 0$, then $c$ is a finite zero of $R(\lambda)$.
- **Remark:** a number $c$ may be simultaneously a finite zero and a finite pole.

The matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

has the Smith-McMillan form

$$R(\lambda) \sim \begin{bmatrix} \frac{1}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & (\lambda-1)^2\lambda & 0 \end{bmatrix},$$

and the sequences of structural indices ($\text{rank}(R) = 5$)

- $S(R, 1) = (-1, -1, 0, 0, 2)$.
- $S(R, 0) = (0, 0, 0, 0, 1)$.

**Definition**

Let $R(\lambda)$ be a rational matrix. Then, the pole-zero structure of $R(\lambda)$ at $\lambda = \infty$ is the pole-zero structure of $R(1/\lambda)$ at $\lambda = 0$.

More precisely, the sequence of structural indices of $R(\lambda)$ at $\lambda = \infty$ is the sequence of structural indices of $R(1/\lambda)$ at $\lambda = 0$.

- The structural indices of $R(\lambda)$ at $\lambda = \infty$ are also called "the invariant orders at infinity of $R(\lambda)$".
- A modern, rigorous, and complete reference on this topic is: A. Amparán, S. Marcaida, and I. Zaballa, "Finite and infinite structures of rational matrices: a local approach", ELA, 2015.

## Example (continued): sequence of structural indices at infinity (I)

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

and

$$\widetilde{R}(\lambda) := R(1/\lambda) = \begin{bmatrix} \frac{1}{1-\lambda} & & & & & \\ & \frac{\lambda}{1-\lambda} & & & & \\ & & \frac{(\lambda-1)^2}{\lambda^2} & & & \\ & & & 1 & \frac{1}{\lambda^2} & \\ & & & & 1 & \frac{1}{\lambda^7} \end{bmatrix},$$

whose Smith-McMillan form is

$$\widetilde{R}(\lambda) := R(1/\lambda) \sim \begin{bmatrix} \frac{1}{\lambda^7(\lambda-1)} & & & & & \\ & \frac{1}{\lambda^2(\lambda-1)} & & & & \\ & & \frac{1}{\lambda^2} & & & \\ & & & 1 & & \\ & & & & \lambda(\lambda-1)^2 & 0 \end{bmatrix}$$

Thus, the sequence of structural indices at infinity of $R(\lambda)$ is

$$S(R,\infty) = S(\widetilde{R},0) = (-7,-2,-2,0,1).$$

## Proposition

Let us express the rational matrix $R(\lambda)$ as

$$R(\lambda) = P(\lambda) + R_{sp}(\lambda), \quad \text{where}$$

$P(\lambda)$ is its polynomial part and $R_{sp}(\lambda)$ is its strictly proper part.

1. If $P(\lambda) \neq 0$, then $-\deg(P)$ is the smallest structural index of $R(\lambda)$ at infinity.
2. If $P(\lambda) = 0$, then the smallest structural index of $R(\lambda)$ at infinity is positive.

## Remark

This result has an impact on how to define and to state the properties of strong linearizations of rational matrices (see talk in ILAS by **Marcaida**) since rational matrices whose polynomial parts have different degrees cannot have the same structural indices at infinity.

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$S(R,\infty) = ({\color{red}-7}, -2, -2, 0, 1).$$

Note

$$R(\lambda) = \underbrace{\begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & {\color{red}\lambda^7} \end{bmatrix}}_{{\color{red}P(\lambda)}} + \underbrace{\begin{bmatrix} \frac{1}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & 0 & & & \\ & & & 0 & 0 & \\ & & & & 0 & 0 \end{bmatrix}}_{R_{sp}(\lambda)}$$

## Polar and zero degrees (total numbers of poles and zeros)

**Theorem (Paul's Index Sum Theorem for Rational Matrices)**

*Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then*

$$\delta_p(R) = \delta_z(R) + \alpha(R) \, .$$

**Definition (total numbers of poles and zeros)**

- The total number of poles of $R(\lambda)$ (more formally known as the polar degree or the McMillan degree of $R(\lambda)$) is minus the sum of all negative structural indices of $R(\lambda)$ (including those at $\infty$).
- The total number of zeros of $R(\lambda)$ (more formally known as the zero degree of $R(\lambda)$) is the sum of all positive structural indices of $R(\lambda)$ (including those at $\infty$).

