Matrix polynomials with bounded rank and degree: generic eigenstructures and explicit descriptions

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Setting (I): The ambient vector space and its metric

- We will consider the vector space

$$\mathrm{POL}_d^{m\times n} := \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \end{array} \right\}.$$

- The Euclidean distance in $\mathrm{POL}_d^{m\times n}$ is defined as follows. Given

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0 \in \mathrm{POL}_d^{m\times n}, \quad (P_i \in \mathbb{C}^{m\times n})$$

$$Q(\lambda) = \lambda^d Q_d + \cdots + \lambda Q_1 + Q_0 \in \mathrm{POL}_d^{m\times n}, \quad (Q_i \in \mathbb{C}^{m\times n}),$$

$$\rho(P,Q) := \sqrt{\sum_{i=0}^{d} \|P_i - Q_i\|_F^2}\,.$$

- It makes $\mathrm{POL}_d^{m\times n}$ a metric space and we can consider closures of subsets of $\mathrm{POL}_d^{m\times n}$, as well as any other topological concept.
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- where r is a fixed positive integer such that
  - $r \leq \min\{m,n\}$, if $m \neq n$,
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- This means that we consider sets of singular polynomials.
- The set $\mathrm{POL}_{d,r}^{m\times n}$ contains matrix polynomials with many different properties, but **generically** (most of the times) the matrix polynomials of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures.
- In this talk, **generically** means that "all the matrix polynomials in an open dense subset of $\mathrm{POL}_{d,r}^{m\times n}$ have just a few possible eigenstructures",
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Generically a matrix polynomial in $\mathrm{POL}_{d,r}^{m\times n}$

- does not have eigenvalues,
- that is, its eigenstructure has only minimal indices,
- its left minimal indices differ at most by one (they try to be as equal as possible),
- and, its right minimal indices also differ at most by one (they also try to be as equal as possible).

Of course, these properties do not tell the whole story since the generic possible values of the minimal indices have to be determined.

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- from many different perspectives: generic, nongeneric, for several classes of structured matrices and pencils, etc.
- Some names involved in this research area are: Batzke, De Terán, Dodig, D., Hörmander, Mehl, Mehrmann, Melin, Moro, Ran, Rodman, Savchenko, Wojtylak,...
- However, there are essentially no papers on the effect of "low rank" perturbations on the eigenstructure of **matrix polynomials of given degree** (for instance, quadratic polynomials),
- which is a more difficult problem.
- I think that such difficulty is related to the fact that for any fixed low rank the structure of the set of matrix polynomials that have (at most) that given rank and a certain bounded degree is not well understood.
- The results in this talk are a first step in this direction.

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Outline

1. Preliminaries: the result for pencils
2. The main results for matrix polynomials of degree at most d
3. Full rank rectangular matrix polynomials of degree at most d
4. Skew-symmetric matrix polynomials of degree at most d (d odd)
5. Explicit descriptions as products of two factors

Outline

- **All the pencils with the same complete eigenstructure form an orbit** under strict equivalence:

$$\mathrm{O}(\lambda A + B) := \{P(\lambda A + B)Q \mid \det P \cdot \det Q \neq 0\}.$$

- The complete eigenstructure of a pencil is determined by its **Kronecker canonical form (KCF)** under strict equivalence, which a direct sum of four types of canonical matrix pencils:
- the regular $k \times k$ Jordan blocks for **finite and infinite eigenvalues**

$$\mathcal{J}_k(\mu) := \begin{bmatrix} \lambda - \mu & 1 & & \\ & \lambda - \mu & \ddots & \\ & & \ddots & 1 \\ & & & \lambda - \mu \end{bmatrix}, \quad \mathcal{J}_k(\infty) := \begin{bmatrix} 1 & \lambda & & \\ & 1 & \ddots & \\ & & \ddots & \lambda \\ & & & 1 \end{bmatrix} \quad k = 1, 2, 3, \dots$$