## Example (continued): polar and zero degrees

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$\begin{aligned} S(R,1) &= (-1,-1,0,0,2), \\ S(R,0) &= (0,0,0,0,1), \\ S(R,\infty) &= (-7,-2,-2,0,1). \end{aligned}$$

- $\delta_p(R) = 13$ **(polar degree)**.
- $\delta_z(R) = 4$ **(zero degree)**.

An $m \times n$ rational matrix $R(\lambda)$ whose rank $r$ is smaller than $m$ and/or $n$ has non-trivial left and/or right null-spaces over the field $\mathbb{C}(\lambda)$ of rational functions:

$$\begin{aligned} \mathcal{N}_\ell(R) &:= \left\{ y(\lambda)^T \in \mathbb{C}(\lambda)^{1\times m} : y(\lambda)^T R(\lambda) \equiv 0^T \right\}, \\ \mathcal{N}_r(R) &:= \left\{ x(\lambda) \in \mathbb{C}(\lambda)^{n\times 1} : R(\lambda) x(\lambda) \equiv 0 \right\}. \end{aligned}$$

$\mathcal{N}_\ell(R)$ and $\mathcal{N}_r(R)$ have bases consisting entirely of vector polynomials.

**Definition (Minimal bases)**

A **right minimal basis** of $R(\lambda)$ is a basis of $\mathcal{N}_r(R)$

1. consisting of vector polynomials
2. whose sum of degrees is minimal among all bases of $\mathcal{N}_r(R)$ consisting of vector polynomials.

Analogous definition for **left minimal basis** of $R(\lambda)$.

There are infinitely many right minimal bases of $R(\lambda)$, but...

**Theorem (Forney, 1975)**

*The ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{N}_r(R)$ is always the same.*

**Definition**

These degrees are called the **right minimal indices** of $R(\lambda)$.

Analogous definition for **left minimal indices** of $R(\lambda)$.

Consider again the matrix

$$R(\lambda) = \begin{bmatrix} \frac{\lambda}{\lambda-1} & & & & & \\ & \frac{1}{\lambda-1} & & & & \\ & & (\lambda-1)^2 & & & \\ & & & 1 & \lambda^2 & \\ & & & & 1 & \lambda^7 \end{bmatrix} \in \mathbb{C}(\lambda)^{5\times 6}$$

$$\text{rank}(R) = 5 \implies \dim \mathcal{N}_\ell(R) = 0 \quad \text{and} \quad \dim \mathcal{N}_r(R) = 1$$

- $\{[0,0,0,\lambda^9,-\lambda^7,1]^T\}$ right minimal basis of $R(\lambda)$, which has only one right minimal index equal to $9$.
- Sum of all minimal indices of $R(\lambda)$ is $\alpha(R) = 9$.
- Since $\delta_p(R) = 13$ (polar degree) and $\delta_z(R) = 4$ (zero degree),
- $$\delta_p(R) = \delta_z(R) + \alpha(R)\,,$$

  which is Paul's Index Sum Theorem in action.

# Outline

Polynomial matrices are simpler than rational matrices:

- The Smith-McMillan form reduces to the Smith form and
- the finite invariant rational functions to the invariant polynomials.
- Therefore, polynomial matrices do not have finite poles, so
- every sequence of structural indices at a finite point $\beta \in \mathbb{C}$ is nonnegative,
- and coincides with the corresponding sequence of partial multiplicities at $\beta$
- (whose nonzero entries are the degrees of the elementary divisors at $\beta$).
- The finite zeros are usually called in the polynomial context finite eigenvalues.