- the singular $k \times (k+1)$ and $(k+1) \times k$ blocks for right and left **minimal indices** of value k

$$\mathcal{L}_k := \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \end{bmatrix}, \quad \mathcal{L}_k^T, \quad k = 0, 1, 2, \dots$$

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The set of matrix pencils with rank at most r

Theorem (De Terán and D., SIMAX, 2008)

Let m, n, and r be integers such that $m, n \geq 2$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{1,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix pencils} \\ \text{with rank at most } r \end{array} \right\} = \bigcup_{0\leq a\leq r} \overline{\mathrm{O}}(\mathcal{K}_a),$$

where the $m \times n$ complex matrix pencils $\mathcal{K}_a, a = 0, 1, \ldots, r$, have rank r and the KCF

$$\mathcal{K}_a = \mathrm{diag}\Bigg(\overbrace{\underbrace{\underbrace{\mathcal{L}_{\alpha+1}, \ldots, \mathcal{L}_{\alpha+1}}_{s}, \underbrace{\mathcal{L}_{\alpha}, \ldots, \mathcal{L}_{\alpha}}_{n-r-s}}_{\text{rank}=a}}^{\text{right minimal indices}}, \overbrace{\underbrace{\underbrace{\mathcal{L}_{\beta+1}^T, \ldots, \mathcal{L}_{\beta+1}^T}_{t}, \underbrace{\mathcal{L}_{\beta}^T, \ldots, \mathcal{L}_{\beta}^T}_{m-r-t}}_{\text{rank}=r-a}}^{\text{left minimal indices}} \Bigg)$$

with $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
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Moreover, $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \mathrm{O}(\mathcal{K}_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(\mathcal{K}_a) \bigcap \overline{\mathrm{O}}(\mathcal{K}_{a'}) \neq \varnothing$).

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496

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$\bigcup_{0\leq a\leq r} \mathrm{O}(\mathcal{K}_a)$ is an open dense subset of $\mathrm{POL}_{1,r}^{m\times n}$ (in the topology of $\mathrm{POL}_{1,r}^{m\times n}$). So, **generically, the $m \times n$ pencils with rank at most r have only $r+1$ possible KCFs given by $\mathcal{K}_a$ for $a = 0, 1, \ldots, r$.**

Outline

$$P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0, \quad P_i \in \mathbb{C}^{m\times n} \text{ and } P_d \neq 0.$$

- **Essentially the same as in pencils** but definitions more complicated since there is NOT KCF.
- **Finite and infinite eigenvalues and their elementary divisors** defined with Smith Form under unimodular equivalence of $P(\lambda)$ and rev$P(\lambda)$:

$$U(\lambda)P(\lambda)V(\lambda) = \text{diag}\,(g_1(\lambda), \dots, g_r(\lambda)) \oplus 0_{(m-r)\times(n-r)}, \quad g_j(\lambda) \mid g_{j+1}(\lambda).$$

  Invariant polynomials: $g_j(\lambda) = (\lambda - \alpha_1)^{\delta_{j1}} \cdot (\lambda - \alpha_2)^{\delta_{j2}} \cdot \ldots \cdot (\lambda - \alpha_{l_j})^{\delta_{jl_j}}$.
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- **The definition of orbit does not involve a group action**

$$\mathrm{O}(P) = \left\{ \begin{array}{c} \text{matrix polynomials of the same size, degree,} \\ \text{and with the same complete eigenstructure as } P(\lambda) \end{array} \right\}$$

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The set of matrix polynomials with degree at most d and rank at most r

Theorem (Dmytryshyn and D., submitted, 2016)

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$ and $1 \leq r \leq \min\{m, n\} - 1$. Then

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m \times n \text{ complex matrix polynomials} \\ \text{with degree at most } d \text{, with rank at most } r \end{array} \right\} = \bigcup_{0 \leq a \leq rd} \overline{\mathrm{O}}(K_a),$$

where the $m \times n$ complex matrix polynomial K_a, $a = 0, 1, \ldots, rd$, has

- *degree exactly d, rank exactly r, and*
- *the complete eigenstructure*

$$\mathbf{K}_a : \{\overbrace{\underbrace{\alpha+1, \ldots, \alpha+1}_{s}, \underbrace{\alpha, \ldots, \alpha}_{n-r-s}}^{\text{right minimal indices}}, \overbrace{\underbrace{\beta+1, \ldots, \beta+1}_{t}, \underbrace{\beta, \ldots, \beta}_{m-r-t}}^{\text{left minimal indices}}\},$$

where $\alpha = \lfloor a/(n-r) \rfloor$ and $s = a \mod (n-r)$,
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Moreover, $\overline{\mathrm{O}}(K_a) \cap \mathrm{O}(K_{a'}) = \varnothing$ whenever $a \neq a'$ (but $\overline{\mathrm{O}}(K_a) \cap \overline{\mathrm{O}}(K_{a'}) \neq \varnothing$).

Theorem (Dmytryshyn and D., submitted, 2016)

Let m, n, r *and* d *be integers such that* $m, n \geq 2$, $d \geq 1$ *and* $1 \leq r \leq \min\{m, n\} - 1$. *Then*

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generically, the $m \times n$ matrix polys with degree at most d and with rank at most r have only $rd + 1$ possible complete eigenstructures given by $\mathbf{K}_a$ for $a = 0, 1, \ldots, rd$.

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$, and $1 \leq r \leq \min\{m, n\} - 1$.

Corollary

For every $M \in \mathrm{POL}_{d,r}^{m \times n}$ and every $\varepsilon > 0$ there exists $M' \in \mathrm{POL}_{d,r}^{m \times n}$ such that

1. *M' has the complete eigenstructure $\mathbf{K}_a$ for some $a \in \{0, 1, \dots, rd\}$ and*
2. *$d(M, M') < \varepsilon$.*

Corollary

Let $a \in \{0, 1, \dots, rd\}$. Then for every $M' \in \mathrm{POL}_{d,r}^{m \times n}$ with the complete eigenstructure $\mathbf{K}_a$, there exists $\varepsilon > 0$ such that all the matrix polynomials in

$$\mathcal{B}_r(M'; \varepsilon) := \left\{ M \, : \, M \in \mathrm{POL}_{d,r}^{m \times n} \quad \text{and} \quad d(M', M) < \varepsilon \right\}$$

have complete eigenstructure $\mathbf{K}_a$.

Corollary 1 of previous MAIN theorem: Analytical interpretation

Let m, n, r and d be integers such that $m, n \geq 2$, $d \geq 1$, and $1 \leq r \leq \min\{m, n\} - 1$.

Corollary

For every $M \in \mathrm{POL}_{d,r}^{m \times n}$ and every $\varepsilon > 0$ there exists $M' \in \mathrm{POL}_{d,r}^{m \times n}$ such that

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Let $a \in \{0, 1, \ldots, rd\}$. Then for every $M' \in \mathrm{POL}_{d,r}^{m \times n}$ with the complete eigenstructure $\mathbf{K}_a$, there exists $\varepsilon > 0$ such that all the matrix polynomials in

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Corollary 2 of MAIN theorem: the set of SQUARE singular matrix polynomials with degree at most d

Remark: an $n \times n$ matrix polynomial is singular if and only if its rank is a most $n-1$.

Corollary (The main theorem with $m = n$ and $r = n-1$)

$$\left\{ \begin{array}{c} \text{singular } n \times n \text{ complex matrix} \\ \text{polynomials of degree at most } d \end{array} \right\} = \bigcup_{0 \le a \le (n-1)d} \overline{\mathrm{O}}(K_a),$$

where the complete eigenstructure of each of the matrix polynomials $K_a, a = 0, 1, \dots, (n-1)d$, has

- *no elementary divisors (no eigenvalues);*
- *only one left minimal index equal to $nd - d - a$;*
- *only one right minimal index equal to a.*

This corollary extends the classical result for pencils:

W. Waterhouse, The codimension of singular matrix pairs, Linear Algebra Appl., 57 (1984) 227–245

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where the complete eigenstructure of each of the matrix polynomials $K_a, a = 0, 1, \ldots, (n-1)d$, *has*

- *no elementary divisors (no eigenvalues);*
- *only one left minimal index equal to* $nd - d - a$*;*
- *only one right minimal index equal to* a.