- However, **a polynomial matrix $P(\lambda)$ of degree $d$ has always at least one pole of order $d$ at infinity**, i.e.,
- if $\text{rank}(P) = r$, then the sequence of structural indices at infinity is

$$S(P,\infty) = \underbrace{(-d, s_2, \ldots, s_r)}_{r}, \quad \text{with} \quad -d \leq s_2 \leq \cdots \leq s_r.$$

- $P(\lambda)$ may also have zeros at infinity.

In the "community of polynomial matrices", the structure at infinity is usually defined as follows.

**Definition (Reversal polynomial)**

Let

$$P(\lambda) = P_d\lambda^d + P_{d-1}\lambda^{d-1} + \cdots + P_0, \qquad P_d \neq 0$$

be a matrix polynomial of **degree** $d$, the **reversal** of $P(\lambda)$ is

$$\mathrm{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right) = P_d + P_{d-1}\,\lambda + \cdots + P_0\,\lambda^d .$$

**Definition (Eigenvalues at $\infty$)**

$P(\lambda)$ **has an eigenvalue at $\infty$ if $0$ is an eigenvalue of $\mathrm{rev}P(\lambda)$.**

The partial multiplicity sequence of $P(\lambda)$ at $\infty$ is the same as that of $0$ in $\mathrm{rev}P(\lambda)$.

The elementary divisors for $0$ of $\mathrm{rev}P(\lambda)$ are the **elementary divisors for $\infty$** of $P(\lambda)$.

## Relation between "both concepts" of structure at infinity

**Trivial remark:** Since $\mathrm{rev}P(\lambda) := \lambda^d P\left(\frac{1}{\lambda}\right)$, the Smith form of the polynomial matrix $\mathrm{rev}P(\lambda)$ is the Smith-McMillan form of the rational matrix $P(1/\lambda)$ multiplied by $\lambda^d$.

### Proposition

Let $P(\lambda)$ be a polynomial matrix of degree $d$ and let its sequence of structural indices at infinity (the sequence at $0$ of $P(1/\lambda)$) be

$$S(P,\infty) = \underbrace{(-d, s_2, \ldots, s_r)}_{r}, \quad \text{with} \quad -d \le s_2 \le \cdots \le s_r.$$

Then:

- the partial multiplicity sequence of $P(\lambda)$ at infinity is

$$S(P,\infty) + d := (0, s_2 + d, \ldots, s_r + d), \quad \text{with} \quad 0 \le s_2 + d \le \cdots \le s_r + d.$$

- $P(\lambda)$ has an eigenvalue at infinity if and only if "$S(P,\infty) + d$" is not the zero sequence and the degrees of the elementary divisors of $P(\lambda)$ at infinity are the nonzero entries of "$S(P,\infty) + d$".

# Outline

**Theorem (Paul's Index Sum Theorem for Rational Matrices)**

*Let $\delta_p(R)$ and $\delta_z(R)$ denote the total number of poles and zeros (finite and infinite) respectively of an arbitrary rational matrix $R(\lambda)$, and let $\alpha(R)$ denote the sum of its left and right minimal indices. Then*

$$\delta_p(R) = \delta_z(R) + \alpha(R)\,.$$

If $R(\lambda) = P(\lambda)$ is a polynomial matrix of degree $d$, $r = \operatorname{rank}(P)$, and with sequence of structural indices at $\infty$ given by

$$S(P,\infty) = \underbrace{(-d, s_2, \ldots, s_r)}_{r}, \quad \text{with} \quad -d \leq s_2 \leq \cdots \leq s_k < 0 \leq s_{k+1} \leq \cdots \leq s_r,$$

then, since $P(\lambda)$ has poles only at infinity,

$$\delta_p(P) = -\left(-d + \sum_{i=2}^{k} s_i\right),$$

$$\delta_z(P) = \sum_{i=k+1}^{r} s_i + \delta_z^{finite}(P)\,.$$

Therefore,

$$\begin{aligned}
\delta_p(P) = \delta_z(P) + \alpha(P) &\Longrightarrow -\left(-d + \sum_{i=2}^{k} s_i\right) = \sum_{i=k+1}^{r} s_i + \delta_z^{finite}(P) + \alpha(P) \\
&\Longrightarrow 0 = \left(-d + \sum_{i=2}^{r} s_i\right) + \delta_z^{finite}(P) + \alpha(P) \\
&\Longrightarrow d\,r = \left(0 + \sum_{i=2}^{r} (s_i + d)\right) + \delta_z^{finite}(P) + \alpha(P)\,.
\end{aligned}$$