This corollary extends the classical result for pencils:

W. Waterhouse, The codimension of singular matrix pairs, Linear Algebra Appl., 57 (1984) 227–245

Corollary 2 of MAIN theorem: the set of SQUARE singular matrix polynomials with degree at most d

Remark: an $n \times n$ matrix polynomial is singular if and only if its rank is a most $n-1$.

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Example: Singular 2×2 complex matrix polynomials of degree at most 3

Example ($m = n = 2, \quad r = 1, \quad d = 3$)

$$\left\{ \begin{array}{c} \text{singular } 2 \times 2 \text{ complex matrix} \\ \text{polynomials of degree at most } 3 \end{array} \right\} = \bigcup_{0 \le a \le 3} \overline{\mathrm{O}}(K_a)$$

each of the matrix polynomials K_0, K_1, K_2, and K_3 has

- no elementary divisors;
- one left minimal index equal to $3 - a$;
- one right minimal index equal to a.

$$\mathbf{K}_0 : \{ \underbrace{3}_{left}, \underbrace{0}_{right} \} \qquad \mathbf{K}_1 : \{ \underbrace{2}_{left}, \underbrace{1}_{right} \}$$

$$\mathbf{K}_2 : \{ \underbrace{1}_{left}, \underbrace{2}_{right} \} \qquad \mathbf{K}_3 : \{ \underbrace{0}_{left}, \underbrace{3}_{right} \}$$

Comments on the proof of the main theorem (I)

- The proof is delicate.
- Of course, it relies on the corresponding result for pencils (De Terán & D., SIMAX, 2008) and uses heavily the first Frobenius companion strong linearization of matrix polynomials,
- but also several key results for matrix polynomials that have been developed very recently (or, rescued and improved from "old" references). We emphasize the following ones:
- Necessary and sufficient conditions for a matrix polynomial with prescribed degree and complete eigenstructure to exist.

F. De Terán, F.M. Dopico, and P. Van Dooren, Matrix polynomials with completely prescribed eigenstructure, SIAM J. Matrix Anal. Appl., 36 (2015) 302–328.

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F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations for rectangular matrix polynomials, Linear Algebra Appl., 437 (2012) 957–991.

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- Correspondence between perturbations of certain strong linearizations of matrix polynomials and the matrix polynomials themselves.

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- In addition to all these results, it is needed a delicate translation of results in the Euclidean topology of the set of $(m+n(d-1)) \times nd$ pencils and of the subset of pencils formed by the first Frobenius companion forms of all the $m \times n$ matrix polynomials of grade d into the Euclidean topology of the set of $m \times n$ matrix polynomials of degree at most d.

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Outline

The set $\mathrm{POL}_d^{m\times n}$ when $m < n$

- In this case, the set $\mathrm{POL}_d^{m\times n}$ is equal to $\mathrm{POL}_{d,m}^{m\times n}$, i.e., the set of matrix polynomials of rank at most m,
- but main result assumes (and uses) $r \leq \min\{m,n\} - 1$. Nevertheless,
- since all the matrix polynomials in $\mathrm{POL}_d^{m\times n}$ are singular, this set can be described using techniques similar to those in main result, but
- a **very important difference** appears: **there is only one generic complete eigenstructure**.