We have obtained easily

$$d\,r = \underbrace{\underbrace{\left(0 + \sum_{i=2}^{r} (s_i + d)\right)}_{\text{Sum degrees } \infty \text{ element. divisors}} + \delta_z^{finite}(P) + \alpha(P)}_{\text{Sum degrees ALL element. divisors}}\,,$$

i.e., we have obtained

### Theorem (Index Sum Theorem for Polynomial Matrices)

*Let $\delta(P)$ be the sum of the degrees of all the elementary divisors (finite and infinite) of an arbitrary polynomial matrix $P(\lambda)$, and let $\alpha(P)$ denote the sum of its left and right minimal indices. Then*

$$\delta(P) + \alpha(P) = \textit{degree}(P) \cdot \textit{rank}(P).$$

### Remarks on degree$(P)$ and rank$(P)$

- They do not appear explicitly in Paul's formulation $\delta_p(R) = \delta_z(R) + \alpha(R)$
- simply because of the differences in the definitions of structures at $\infty$.
- In Paul's formulation, the degree of the polynomial part is minus the smallest structural index at infinity, so the degree is present there.
- Where is the rank? In the length of the lists of structural indices and/or in the lengths of the lists of left and right minimal indices (by using the size of $R$).

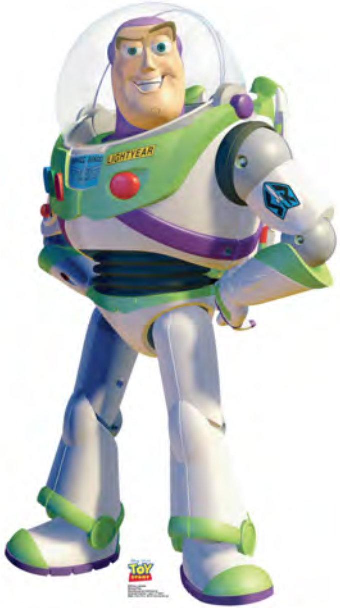
## Outline

1. Basic concepts on rational matrices
2. A few words on polynomial matrices
3. From Paul's to Polynomial Index Sum Theorem
4. **...and beyond**

# There are still many challenging adventures to live related to Paul's result

and I guess that in all of them the structure at infinity of rational matrices will pose very hard and difficult obstacles. Therefore, now, that Paul will have more time, we need his help to travel and I guess that in all of them the structure at infinity of rational matrices will pose very hard and difficult obstacles. Therefore, now, that Paul will have more time, we need his help to travel

**To infinity... and beyond!**



The Index Sum Theorem for Polynomial Matrices has been fundamental

1. for solving the most general possible form of inverse eigenstructure problem for polynomial matrices with prescribed degree (De Terán, D, Van Dooren, SIMAX, 2015), leading to the Fundamental Realization Theorem (recently baptized by Steve Mackey);
2. for describing a complete stratification of arbitrary polynomial matrices (related to the Fundamental Realization Theorem above, ), i.e., for characterizing all possible eigenstructures nearby a given matrix polynomial and their hierarchy of "genericities" (A. Dmytryshyn, S. Johansson, B. Kågström, and P. Van Dooren, in preparation; S. Johansson, B. Kågström, and P. Van Dooren, LAA, 2013);
3. for proving that many structured classes of matrix polynomials with **even degree** contain polynomials that cannot be "strongly linearized" via a pencil having the same structure (De Terán, D, Mackey, LAA, 2014).

**The proofs of 1 and 2 are very far from trivial.**

- Since we have Paul's Index Sum Theorem available for arbitrary rational matrices, the following question arises naturally:
- Can the results in the previous slide be generalized to **arbitrary rational matrices** by replacing the conditions coming from the Index Sum Theorem for polynomial matrices just by Paul's Index Sum Theorem?

**Let us develop a bit more only one of such potential generalizations. For instance:**

## The fundamental realization theorem for polynomial matrices is

### Theorem (De Terán, D, Van Dooren, SIMAX, (2015))

*Consider that the following data*

- $m$, $n$, $d$, *and* $r \leq \min\{m, n\}$ *positive integers,*
- $r$ *scalar monic polynomials such that* $p_1(\lambda) | p_2(\lambda) | \cdots | p_r(\lambda)$,
- $0 = \gamma_1 \leq \cdots \leq \gamma_r$ *integers,*
- $0 \leq \varepsilon_1 \leq \cdots \leq \varepsilon_{n-r}$ *and* $0 \leq \eta_1 \leq \cdots \leq \eta_{m-r}$ *integers*

*are prescribed. Then, there exists an* $m \times n$ *polynomial matrix* $P(\lambda)$, *with rank* $r$, *with degree* $d$, *with invariant polynomials* $p_1(\lambda), \ldots, p_r(\lambda)$, *with partial multiplicities at infinity* $\gamma_1, \cdots, \gamma_r$, *and with right and left minimal indices equal to* $\varepsilon_1, \cdots, \varepsilon_{n-r}$ *and* $\eta_1, \cdots, \eta_{m-r}$, *respectively, if and only if*

$$\sum_{j=1}^{r} \mathrm{degree}(p_j) + \sum_{j=1}^{r} \gamma_j + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j = dr.$$

## or, in plain words,

**if and only if the prescribed data satisfy the Index Sum Theorem for polynomial matrices.**

## Conjectured!!! The fundamental realization theorem for rational matrices

**Theorem (???, ???, Van Dooren, ???, (201?))**

*Consider that the following data $\mathcal{L}$*

- $m$, $n$, *and* $r \le \min\{m, n\}$ *positive integers,*
- $r$ *(monic) irreducible fractions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$, *such that* $\varepsilon_1(\lambda) \mid \cdots \mid \varepsilon_r(\lambda)$ *and* $\psi_r(\lambda) \mid \cdots \mid \psi_1(\lambda)$,
- $\gamma_1 \le \cdots \le \gamma_r$ *integers (sequence of potential structural indices at infinity),*
- $0 \le \varepsilon_1 \le \cdots \le \varepsilon_{n-r}$ *and* $0 \le \eta_1 \le \cdots \le \eta_{m-r}$ *integers*

*are prescribed and let* $\delta_p(\mathcal{L})$ *and* $\delta_z(\mathcal{L})$ *be the polar and the zero degrees of* $\mathcal{L}$. *Then, there exists an* $m \times n$ *rational matrix* $R(\lambda)$, *with rank* $r$, *with finite invariant rational functions* $\frac{\varepsilon_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\varepsilon_r(\lambda)}{\psi_r(\lambda)}$, *with sequence of structural indices at infinity* $\gamma_1, \cdots, \gamma_r$, *and with right and left minimal indices equal to* $\varepsilon_1, \cdots, \varepsilon_{n-r}$ *and* $\eta_1, \cdots, \eta_{m-r}$, *respectively, if and only if*

$$\delta_p(\mathcal{L}) = \delta_z(\mathcal{L}) + \sum_{j=1}^{n-r} \varepsilon_j + \sum_{j=1}^{m-r} \eta_j .$$

**if and only if the prescribed data satisfy Paul Van Dooren's (established 1979) Index Sum Theorem for arbitrary rational matrices.**

- This would be an extremely elegant, easy to state, and powerful result,
- one of those results that many researchers in Mathematics would like to prove,
- and that, probably, would be the starting point to obtain several other fundamental results.
- However, to realize **the structure at infinity** seems very hard,
- therefore, to attack this problem is recommendable only for very strong researchers, with a deep background on rational and polynomial matrices, having considerable time, and a lot of energy..., i.e.,

**it is a problem for Paul in his new life as Emeritus Professor.**