Theorem (Dmytryshyn and D., submitted, 2016)

$$\mathrm{POL}_d^{m\times n} = \overline{\mathrm{O}}(K_{rp}),$$

where K_{rp} is an $m \times n$ complex matrix polynomial of degree exactly d and rank exactly m with the complete eigenstructure

$$\mathbf{K}_{rp} : \{\overbrace{\underbrace{\alpha+1,\ldots,\alpha+1}_{s},\underbrace{\alpha,\ldots,\alpha}_{n-m-s}}^{\text{right minimal indices}}\},$$

with $\alpha = \lfloor md/(n-m) \rfloor$ and $s = md \mod (n-m)$.

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The two previous theorems extend classical results for matrix pencils:

J. Demmel and A. Edelman, The dimension of matrices (matrix pencils) with given Jordan (Kronecker) canonical forms, Linear Algebra Appl., 230 (1995) 61–87

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A few properties of skew symmetric matrix polynomials

- **Definition:** $P(\lambda) = \lambda^d P_d + \cdots + \lambda P_1 + P_0$ with $P_i^T = -P_i \in \mathbb{C}^{n\times n}$.
- Skew-symmetric matrix polynomials with degree at most d form a vector space and we can define on it an Euclidean distance.
- Their rank is always even.
- Their invariant polynomials are paired-up and their left minimal indices are equal to the right ones.
- When the degree is odd, they can be always strongly linearized through a skew-symmetric block-tridiagonal companion form (Mackey et al, LAA, 2013) that allows us to recover via a shift the minimal indices of the polynomial (Dmytryshyn, LAA, 2017).
- The description of the sets of skew-symmetric matrix pencils with bounded ranks was not previously available.
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Theorem (Dmytryshyn and D., submitted, 2017)

Let m, r and d be integers such that $m \geq 2$, $d \geq 1$ is odd, and $2 \leq 2r \leq (m-1)$. Then

$$\left\{ \begin{array}{c} m \times m \text{ complex skew-symmetric matrix polynomials} \\ \text{with degree at most } d \text{ and with rank at most } 2r \end{array} \right\} = \overline{\mathrm{O}}(W),$$

where the $m \times m$ complex skew-symmetric matrix polynomial W has degree exactly d, rank exactly $2r$, and the complete eigenstructure

$$\mathbf{W}: \Big\{ \overbrace{\underbrace{\beta+1, \dots, \beta+1}_{t}, \underbrace{\beta, \dots, \beta}_{m-2r-t}}^{\text{left minimal indices}}, \overbrace{\underbrace{\beta+1, \dots, \beta+1}_{t}, \underbrace{\beta, \dots, \beta}_{m-2r-t}}^{\text{right minimal indices}} \Big\}$$

with $\beta = \lfloor rd/(m-2r) \rfloor$ and $t = rd \mod (m-2r)$.

The effect of imposing structure is dramatic since in the skew-symmetric case **there is only one** generic eigenstructure compared to the $(2r)d+1$ generic eigenstructures of the unstructured case.

The set of skew-symm polys of degree at most d and rank at most $2r$

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Generic eigenstructures of sets of STRUCTURED matrix polynomials with bounded rank and degree: Solved and Open problems

	Pencils	**Polynomials** $\deg > 1$
General	De Terán and D., 2008	Dmytryshyn and D., 2016
Skew-Symmetric	Dmytryshyn and D., 2017	Dmytryshyn and D., 2017 ($\deg$ odd)
T-(anti)palindromic	De Terán, 2017	open
T-even and odd	De Terán, 2017	open
Symmetric	open	open
Hermitian	open	open

F. De Terán and F.M. Dopico, A note on generic Kronecker orbits of matrix pencils with fixed rank, SIAM J. Matrix Anal. Appl., 30 (2008) 491–496.

A. Dmytryshyn and F.M. Dopico, Generic matrix polynomials with fixed rank and fixed degree, Report UMINF 16.19, Department of Computing Science, Umeå University, submitted, 2016.

A. Dmytryshyn and F.M. Dopico, Generic skew-symmetric matrix polynomials with fixed rank and fixed odd grade, Report UMINF 17.07, Department of Computing Science, Umeå University, submitted, 2017.

F. De Terán, A geometric description of sets of structured matrix pencils with bounded rank, submitted, 2017.

Outline

Ideas, difficulties, and simple statement of results

- Any $m \times n$ constant matrix A of rank r is

$$A = LR, \quad \text{where } \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc. Nevertheless,

- generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

$$P(\lambda) = L(\lambda)R(\lambda),$$

where

1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

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1. $L(\lambda)$ is an $m \times r$ matrix polynomial, $\operatorname{rank} L(\lambda) = r$, and degrees of its columns differ at most by one,
2. $R(\lambda)$ is an $r \times n$ matrix polynomial, $\operatorname{rank} R(\lambda) = r$, and degrees of its rows differ at most by one, and
3. $\deg \mathrm{col}_i(L(\lambda)) + \deg \mathrm{row}_i(R(\lambda)) = d$, for $i = 1, \ldots, r$.

- Any $m \times n$ constant matrix A of rank r is

$$A = LR, \quad \text{where} \begin{cases} L \text{ is } m \times r \text{ and } \operatorname{rank} L = r, \\ R \text{ is } r \times n \text{ and } \operatorname{rank} R = r. \end{cases}$$

- The idea is to get a similar description of $\mathrm{POL}_{d,r}^{m\times n}$ but the degree of the factors makes the problem not trivial: it might be cancellations of "high degrees", how to distribute degrees between the factors, etc. Nevertheless,
- generically, i.e., using closures of open dense sets, we can prove that if $P(\lambda) \in \mathrm{POL}_{d,r}^{m\times n}$

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where

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Theorem (Dmytryshyn, D., and Van Dooren, in progress, 2017)

$$\mathrm{POL}_{d,r}^{m\times n} = \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a},$$

where, for $a = 0, 1, \ldots, rd$,

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) : \begin{array}{l} L(\lambda)\in\mathbb{C}[\lambda]^{m\times r},\ R(\lambda)\in\mathbb{C}[\lambda]^{r\times n}, \\ \deg \mathrm{row}_i(R) = d_R + 1, \quad \text{for } i = 1,\ldots,t_R, \\ \deg \mathrm{row}_i(R) = d_R, \quad \text{for } i = t_R+1,\ldots,r, \\ \deg \mathrm{col}_i(L) = d - \deg(R_{i*}), \quad \text{for } i = 1,\ldots,r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$. *Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(\mathcal{K}_a),$$

where $\mathcal{K}_a$ *are the* $m\times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the first part of the talk.*

...and more next year. **Thank you for your attention!**

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The precise result

Theorem (Dmytryshyn, D., and Van Dooren, in progress, 2017)

$$\mathrm{POL}^{m\times n}_{d,r} = \left\{ \begin{array}{c} m\times n \text{ complex matrix polynomials} \\ \text{with degree at most } d, \text{ with rank at most } r \end{array} \right\} = \bigcup_{0\le a\le rd} \overline{\mathcal{B}_a},$$

where, for $a = 0, 1, \ldots, rd$,

$$\mathcal{B}_a := \left\{ L(\lambda)R(\lambda) \; : \; \begin{array}{l} L(\lambda) \in \mathbb{C}[\lambda]^{m\times r},\; R(\lambda) \in \mathbb{C}[\lambda]^{r\times n}, \\ \deg \textit{row}_i(R) = d_R + 1, \quad \text{for } i = 1, \ldots, t_R, \\ \deg \textit{row}_i(R) = d_R, \quad \text{for } i = t_R + 1, \ldots, r, \\ \deg \textit{col}_i(L) = d - \deg(R_{i*}), \quad \text{for } i = 1, \ldots, r \end{array} \right\},$$

with $d_R = \lfloor a/r \rfloor$ *and* $t_R = a \bmod r$. *Moreover,*

$$\overline{\mathcal{B}_a} = \overline{\mathrm{O}}(K_a),$$

where K_a *are the* $m\times n$ *matrix polynomials of degree exactly* d *and rank exactly* r *with the generic eigenstructures defined in the first part of the talk.*

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